1926/1927

# ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)


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PUBLISHED BY THE

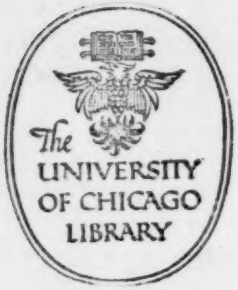
PRINCETON UNIVERSITY PRESS

SECOND SERIES, VOL. 28

PRINCETON, N. J.

1927

QA1
.A65
ser. 2
v. 28
cop. 2

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LÜTCKE & WULFF, HAMBURG, GERMANY

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# ON THE PROBLEM OF COLORING MAPS IN FOUR COLORS, I.*

By C. N. Reynolds, Jr.

**Introduction.** The problem of coloring in four colors the map of a simply connected closed surface has been reduced to the problem of coloring maps in which certain configurations, known as reducible configurations, are absent.† In this paper we shall develope some methods of so analysing the known geometric reductions of our problem as to discover and to prove some of their more important implications.

We shall use the reductions due to Kempe, Birkhoff and Franklin together with Errera's reduction of pairs of adjacent pentagons surrounded by hexagons. The reductions of Kempe, Birkhoff and Franklin are sufficient hypotheses for our final theorem that any irreducible map has at least twenty-eight regions. The use of one of Errera's reductions simplifies the proof of that theorem by eliminating much of the case making which would otherwise be necessary.

Our fundamental method will be a systematic study of a set of geometric operations which suffice to build any connected configuration of pentagons which may exist in an irreducible map. Under these operations certain numerical topological characteristics are found to undergo well defined increments. Linear relations between these increments imply homogeneous linear difference equations which yield certain homogeneous relations between our topological characteristics. These relations were originally found inductively from a study of particular configurations.

Secondly we shall prove synthetically certain inequalities between the topological characteristics of an irreducible map, making use of the linear relations mentioned above to reduce them to more serviceable forms.

We shall then apply these inequalities to the proof of the theorem that any irreducible map must have at least twenty-eight regions.

Finally we shall describe: (a) A map of twenty-eight regions which is irreducible with respect to the reductions of Kempe, Birkhoff and Franklin, but reducible with respect to the reduction due to Errera which we mentioned above. This map proves that what we may call the pre-Errera

* Presented, in part, to the American Mathematical Society, Feb. 28, 1925.

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We shall use the reductions due to Kempe, Birkhoff and Franklin together with Errera's reduction of pairs of adjacent pentagons surrounded by hexagons. The reductions of Kempe, Birkhoff and Franklin are sufficient hypotheses for our final theorem that any irreducible map has at least twenty-eight regions. The use of one of Errera's reductions simplifies the proof of that theorem by eliminating much of the case making which would otherwise be necessary.

Our fundamental method will be a systematic study of a set of geometric operations which suffice to build any connected configuration of pentagons which may exist in an irreducible map. Under these operations certain numerical topological characteristics are found to undergo well defined increments. Linear relations between these increments imply homogeneous linear difference equations which yield certain homogeneous relations between our topological characteristics. These relations were originally found inductively from a study of particular configurations.

Secondly we shall prove synthetically certain inequalities between the topological characteristics of an irreducible map, making use of the linear relations mentioned above to reduce them to more serviceable forms.

We shall then apply these inequalities to the proof of the theorem that any irreducible map must have at least twenty-eight regions.

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reductions do not imply that the number of regions in an irreducible map is greater than twenty-eight.

(b) A map of thirty regions, reducible only by means of a reduction of Errera's which we do not use in our paper. This map proves that, under the reductions which we use in the proof of this paper, we have come within two units of determining the minimum number of regions in an irreducible map.

(c) A map of thirty-six regions, irreducible with respect to all of the reductions now published (May 1926).

1. **Notation.** Since Birkhoff has shown that a boundary completely surrounded by pentagons is reducible,* no pentagon may be in contact with more than three other pentagons. Therefore every pentagon of a connected configuration of pentagons must have at least two "exposed" sides, i. e. sides in contact with non-pentagons. This implies that a connected configuration of pentagons may be treated, in some respects, as a linear graph and, like linear graphs, such a configuration or set of configurations will have a cyclomatic number, $\mu$, in the sense defined by Veblen.† In addition to the symbol $\mu$ we shall use certain other symbols to be defined below. In general we shall use capital letters to denote certain types of configurations and the corresponding small letters to denote the number of such configurations in an irreducible map.

$A$ shall denote a region of our map.

$A_n$ shall denote a region of $n$ sides.

$E_{mn}$ shall denote a boundary between an $m$-sided region and an $n$-sided region.

$P_1$ shall denote a pentagon in contact with no other pentagon.

$P_2$ shall denote a pentagon in contact with one other pentagon.

$P_3$ shall denote a pentagon meeting two other pentagons at a point.

$P_4$ shall denote a pentagon meeting two other pentagons along non-consecutive edges.

$P_5$ shall denote a pentagon in contact with three other pentagons, no edge being surrounded by four pentagons.

$T$ shall denote a point where three pentagons meet.

$T_n$ shall denote a $T$ where $n$ $P_5$'s and $(3-n)$ $P_3$'s meet $(n = 0, 1, 2, 3)$.

$D$ shall denote a pair of $P_2$'s in contact with one another.

$Y_n$ shall denote a contact between a non-pentagon and the edges of $n$ consecutive pentagons of a connected configuration of pentagons.

$J$ shall denote an $E_{5,5}$ joining a $Y_n (n \geqq 3)$ to another $Y_{n'} (n' \geqq 3)$.

$J_{5,5}$ shall denote a $J$ lying between two $P_5$'s.

* cf. Franklin, l. c. Reduction number 4, p. 225.

† "Analysis Situs", Cambridge Colloquium Lectures, p. 9.

$J'_{5,5}$ shall denote a $J_{5,5}$ lying between two $T_1$'s.
$P_{5,4,5}$ shall denote a $P_4$ lying between two $P_5$'s.
$P'_{5,4,5}$ shall denote a $P_{5,4,5}$ lying between two $T_1$'s.
$c$ shall denote the number of connected configurations of pentagons in our map.
$\lambda$ shall denote the expression: $3p_1+p_2+2y_3+y_4-2(c-\mu)$.

With the exception of the last two, these definitions are arranged approximately in the order in which the symbols appear in this article.

**2. The systematic construction of sets of irreducible connected configurations of pentagons.** Any set of $c$ connected configurations of pentagons can be built up by some, not uniquely defined, sequence of operations of the types given below, each operation $X_n$ being repeated $x_n$ times during the process.

$X_1$: The initial operation of taking a $P_1$ with which to start our construction. This operation must be performed $c$ times.
$X_2$: The adjunction of a $P_2$ to a given $P_1$. This operation must follow $X_1$ if our configuration is not to be an isolated pentagon.
$X_3$: The adjunction of a $P_3$ to a pair of $P_2$'s. This operation cannot occur later than the third step in the construction of any connected configuration of pentagons.
$X_4$: The adjunction of a $P_2$ to a $P_2$.
$X_5$: The adjunction of a $P_2$ to a $P_3$.
$X_6$: The adjunction of a $P_3$ to a pair of $P_4$'s.
$X_7$: The adjunction of a $P_3$ to a $P_2$ and an adjoining $P_4$.

If we are building a simply connected configuration the other conceivable adjunctions are precluded by the reducibility of an edge surrounded by pentagons.* If, however, we are building one or more multiply connected configurations then other operations are, at times, necessary. Any configuration of cyclomatic number $\mu$ may be rendered simply connected by means of $\mu$ cross cuts which may always be taken between two $P_4$'s between a $P_4$ and a $P_5$, or between two $P_5$'s. After this cutting process the $P_4$'s become $P_2$'s and the $P_5$'s become $P_3$'s in a simply connected configuration. The inverses of these cutting processes will be the following operations:

$X_8$ : The reunion of two $P_2$'s.
$X_9$ : The reunion of a $P_2$ and a $P_3$.
$X_{10}$: The reunion of two $P_3$'s.

For any set of configurations we obviously have $x_8+x_9+x_{10}=\mu$.

Under each of our ten operations, $X_n (n=1, 2, \cdots, 10)$, the previously defined numbers $c$, $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, $\mu$, $a_5$, $e_{5,5}$, $t$, $y_1$, $\sum_1 y_n$, and $\sum_1 n y_n$ will

* Franklin, l. c. Reduction number 4, p. 225.

take on certain increments, indicated by the symbolic operator $\Delta_n$, which are given in the following table:

| $n$ | $\Delta_n c$ | $\Delta_n p_1$ | $\Delta_n p_2$ | $\Delta_n p_3$ | $\Delta_n p_4$ | $\Delta_n p_5$ | $\Delta_n \mu$ | $\Delta_n a_5$ | $\Delta_n e_{5,5}$ | $\Delta_n t$ | $\Delta_n y_1$ | $\Delta_n \sum_1 y_n$ | $\Delta_n \sum_1 n y_n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 5 | 5 | 5 |
| 2 | 0 | −1 | 2 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | −1 | 1 | 3 |
| 3 | 0 | 0 | −2 | 3 | 0 | 0 | 0 | 1 | 2 | 1 | −1 | 0 | 1 |
| 4 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 3 |
| 5 | 0 | 0 | 1 | −1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 3 |
| 6 | 0 | 0 | 0 | 1 | −2 | 2 | 0 | 1 | 2 | 1 | 1 | 0 | 1 |
| 7 | 0 | 0 | −1 | 2 | −1 | 1 | 0 | 1 | 2 | 1 | 0 | 0 | 1 |
| 8 | 0 | 0 | −2 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | −4 | −4 | −2 |
| 9 | 0 | 0 | −1 | −1 | 1 | 1 | 1 | 0 | 1 | 0 | −3 | −4 | −2 |
| 10 | 0 | 0 | 0 | −2 | 0 | 2 | 1 | 0 | 1 | 0 | −2 | −4 | −2 |

These increments follow immediately from the definitions of our operations and symbols. In a series of operations of these ten types we will have a total increment indicated by the symbolic operator $\Delta = \sum_1^{10} x_n \Delta_n$. In our table we have a matrix of ten rows and thirteen columns. The last seven columns are each linearly dependent upon the first six columns. This linear dependence gives rise to the following set of difference equations:

$$
\begin{aligned}
6\Delta(c-\mu) &= 6\Delta p_1 + 3\Delta p_2 + 2\Delta p_3 - \Delta p_5, \\
\Delta a_5 &= \Delta p_1 + \Delta p_2 + \Delta p_3 + \Delta p_4 + \Delta p_5, \\
2\Delta e_{55} &= \Delta p_2 + 2\Delta p_3 + 2\Delta p_4 + 3\Delta p_5, \\
3\Delta t &= \Delta p_3 + \Delta p_5, \\
\Delta y_1 &= 5\Delta p_1 + 2\Delta p_2 + \Delta p_3, \\
\Delta \sum_1 y_n &= 5\Delta p_1 + 3\Delta p_2 + 2\Delta p_3 + \Delta p_4, \\
\Delta \sum_1 n y_n &= 5\Delta p_1 + 4\Delta p_2 + 3\Delta p_3 + 3\Delta p_4 + 2\Delta p_5.
\end{aligned}
$$

Since any set of one or more configurations of pentagons can be formed by means of a sequence of the operations, $X_n$, we may therefore sum our difference equations over such a sequence of operations. The initial values of each of our variables being zero, the constants of summation will vanish, and we have:

$$
\begin{aligned}
&(1) \quad 6(c-\mu) = 6p_1 + 3p_2 + 2p_3 - p_5, \\
&(2) \quad a_5 = p_1 + p_2 + p_3 + p_4 + p_5, \\
&(3) \quad 2e_{55} = p_2 + 2p_3 + 2p_4 + 3p_5, \\
&(4) \quad 3t = p_3 + p_5,
\end{aligned}
$$

$$y_1 = 5p_1 + 2p_2 + p_3, \tag{5}$$

$$\sum_1 y_n = 5p_1 + 3p_2 + 2p_3 + p_4, \tag{6}$$

$$\sum_1 n y_n = 5p_1 + 4p_2 + 3p_3 + 3p_4 + 2p_5. \tag{7}$$

Whence we have:

$$t = (c-\mu) - a_5 + e_{55}, \tag{8a}$$

$$\sum_1 y_n = 3(c-\mu) + 2a_5 - e_{55}, \tag{8b}$$

$$\sum_1 n y_n = 5a_5 - 2e_{55}. \tag{8c}$$

There also exist four linear relations between the rows of our matrix, which is of rank six, the first five and the eigth rows being linearly independent. These four relations may be written in the form:

$$\Delta_5 - \Delta_4 \equiv \Delta_6 - \Delta_7 \equiv \Delta_7 - \Delta_3 \equiv \Delta_{10} - \Delta_9 \equiv \Delta_9 - \Delta_8.$$

The total increment due to any set of $x_n$ operations of the types $X_n$ $(n = 1, 2, 3, \cdots, 10)$ will then be

$$\Delta = \sum_1^{10} x_n \Delta_n = x_1\Delta_1 + x_2\Delta_2 + (x_3 + x_6 + x_7)\Delta_3 + (x_4 - 2x_6 - x_7 - x_9 - 2x_{10})\Delta_4 + (x_5 + 2x_6 + x_7 + x_9 + 2x_{10})\Delta_5 + (x_8 + x_9 + x_{10})\Delta_8.$$

By properly restricting the order in which our ten operations are performed, it would doubtless be possible to restrict ourselves to six linearly independent sequences of them. I have not done this.

Solving equations (1), (2), (3) and (5) for $p_2$, $p_3$, $p_4$ and $p_5$ in terms of $(c-\mu-p_1)$, $(a_5-p_1)$, $e_{55}$ and $(y_1-5p_1)$, we have:

$$p_2 = -3(c-\mu-p_1) + (a_5-p_1) - e_{55} + (y_1-5p_1), \tag{9a}$$

$$p_3 = 6(c-\mu-p_1) - 2(a_5-p_1) + 2e_{55} - (y_1-5p_1), \tag{9b}$$

$$p_4 = 3(a_5-p_1) - 2e_{55} - (y_1-5p_1), \tag{9c}$$

$$p_5 = -3(c-\mu-p_1) - (a_5-p_1) + e_{55} + (y_1-5p_1). \tag{9d}$$

This set of relations will be found useful later in reducing inequalities to convenient forms.

3. **Derivation of inequalities, applicable to all irreducible maps.** In this section we shall derive, synthetically, a group of inequalities which hold between the topological characteristics of an irreducible map. It will be found convenient to write inequalities in the form $x = y + c_n$, where $c_n$ is an integer, positive or zero, defined in relation number $(n)$. In this way any inequality which is a linear combination of two ore more fundamental inequalities will indicate its own derivation in the linear combination of $c$'s which is involved. These will be marked $(A)$, $(B)$, etc.

We shall have occasion to apply Euler's theorem concerning the vertices edges and faces of a simply connected polyhedron with simply connected faces, and also Kempe's reduction of our problem to maps in which three edges meet at each vertex. These may be expressed by means of the equations:*

$$a_5 = 12 - \sum_7 (n-6)\, a_n, \tag{10a}$$

$$a_6 = a - 12 - \sum_7 (n-5)\, a_n. \tag{10b}$$

We shall use these to eliminate $a_5$ and $a_6$ from future relations.

Franklin has proved two inequalities† which may be rewritten, by eliminating $a_5$ and $a_6$, in the form:

$$4a = 72 + 5\sum_7 (n-6)\quad a_n + c_{11}, \tag{11}$$

$$3a = 84 + \sum_7 (5n-37)\, a_n + c_{12}. \tag{12}$$

If in the construction of an irreducible map we suppose that the connected configurations of pentagons are constructed before any non-pentagons are introduced, then the pentagons may be so arranged as to impose restraints upon the non-pentagons. For example, Franklin has proved that a boundary of a hexagon surrounded by the hexagon and three pentagons is a reducible configuration‡, and that a polygon of $n$ sides in contact whith more than $n-2$ pentagons is reducible§. These reductions imply that polygons making $Y_3$ or $Y_4$ contacts must have at least seven sides, and that one making a $Y_n$ contact ($n \geqq 5$) must have at least $n+2$ sides. It may be, however, that a single polygon makes two or more contacts, $Y_{n'}$, $Y_{n''}$, $Y_{n'''}$, $\cdots$, ($n'$, $n''$, $n'''$, $\cdots$, $\geqq 3$), with different configurations of pentagons, or with non-consecutive portions of the boundary of a single configuration of pentagons. In this case our polygon could not have fewer than eight sides.

Thus these reductions of Franklin's impose minima upon the number of sides which a region making given contacts with configurations of pentagons may have. These minima will be called simply or multiply imposed minima according as they are derived from one or more than one $Y_n$. Therefore no multiply imposed minima may exist in a map unless the map contains at least one region with eight or more sides, and no octagon may be required by a multiply imposed minimum unless it makes two $Y_3$ contacts; that is to say, in a map having no multiply imposed minima the reductions mentioned above may be satisfied by completing our map with a set of

* Franklin, l. c., (5), p. 226.

† Franklin, l. c., (22) and (23), p. 235.

‡ Franklin, l. c., p. 229.

§ Franklin, l. c., p. 233.

non-pentagons provided that the number of heptagons is at least equal to $y_8+y_4+y_5$ and the number of polygons of more than seven sides is at least equal to $\sum_6 y_n$.

If we examine the connected configurations of pentagons for which $y_3+y_4+y_5$ is at most equal to four and $y_n = 0$ $(n > 5)$, then we find empirically that in all cases except the $T_0$ configuration, formed by three pentagons meeting at a point, $e_{55}$ does not exceed $3(y_3+y_4+y_5)+1$. We may prove a generalization of this from relations (1), (3), (5), (6), (7) and our definition of $\lambda$. These imply that

$$e_{55} = (c-\mu)+3(y_3+y_4+y_5)+\sum_6(n-2)y_n-\lambda.$$

Then, since each $Y_n(n>4)$ must be associated with a polygon of at least $n+2$ sides, since $Y_3$'s and $Y_4$'s are associated with polygons of at least seven sides, and since in a map having no "multiply imposed minima" the replacement of a polygon with the minimum number of sides imposed by these restraints by some other type of polygon increases $\sum_7(n-4)a_n$, we have the theorem:

*In any irreducible map involving no "multiply imposed minima",*

$$e_{55} = (c-\mu)+\sum_7(n-4)a_n-\lambda-c_{13}. \tag{13}$$

The total number of contacts made by the non-pentagons of any irreducible map will include the exposed edges of pentagons, $\sum_1 n y_n$, and also those edges, lying between non-pentagons, which terminate on pentagons, $\sum_1 y_n$. These edges would be each be counted twice, once for each of the two polygons on either side of it, were it not for the possibility that an edge might have both ends at vertices of pentagons. Hence the most that we can conclude is that $\sum_6 n a_n$ is at least equal to $\sum_1(n+1)y_n$, or, upon applying our earlier identities as before,

$$e_{55} = 52+(c-\mu)+\tfrac{1}{4}\sum_7(n-6)a_n+c_{14}, \tag{14}$$

We shall now define a type of association between polygons of more than six sides and $E_{55}$'s (edges separating pentagons) by imposing restraints upon our idea of association. Associated with a given assortment of polygons of more than six sides there shall be only a limited number of $E_{55}$'s. Whenever a connected configuration of pentagons makes a contact of the type $Y_n(n \geqq 3)$ with a polygon of more than six sides the $n$ pentagons making such a contact are separated by $n-1$ $E_{55}$'s which are to be associated with the non-pentagon making the $Y_n$ contact, The configuration of three pentagons meeting at a point (a $T_0$ in a our notation) must be

so placed that each of its pentagons touches at least one polygon of more than six sides.* Such a contact may be made in either of two ways. A polygon of more than six sides may touch one, or it may touch two pentagons of the configuration. It shall accordingly be associated with one, or with two, of the three $E_{55}$'s of the configuration provided that this may be possible without associating an $E_{55}$ with more than one polygon of more than six sides. Furthermore, Errera has proved† that a pair of adjacent pentagons (a $D$ in our notation) surrounded by hexagons is reducible. In an irreducible map the $E_{55}$ lying between the pentagons of our $D$ will be associated with one of the polygons of more than six sides which must touch such a $D$. Using these restraints as an effective definition of the word "association", when used in this connection, and remembering that no $n$-sided region of our irreducible map may touch more than $n-2$ pentagons,‡ we see that no more than $n-3$ $E_{55}$'s may be associated with any $A_n$ $(n \geqq 7)$ in an irreducible map. Therefore

$$\sum_7 (n-3)\, a_n = d + 3\,t_0 + \sum_3 (n-1)\, y_n + c_{15}.$$

In reducing this relation to a more useful form we first remind the reader that the definition of $t_n$ implies that

$$\begin{aligned} t &= t_0 + t_1 + t_2 + t_3, \\ p_3 &= 3\,t_0 + 2\,t_1 + t_2, \\ p_5 &= t_1 + 2\,t_2 + 3\,t_3. \end{aligned}$$

We also see from the definition of a $J$ that there are no $J$'s in $D$'s nor in $T_0$'s. Again, no $J$ may terminate in a $T_n (n = 1, 2, 3)$ nor in a $Y_2$. Therefore $j$ is equal to the number of $E_{55}$'s in configurations other than $D$'s and $T_0$'s, minus the number of $E_{55}$'s radiating from $T_n$'s $(n = 1, 2, 3)$, minus the number of $Y_2$'s in configurations other than $D$'s and $T_0$'s, plus a term added to correct a double subtraction viz., $t_1$, the number of $E_{55}$'s joining $Y_2$'s and $T_n$'s $(n = 1, 2, 3)$. Hence,

$$j = e_{55} + d - 2\,t_1 - 3\,t_2 - 3\,t_3 - y_2.$$

The last five relations, identities (6) and (7), and the identity

$$\sum_3 (n-1)\, y_n \equiv \sum_1 (n-1) y_n - y_2,$$

imply that

$$e_{55} = \sum_7 (n-3)\, a_n + t_1 - j - c_{15}. \tag{15}$$

* Franklin, l. c., p. 231.

† Errera, l. c., paragraph (2, 4, 2), p. 49.

‡ Franklin, l. c., p. 233.

Except for cases where two $P_5$'s bound one another we have $n$ $J$'s in the immediate neigborhood of each $T_n (n = 0, 1, 2, 3)$ with no $J$'s shared by two $T$'s. Since $j_{55}$ denotes the number of $J$'s lying between two $P_5$'s we have:

(16) $$j = t_1 + 2t_2 + 3t_3 - j_{55} + c_{16}.$$

On the other hand we may in no case have more than two $J_{55}$'s for each $T_2$, three for each $T_3$, and one apiece for each $J'_{55}$ configuration. Therefore,

(17) $$j_{55} = 2t_2 + 3t_3 + j'_{55} - c_{17}.$$

From the last four inequalities we may prove that for all irreducible maps,

(A) $$2a = 52 + c - \mu + 3\sum_7 (n-7)a_n - j'_{55} + c_{14} + c_{15} + c_{16} + c_{17}.$$

If it were possible for a map with fewer than twenty-nine regions to contain polygons of more than eight sides then (A) would imply that the cyclomatic number, $\mu$, of our map is at least equal to $c - j'_{55} + 2$. Since a $J'_{55}$ is a simply connected configuration of pentagons, $c$ must exceed $j'_{55}$ by at least one if there are to be any multiply connected configurations. Since configurations for which $\mu = 1$ contribute a unit increment to $c$ as well as to $\mu$ they are not effective in satisfying our inequality. Therefore $\mu$ exceeds the number of configurations for which $\mu = 1$ by at least three. Hence, there is either a configuration for which $\mu \geqq 3$ or there are two or more configurations for each of which $\mu \geqq 2$. In either case our map will be divided into at least four parts by rings of pentagons.

As one travels around such a ring one turns to the right or to the left in crossing each pentagon, since pentagons do not have opposite sides. In going around a ring of pentagons on a simply connected orientable surface one may cross an even number of "flex $J$'s" each of which separates a pentagon in which one turns to the right from a pentagon in which one turns to the left. If such a ring had no flex $J$'s it would surround a single polygon, thus forming a reducible configuration.* If there are any flex $J$'s, then any polygon with more than six sides making a $Y_n$ $(n \geqq 3)$ contact with our ring will be the terminus of at least two of the flex $J$'s. If there are two and only two flex $J$'s in a ring, then our map would contain a ring of four polygons, two pentagons of our ring alternating with two polygons of more then six siedes, which would be reducible.† If our ring had more than two flex $J$'s and only one polygon of more than six sides on either side of the ring, then our flex $J$'s would

* Franklin, l. c., Reduction number 5, p. 225.

† Franklin, l. c., Reduction number 2, p. 225.

all terminate in the boundary of this polygon and one side of our ring would make one $Y_n (n \geqq 3)$ contact and several $Y_2$ contacts. On the opposite side of the ring, directly opposite the sequence of $Y_2$'s, we would have a single polygon making a $Y_n$ contact, thus forming a reducible ring of four regions as before. Therefore, in each section of our map bounded by a closed ring of pentagons there must be at least two polygons of more than six sides.

Applying this result to the preceding paragraph, we find that in an irreducible map for which $a \leqq 28$ and $\sum_9 a_n \geqq 1$, there would have to be at least eight polygons of more than six sides. This result is incompatible with (11). Therefore,

*No irreducible map having fewer than twenty-nine regions may have polygons of more than eight sides.*

**4. Derivation of inequalities applicable to irreducible maps in which no region is in contact with more than eight other regions.** If, in an irreducible map, $a_n = 0$ $(n \geqq 9)$, then there may not be more three pentagons between any two flex $J$'s of a ring of pentagons*. If there were two or four flex $J$'s in our ring, then we would be restricted to twenty-seven homeomorphically distinct rings which may classified by the cyclic permutations, admitting repetitions, of the numbers 1, 2 and 3, taken two or four at a time, which give the number of pentagons between consecutive flex $J$'s in all rings of the type described. Constructing each one of these rings and such adjoining polygons as may be required by our reductions, we find, in each case either a ring of fewer than five polygons†, or a ring of five regions including three pentagons‡. Birkhoff's reductions reduce the entire list of twenty-seven rings. From this it follows that if $\mu \geqq 1$ then the boundary of each of the $\mu + 1$ sections into which our multiply connected configurations of pentagons divide our map must be the terminus of at least six $J$'s. Two such sections, however, share each $J$. Hence the multiply connected configurations of pentagons, if any exist, contribute at least $3(\mu + 1)$ to $j$. We have one minimum for $j$ in (16). The $3(\mu + 1)$ contribution to $j$ just established may overlap the terms $2t_2 + 3t_3$ of (16) but it will have nothing in common with the term $t_1$ nor with that part of $j_{55}$ (denoted by $j'_{55}$) which arises from the simply connected configurations formed by two $T_1$'s, one on either side of a $J_{55}$. Therefore,

(18) $j = 3(\mu + 1) + t_1 - j'_{55} + c_{18}$, whenever $\mu \geqq 1$ and $a_n = 0 (n \geqq 9)$.

* Franklin, l. c., p. 233; Errera, l. c., Paragraph (2, 2, 2), p. 48.

† Franklin, l. c., Reduction number 2, p. 225.

‡ Franklin, l. c., Reduction number 3. 4, p. 225.

If an irreducible map has no more than twenty eight regions then it contains no region of more than eight sides. If such a map has one or more multiply connected configurations of pentagons then the hypotheses of (14), (15) and (18) are satisfied and we have

$$2a = 55 + 2\mu + c - j'_{55} + 3a_8 + c_{14} + c_{15} + c_{18}. \tag{B}$$

Here the $J'_{55}$'s are simply connected and there is a least one multiply connected configuration. Hence $c - j'_{55}$ is at least equal to one and (B) contradicts our hypothesis. Therefore

*No irreducible map with fewer than twenty-nine regions may have multiply connected configurations of pentagons.*

Any $T_0$, $T_1$, $T_2$, $T_3$ of our map may contribute to our map portions of not more than 0, 2, 3, 3 distinct $Y_n$'s ($n \geqq 3$). Whenever we have a $J_{55}$ each of two pairs of these $Y_n$'s coalesce. Whenever we have a $P_{545}$ then one pair of $Y_n$'s coalesce. Whenever $a_n = 0$, $n \geqq 9$, we have $y_n = 0$, $n \geqq 6$, and no two $T$'s may be so separated by $P_4$'s as to cause any $Y_n$'s to coalesce except under the conditions just mentioned. Therefore

$$\sum_3 y_n = 2t_1 + 3t_2 + 3t_3 - 2j_{55} - p_{545} + c_{19}. \tag{19}$$

All connected configurations of pentagons in an irreducible map with the exception of the $P_1$'s, $D$'s and $T_0$'s consist of one or more $P_3$'s and one or more sequences of zero or more $P_4$'s included between two $P_2$'s, a $P_2$ and a $P_5$ or two $P_5$'s. The number of these sequences will be one half the number of their termini, i. e. $\frac{1}{2}(p_2 + p_5 - 2d)$. If the map contains no region of more than eight sides then reductions due to Franklin* and Errera† imply that $y_n = 0 (n \geqq 6)$. Therefore we cannot have a sequence of more than three $P_4$'s lying between consecutive $J$'s, nor may we have more than three $P_4$'s between a terminal $P_2$ and the nearest $J$. Therefore we will have at least $\frac{1}{3}p_4 - 1$ $J$'s between the first and last $P_4$ of a sequence. It will be noted that, if $p_4 = 0$ in a given sequence, this gives a lower limit of minus one for the contribution of this sequence toward the value of $j$ for the whole map. This, in turn, justifies us in summing $j$ over a map by counting the contributions of the $T_n$'s, and the sequences of $P_4$'s separately; since a $J_{55}$ may be counted once for each of the neighboring $P_5$'s and negatively once for the "sequence" of zero $P_4$'s between them. Hence $j$ must be at least equal to one third the number of $P_4$'s

* Franklin, l. c., p. 233.

† Errera, l. c., Paragraph (2, 2, 2), p. 48.

minus the number of sequences of $P_4$'s plus the number of $P_5$'s which serve as termini of sequences of $P_4$'s, or

$$j = \frac{1}{3}p_4 - \frac{1}{2}(p_2 + p_5 - 2d) + p_5 + \frac{1}{3}c_{20}.$$

On applying equations (9) and the value of $p_5$ in terms of the $t_n$'s we have

$$(20)\quad 3j = 12 + a_7 + 2a_8 + 4p_1 + 3d - 5(c-\mu) + 2t_0 + t_1 - t_3 + c_{20}, \quad \text{whenever } a_n = 0,\ n > 9.$$

In defining simply and multiply imposed minima for the number of sides a polygon making given $Y_n$ contacts might have, we showed that no heptagons may be multiply imposed nor are octagons except when in contact with two $Y_3$'s. Therefore

$$(21)\qquad a_7 + 2a_8 = \sum_3 y_n + c_{21}, \quad \text{whenever } a_n = 0,\ n \geqq 9.$$

When there are no polygons of more than eight sides in an irreducible map then neither $T_2$'s nor $T_3$'s may have more than one adjoining $J_{55}$, nor may a $T_2$ ($T_3$) have more than two (zero) adjoining $P_{545}$'s. If a $T_2$ has an adjoining $J_{55}$ than that displaces a $P_{545}$ which would otherwise be a possible neighbor. Therefore

$$(22)\quad 2j_{55} + p_{545} = 3t_2 + 2t_3 + 2j''_{55} + p'_{545} - c_{22}, \quad \text{whenever } a_n = 0,\ n \geqq 9.$$

Combining several of our inequalities we now have

$$(C)\quad 4j'_{55} + p'_{545} = 272 - 10a + 15a_8 + 4p_1 + 3d + 2t_0 + 5c_{14} + 5c_{15} + 2c_{16} + 2c_{17} + c_{19} + c_{20} + c_{21} + c_{22},$$

an inequality which is valid whenever $a_n = 0$, $n \geqq 9$.

In each $J'_{55}$ there are six pentagons and in each $P'_{545}$ there are seven pentagons. ($C$) implies, therefore, that if an irreducible map of not more than twenty-seven regions contained even one octagon our map would contain at least five of these configurations and, therefore, at least thirty pentagons, which contradicts our hypothesis. Hence

*An irreducible map of fewer than twenty-eight regions may contain no polygons with more than seven sides.*

From the definition of the $\lambda$ occurring in (13) and from our identity (1) it follows that

$$\lambda = p_1 - 2t_0 - t_1 + t_3 + 2y_3 + y_4.$$

Hence $\lambda = -2$ for a $T_0$. If $\lambda$ is to be negative for any other configuration of pentagons, then it must be negative because of the presence of $T_1$'s

whose effect is uncanceled by $T_3$'s, $Y_3$'s or $Y_4$'s, If a $T_1$ were to adjoin a $P_4$, then its effect would be canceled by the accompanying $Y_3$. If two $T_1$'s lie on opposite sides of a $J_{55}$, then the two accompanying $Y_4$'s cancel the effect of both $T_1$'s. If the $T_1$ and a $T_2$ lie on opposite sides of a $J_{55}$, then there will be an accompanying $Y_4$ to cancel the effect of the $T_1$ If a $T_1$ and a $T_3$ lie on opposite sides of a $J_{55}$ then, unless there are regions in our map having more than eight sides, it will be impossible for any second $T_1$ to be similarly related to this same $T_3$, and the $T_3$ cancels the effect of our $T_1$. Hence $\lambda \geqq 0$ for every configuration of pentagons except $T_0$'s in an irreducible map containing no regions with more than eight sides.

Hence

(23) $$2t_0 = -\lambda + c_{23}, \quad \text{whenever } a_n = 0,\ n \geqq 9.$$

When an irreducible map contains only twenty seven regions, it must consist of pentagons, hexagons and heptagons. Therefore all minima are simply imposed and the hypothesis of (13) is satisfied. Under these conditions each $J'_{55}$ requires the presence of two and each $P_{545}$ requires the presence of three heptagons. Hence

(24) $$a_7 = 2j''_{55} + 3p'_{545} + c_{24}, \quad \text{whenever } a_n = 0,\ n \geqq 8.$$

Combining some of our inequalities linearly we now have:

(D) $$2j'_{55} = 2p'_{545} + 324 - 12a + 4p_1 + 3d + c_{13} + 6c_{14} + 5c_{15} + 2c_{16} + 2c_{17} + c_{19} + c_{20} + c_{21} + c_{22} + c_{23} + c_{24}, \quad \text{whenever } a_n = 0,\ n \geqq 8.$$

(E) $$2t_0 = 52 - 2a + a_7 + 4a_8 + c_{13} + c_{14} + c_{23},$$

whenever $a_n = 0$, $n \geqq 9$ and no octagon makes two $Y_3$ contacts.

The inequalities in sections three and four of this paper which are marked numerically are independent propositions. Those which are marked with capital letters are linear combinations of these more fundamental inequalities which are important only because of the uses which we make of them in sections three, four and five.

5. THEOREM. **Any irreducible map must contain at least twenty eight regions.**

*Proof:* Let $a$ be less than or equal to twenty-seven. Then (C) and (D) are applicable since we have already proved in the preceding section that any irreducible map with fewer than twenty-eight regions may have no polygons with more than seven sides. Since (C) and (D) contradict each other when $j'_{55} = 0$ we conclude that our map must contain at least one $J'_{55}$.

One $J'_{55}$, with its two $Y_4$'s requires the presence of at least two heptagons. These, by (10a), imply the presence of at least fourteen pentagons. (A) now implies that we have at least eight pentagons to be distributed

between not more than two configurations of pentagons. This cannot be done without providing at least two additional $Y_n$'s ($n \geq 3$), and so at least two additional heptagons and, by (10a), at least two more pentagons. With at least four heptagons in our map (E) implies that at least one of our non-$J'_{55}$ configurations must be a $T_0$. If $t_0 = 1$ then, by (A), there must be at least seven pentagons in a third connected configuration of pentagons. If there are not to be more than four heptagons, then this third configuration cannot have more than two $Y_n$'s ($n \geq 3$), (14) and (8a) imply that it must have at least two $T$'s, and these requirements are easily seen to be incompatible. If $t_0 = 2$, then we have only twelve pentagons in one $J'_{55}$ and two $T'_0$s, (10a) and the presence of at least four heptagons rendering this situation impossible. If there are more than four heptagons then (A) and (E) lead immediately to this same contradiction.

Two $J'_{55}$'s with their four $Y_n$'s ($n \geq 3$), imply the presence of at least four heptagons. These with (E) imply the existence of at last one $T_0$, and therefore of at least five heptagons since four heptagons making $Y_4$ contacts with two $J'_{55}$ connot be so arranged as to make contacts with each of the three pentagons of a $T_0$ wihout making our map reducible.* If we have five or more heptagons, then (A) and (E) imply that we have $t_0 = 2$. Then $a_5 = 18$, and, by (10a), $a_7 = 6$. Under these conditions (14) implies that we have at least twenty-six $E_{55}$'s whereas, in fact we only have twenty.

Three or more $J'_{55}$'s bring at last six heptagons with them, and also, by (A) and (E), two $T_0$'s. The twenty-four pentagons of these configurations must, by (10a), be accompanied by twelve heptagons, contradicting our initial hypothesis that our map was to contain not more than twenty-seven regions.

The assumed falsity of our proposition having lead to a sequence of contradictions, our proposition is proved, and may be restated as follows

*Any map of a simply connected closed surface containing not more than twenty-seven regions may be colored in four colors.*

6. **Conclusion.** Relations (C), (D) and (E) suggest that the configurations whose reduction would be of greatest value in raising the minimum value of $a$ are the $T_0$, $J'_{55}$ and $P'_{545}$ configurations. They also suggest that $J'_{55}$'s and $P'_{545}$'s might be most serviceable in constructing irreducible maps with comparatively few regions. Actually we find it possible to form irreducible maps of only 28, 30 or 36 regions according as we use (1) the reductions of Kempe, Birkhoff and Franklin, (2) the reduction used in this paper, or (3) the reductions (1) plus all of the reductions published by Errera.

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* Franklin, l. c., p. 231. Reduction number 2, p. 225.

As I have already indicated, the theorem of the preceding section may be proved without using any of Errera's reductions. Such a proof involves a considerable amount of case-making. Under such a restricted hypothesis our theorem establishes as high a minimum value of $a$ as is possible, since there exists a map of only 28 regions which is irreducible under this hypothesis. One such map may be constructed from 12 pentagons and 16 hexagons by startig with four sets of three hexagons, the three hexagons of each set meeting at a point. Then six $D$'s and the four remaining hexagons are so arranged that each hexagon of the "sets of three hexagons" makes one $Y_1$ and one $Y_2$ contact with pentagons.

Upon introducing one of Errera's reductions* the number of undetermined integers in (C) and (D) might lead us to suspect that more might be proved from our reductions by other and better methods. It is not possible, however, to raise the number 28 of our theorem more than two units, since there exists a map of 30 regions which is irreducible in this sense. One such map contains three pairs of adjacent heptagons. Each heptagon makes a $Y_4$ and a $Y_1$ contact with two of three $J'_{55}$'s in our map. The remainder of the map is filled in with two sets of three hexagons each, the three hexagons of each set meeting at a point.

Applying our inequalities to this map we find that the hypothesis of (18) is not satisfied, that $c_{11} = 18$, $c_{12} = 18$, $\lambda = c_{13} = 0$, $c_{14} = 2$, $c_{15} = 6$ and that all of the other $c_n$'s vanish. Since the linear combinations of the $c_n$'s occurring in (A), (C) and (D) are, for this map, equal respectively to 4, 4 and 3.5 times the coefficients of $a$ in these relations, we again suspect that these reductions may imply that the number of regions in an irreducible map is at least thirty. The strengthening of (14) and (15) may enable someone to prove this.

Turning now to the third set of reductions which includes all of Errera's published reductions, we find that although he has published† a map of 52 regions as a suggestion of the progress which he has made, there exists a map of only 36 regions which is irreducible in this sense. One such map contains three triads of heptagons, each triad meeting at a point, and each triad making two $Y_3$ and one $Y_5$ contact with two of three $P'_{545}$'s in our map, the remainder of the map being filled in with two triads of hexagons, the hexagons of each triad meeting at a point.

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* Errera, l. c., paragraph (2, 4, 2).

† Comptes Rendus de l'Association Française pour l'Avancement des Sciences, Liége 1924, p. 96.

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# SOME THEOREMS ON DEDUCIBILITY.*

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This paper is concerned with a form of the problem of categoricalness in connection with sets of defining properties for types of dense series. Sets of defining properties for serial relations are given on the base $K$, $R_2$. All of the properties which occur in the determination of dense series are first-order functions. A first-order function is a function whose values are first-order propositions, and a first-order proposition is a proposition which contains variable individuals but does not contain any variable functions.† Given any function $f(x, y, \cdots, n)$, such that when values are assigned to $f, x, y, \cdots, n$ the expression becomes a proposition containing no variable constituents, if we quantify the variable constituents $x, y, \cdots, n$ as in $(x)(\exists y)\cdots(n)\cdot f(x, y, \cdots, n)$, the result is a first-order function, and for any value of $f$ this becomes a first-order proposition. Sets of postulates for three sorts of dense series are to be treated: (1) series having neither a first nor a last element, (2) series having a first but no last element, and (3) those having both a first and a last element.

The class of all first-order functions on the base $K$, $R_2$ is the class of all such functions which are formulated solely in terms of $K$ and $R$. These functions may involve any finite number of variable constituents. It will be shown for each set of postulates that the truth-value of any first-order function on the base $K$, $R_2$ is determined. Given any first-order function $F(x, y, \cdots, n)$ on $K$, $R_2$, then any one of the sets is such that $F$ follows from it or else the contradictory of $F$ follows; no first-order function can be independent of the set.

It will be necessary, as a preliminary to the analysis to be undertaken, to exhibit in some detail the general properties of first-order functions.‡ We are concerned here with quantifications as applied to the variable constituents of a construct which denote individuals and not with variable predicates. In this case it will be shown that any proposition has either the form

$$(x, y, \cdots, t) :: (\exists z, w, \cdots, r) :. (\mu, \nu, \cdots, m) : \cdots \cdots . \varphi(x, y, \cdots, t; z, w, \cdots, r; \mu, \nu, \cdots, m; \cdots)$$

or the form

$$(\exists x, y, \cdots, t) :: (z, w, \cdots, r) :. (\mu, \nu, \cdots, m) : \cdots \cdots . \varphi(x, y, \cdots, t; z, w, \cdots, r; \mu, \nu, \cdots, m; \cdots).$$

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* Presented to the American Mathematical Society, February 27, 1926.

† Compare *Principia Mathematica*, second edition, vol. 1, p. xxiii.

‡ See *Principia Mathematica*, vol. 1, pp. 127–160.

In the simplest case such functions reduce to

$$(x, y, \cdots, t) \cdot \varphi(x, y, \cdots, t) \quad \text{or} \quad (\exists x, y, \cdots, t) \cdot \varphi(x, y, \cdots, t)$$

in which only a single quantifier "some" or "every" is applied. Propositions involving a single applicative "some" or "every" will be called singly-quantified propositions, whereas those involving more than one applicative will be said to be multiply quantified.* There are two factors in the structure of a general proposition which are of especial importance, namely, the applicative which attaches to a given variable and the scope of the variable. The scope of a variable is indicative of a mode of bracketing in the proposition. In a proposition of the form "For every $x$ some $y$ or other is such that $\varphi(x, y)$" which would be written $(x) : (\exists y) \cdot \varphi(x, y)$, $x$ is said to have a wider scope than $y$; whereas in "For some one and the same $y$ every $x$ is such that $\varphi(x, y)$," which would be written $(\exists y) : (x) \cdot \varphi(x, y)$, $y$ is said to have a wider scope than $x$. Wider to narrower scope is indicated by successive parantheses, those variables appearing in the first paranthesis having the widest scope. Variables which appear in the same paranthesis have the same scope. Thus

$$(x, y, \cdots, n) : (\exists z, w, \cdots, m) \cdot \varphi(x, y, \cdots, n; z, w, \cdots, m)$$

is to be read "For every $x, y, \cdots, n$, some or other $z, w, \cdots, m$ is such that $\varphi(x, y, \cdots, n; z, w, \cdots, m)$". In this point we are deviating from customary usage in which each variable is assigned a different scope. Thus we should have

$$(x)(y) \cdots (n)(\exists z)(\exists w) \cdots (\exists m) \cdot \varphi(x, y, \cdots, n; z, w, \cdots, m)$$

which may be read "For every $x$ every $y$, etc. is such that for some $z$ some $w$, etc. is such that $\varphi(x, y, \cdots, n; z, w, \cdots, m)$". These two formulations are clearly equivalent and the latter will be said to be reducible to the former. Such a reduction will be carried out frequently in the present treatment because the classification of general propositions to be used here does not depend upon the number of variables in the propositional form but upon the number of changes in applicative which occur. A singly-quantified proposition involves only one applicative, "some" or "every", though it may involve any number of variables. A proposition which involves a change from "some" to "every" or from "every" to "some" in the quantifier, will be said to be doubly-quantified, and a proposition which involves $n-1$ changes of applicative will be

* The terminology is due to Mr. W. E. Johnson who has used it in a slightly different sense. See *Mind*, N. S., vol. 17, 1892, p. 26 ff.

called an $n$-tuply quantified proposition. It will be seen subsequently that propositions having the same degree of quantification have properties very much alike while those differing as to degree of quantification differ in important respects.

The force of the "scope" of a variable is precisely the same as that of the "order" of a parameter, and a comparison here is of interest because in the parameter we have the factor of "scope" dissociated from the use of the applicatives "some" and "every". Roughly, the order of a parameter in a function is the "degree of fixity" of it as a variable constituent of the function. To use the simplest illustration, $y = kx + l$, as referred to the cartesian plane, will denote systems of parallel lines if $k$ is fixed relatively to $l$, while if $l$ is fixed relatively to $k$ the expression denotes any pencil of lines on a point on the $y$-axis. If neither of these conditions is involved, then the expression denotes individual lines in the plane. In the latter case $k$ and $l$ are parameters of the same order. The three cases may be represented by $y_0 = k_2 x_0 + l_1$, $y_0 = k_1 x_0 + l_2$, and $y_0 = k_1 x_0 + l_1$ respectively, where the highest suffix denotes the parameter or parameters of highest order and a zero suffix the parameters of lowest order, or the variables. This situation, obvious geometrically, requires analytic interpretation and the following analysis seems to be what is needed. The use of parameters involves the notion of a denotative hierarchy. This is seen by asking for a typical value of a given expression. In the first place it is clear that any value of such an expression as $y = zx$, which involves no parameters, is a definite proposition such as $6 = 2 \cdot 3$. $y = zx$ denotes $6 = 2 \cdot 3$ as a value. But $y = kx$ does not denote propositions. It denotes, for example, $y = 2x$ which in turn denotes definite propositions. Again, $y = k_1 x + l_2$ has as a typical value $y = k_1 x + 5$, while $y = k_2 x + l_1$ has as a typical value $y = 4x + l_1$ and this in turn has for value $y = 4x + 5$ for example. But $y = k_1 x + l_1$ has $y = 4x + 5$ for a value though it is not a value of $y = k_2 x + l_1$, nor is $y = 4x + l_1$ a value of it. If we use $\rightarrow$ to denote "has as a typical value", then the following represents the significance of these expressions:

$$y = zx \rightarrow 6 = 2 \cdot 3;$$
$$y = kx \rightarrow y = 2x \rightarrow 6 = 2 \cdot 3;$$
$$y = k_1 x + l_1 \rightarrow y = 3x + 5 \rightarrow 11 = 3 \cdot 2 + 5;$$
$$y = k_1 x + l_2 \rightarrow y = k_1 x + 5 \rightarrow y = 3x + 5 \rightarrow 11 = 3 \cdot 2 + 5;$$
$$y = k_2 x + l_1 \rightarrow y = 3x + l_1 \rightarrow y = 3x + 5 \rightarrow 11 = 3 \cdot 2 + 5.$$

To return to the comparison with the notion of "scope"; following the notation just used, we may have $\varphi(x_3, y_2, z_1)$ in which the suffixes denote

parametric order. Values of this function are $\varphi(a, y_2, z_1)$, $\varphi(b, y_2, z_1,)$, and so on, while each of these latter expressions has a separate set of values, and so on. Now when an applicative, "some" or "every", is applied to $\varphi(x_3, y_2, z_1)$ it is to be applied to the parameter of highest order $x$. If the applicative is universal it makes an assertion about every one of the values of $\varphi(x_3, y_2, z_1)$ and if it is particular it makes an assertion about at least one of these values. The second applicative applies to the values of these values, while the third applicative applies to the values of these values of values. Thus $(x) :. (\exists y) : (z) \cdot \varphi(x, y, z)$ asserts that for any value of $\varphi(x_3, y_2, z_1)$, say $\varphi(a, y_2, z_1)$, $(\exists y) : (z) \cdot \varphi(a, y, z)$, and this in turn asserts for each such function that there is at least one value, say $\varphi(a, b, z_1)$, such that $(z) \cdot \varphi(a, b, z)$, and this latter function entails for any one of its values, say $\varphi(a, b, c)$, that $\varphi(a, b, c)$ holds. The scope of a variable indicates the point in the hierarchy of values at which substitutions for it are to be made.

In the foregoing illustrations the scopes of any two variable constituents of a proposition are given as either wider, narrower, or the same. In many general propositions, however, there are variables which have independent scope. For example, we may have $(x) :. (\exists y) : f_1(x, y) : (z) \,.\, f_2(x, z) \,.\, f_3(y, z) : (\exists w) \,.\, f_4(x, y, w)$. In this expression $z$ and $w$ have independent scope. $(z)$ applies to $f_2(x, z) \,.\, f_3(y, z)$ alone while $(\exists w)$ applies to $f_4(x, y, w)$ alone. It has been pointed out by Russell* that it is always possible to give each variable a scope covering the entire function. This may be seen as follows. There are two ways in which two functions may be connected, conjunctively or disjunctively. All other modes of connection are defined in terms of these. We have accordingly six possible cases:

(1) $\quad (x) \,.\, \varphi x \,.\, \mathbf{V} \,.\, (y) \quad .\, \psi y;$

(2) $\quad (x) \,.\, \varphi x \,.\, \mathbf{V} \,.\, (\exists y) \,.\, \psi y;$

(3) $\quad (\exists x) \,.\, \varphi x \,.\, \mathbf{V} \,.\, (\exists y) \,.\, \psi y;$

(4) $\quad (x) \,.\, \varphi x : \quad (y) \quad .\, \psi y;$

(5) $\quad (x) \,.\, \varphi x : \quad (\exists y) \,.\, \psi y;$

(6) $\quad (\exists x) \,.\, \varphi x : \quad (\exists y) \,.\, \psi y.$

(1) is clearly equivalent to $(x, y) : \varphi x \,.\, \mathbf{V} \,.\, \psi y$; (2) is equivalent to $(x) :. (\exists y) : \varphi x \,.\, \mathbf{V} \,.\, \psi y$ and to $(\exists y) :. (x) : \varphi x \,.\, \mathbf{V} \,.\, \psi y$;† (3) is equivalent to

* *Principia Mathematica*, vol. 1, p. 135 ff.

† Under the premise that there is at least one element within the range of significance of the variables. If this condition is not present a more careful formulation is required. These relations are to be used in connection with a condition for the existence of at least $n$ elements, so that existence conditions will be presupposed here.

$(\exists x, y) : \varphi x \,.\, \mathbf{V} \,.\, \psi y$; (4) is equivalent to $(x, y) \,.\, \varphi x \,.\, \psi y$; (5) is equivalent to $(x) : (\exists y) \,.\, \varphi x \,.\, \psi y$ and to $(\exists y) : (x) \,.\, \varphi x \,.\, \psi y$, and (6) is equivalent to $(\exists x, y) \,.\, \varphi x \,.\, \psi y$. Also,

$$\varphi x \,.\, \mathbf{V} \,.\, (\exists y) \,.\, \psi y : \equiv : (\exists y) \,.\, \varphi x \mathbf{V} \psi y,$$
$$\varphi x \,.\, \mathbf{V} \,.\, (y) \,.\, \psi y : \equiv : (y) \,.\, \varphi x \mathbf{V} \psi y,$$
$$\varphi x \,.\, (\exists y) \,.\, \psi y : \equiv : (\exists y) \,.\, \varphi x \,.\, \psi y,$$
$$\varphi x \,.\, (y) \,.\, \psi y : \equiv : (y) \,.\, \varphi x \,.\, \psi y.$$

Thus

$$(x) :. (\exists y) : f_1(x,y) : (z) \,.\, f_2(x,z) \,.\, f_3(y,z) : (\exists w) \,.\, f_4(x,y,w)$$
$$:: \equiv :: (x) :: (\exists y) :. f_1(x,y) :. (z) : (\exists w) \,.\, f_2(x,z) \,.\, f_3(y,z) \,.\, f_4(x,y,w)$$
$$:: \equiv :: (x) :: (\exists y) :. (z) : f_1(x,y) : (\exists w) \,.\, f_2(x,z) \,.\, f_3(y,z) \,.\, f_4(x,y,w)$$
$$:: \equiv :: (x) :: (\exists y) :. (z) : (\exists w) \,.\, f_1(x,y) \,.\, f_2(x,z) \,.\, f_3(y,z) \,.\, f_4(x,y,w).$$

$$(x) \,.\, \varphi x \,.\, \equiv . \sim (\exists x) \,.\, \sim \varphi x \quad \text{and} \quad (\exists x) \,.\, \varphi x \,.\, \equiv . \sim (x) \,.\, \sim \varphi x$$

Hence,

$$\sim (x) \,.\, \varphi x \,.\, \equiv . \quad (\exists x) \,.\, \sim \varphi x \quad \text{and} \quad \sim (\exists x) \,.\, \varphi x \,.\, \equiv . \quad (x) \,.\, \sim \varphi x.$$

Also,

$$\sim (\varphi x \mathbf{V} \psi y) \,.\, \equiv . \sim \varphi x \,.\, \sim \psi y \quad \text{and} \quad \sim (\varphi x \,.\, \psi y) \,.\, \equiv . \sim \varphi x \mathbf{V} \sim \psi y.$$

By the use of these equivalences the sign of negation can be removed from before any quantifier. Accordingly, any proposition can be expressed in terms of "some" and "every" alone, provided, of course, that $\sim$ be applied within the function itself. It follows that any proposition involving variables having independent scope can be expressed as a proposition all of whose variables have interdependent scopes.

To get the contradictory of a general proposition which involves a single complex quantifier change every universal variable into a particular and every particular into a universal and take the negative of the function. Thus $(x, \cdots, n) :. (\exists y, \cdots, m) : \cdots \,.\, \varphi(x, \cdots, n; y, \cdots, m)$ and $(\exists x, \cdots, n) :. (y, \cdots, m) : \cdots \,.\, \sim \varphi(x, \cdots, n; y, \cdots, m)$ are contradictories.

In formulating sets of defining properties for abstract systems in many cases we have a single class $K$ and an $n$-adic relation $R$ in terms of which all of the properties belonging to the set are formulated. Every postulate is a logical construct on $K$ and $R$ alone. Now the use of a class $K$ is equivalent to the use of a predicate $\varphi\hat{x}$ and the use of an $n$-adic relation $R$ is equivalent to the use of a predicate $\psi(\hat{x}, \hat{y}, \cdots, \hat{n})$. The set is then a logical function, $f(\varphi, \psi)$, of these two predicates, and if we are confined to this base no other predicates can be used except those definable in terms of $\varphi$ and $\psi$. We are concerned in what follows with the class of all first-order functions which can be formulated in terms of a simple case of such a pair of predicates, namely, $\varphi\hat{x}$ and $\psi\hat{x}, \hat{y}$.

The simplest set of properties of any importance on the base $K, R_2$ is the set for a dense series without extreme elements:

(1) $(x) . \sim Rxx.$

(2) $(x, y) :. x \neq y . x, y \varepsilon K : \supset : \quad Rxy . \vee . \quad Ryx.$

(3) $(x, y) :. x \neq y . x, y \varepsilon K : \supset : \sim Rxy . \vee . \sim Ryx.$

(4) $(x, y, z) :. x \neq y . y \neq z . x \neq z . x, y, z \varepsilon K : \supset : Rxy . Ryz . \supset . Rxz.$

(5) $(x) \quad :. (\exists y) : x \varepsilon K . \supset . Rxy.$

(6) $(x) \quad :. (\exists y) : x \varepsilon K . \supset . Ryx.$

(7) $(x, z) :. (\exists y) : Rxz . \supset . Rxy . Ryz.$

(8) $(\exists x) . x \varepsilon K.$

These properties, with some modifications, are the ones usually assigned for this type of order. They are, however, with the exception of (1), confined to assertions relevant to elements in the class $K$, and it is customary to omit any mention of properties belonging to elements not in $K$. But it is necessary in the present case to consider such properties—otherwise some important theorems break down. The following properties are to be added to the list.

(9) $(x, y) :. x \neq y : \sim x \varepsilon K . \vee . \sim y \varepsilon K : \supset : \sim Rxy.$

(10) $(\exists x, y, \cdots, n) . x \neq y ., \cdots, . x \neq n ., \cdots, . y \neq n ., \cdots, . \sim x \varepsilon K . \sim y \varepsilon K ., \cdots, . \sim n \varepsilon K.$*

If $x$ and $y$ do not both belong to $K$, then $Rxy$ fails. Also, there are at least $n$ elements not in $K$. (9) and (10) are properties which are usually tacitly understood. The ten properties may be classified as follows. (1), (2), (3), (4), (8), (9), and (10) are singly quantified functions. They involve the use of a single applicative, "some" or "every". (5), (6), and (7) are doubly quantified. They involve a single change from "every" to "some". Also, (1) and (8) are functions of one variable; (2), (3), (5), (6), and (9) are functions of two variables; (4) and (7) are functions of three variables, while (10) is a function of $n$ variables. (1) and (7) are functions on $R$ alone while (8) is a function on $K$ alone. It is proposed to study the consequences of (1)–(10) in relation to the class of all first-order functions which can be formulated in terms of $K$ and $R_2$.

Properties (1)–(4) have for values singly quantified propositions in which all of the variables are quantified universally. These four properties will be treated first. $R'(abc \cdots n)$ is to mean that any two terms $x, y$ which occur in the order $xy$ in $R'$ are such that $Rxy$ holds. Thus $R'(abc)$ means

* It is to be noted that existence propositions are always empirically significant and that the existence expressions which occur in logic are properties, not propositions.

$Rab \,.\, Rbc \,.\, Rac$. Also, $R''(abc \cdots lmn)$ is to mean that $R'(abc \cdots lmn)$ holds and that any two elements $y, x$ which occur in $R''$ in the order from right to left are such that $Ryx$ fails. This is equivalent to saying that $R''(abc \cdots lmn)$ means $R'(abc \cdots lmn)$ and $R'$ fails for any other permutation of these elements. It will be shown that, as a consequence of (2)–(4), every set of $n$ elements in $K$ ($n>1$ and finite) is such that for some permutation of the elements, $a, b, c, \cdots, n$, $R''(abc \cdots n)$ holds.

THEOREM I. $(x, y) :. x \neq y \,.\, x, y \varepsilon K : \supset : R''xy \,.\, \vee \,.\, R''yx$.

This follows from (2) and (3).

THEOREM II. $(x, y, z) :. x \neq y \,.\, x \neq z \,.\, y \neq z \,.\, x, y, z \varepsilon K : \supset : R''xyz \,.\, \vee \,. R''yzx \,.\, \vee \,. R''zxy \,.\, \vee \,. R''zyx \,.\, \vee \,. R''xzy \,.\, \vee \,. R''yzx$.

That is, for some permutation of any three elements in $K$, say $abc$, $R''abc$ holds. It is to be noted that if $R''abc$ holds the form of the elements with respect to $R$ is completely determined for distinct dyads since $R''$ assigns a determinate validity value to every dyad of distinct elements. When for some permutation of a triad of elements $R''$ holds the elements will be said to have the form $R''(xyz)$. Similarly, when for some permutation of an $n$-ad of elements $R''$ holds the $n$ elements will be said to have the form $R''(xyz \cdots n)$. We have to show that every three elements in $K$ have the form $R''(xyz)$. Since $Rxy \,.\, \equiv \,.\, \sim Ryx$ for every distinct $x, y$ in $K$, any three elements must be such that $R$ holds for three of the six ordered dyads of distinct elements which can be formed of them. There are two possibilities: we may have the form $Rxy \,.\, Ryz \,.\, Rxz$ for some permutation of the three elements, or the form $Rxy \,.\, Ryz \,.\, Rzx$. The latter is contrary to postulate (4). Hence $R'xyz$, which implies $R''xyz$, by (3).

THEOREM III. *Any $n$ distinct $K$-elements have the form $R''(abc \cdots n)$.*

If this property belongs to every subclass of $m$ elemets ($m>2$), then it belongs to every subclass of $m+1$ elements. For consider any $m+1$ elements. Some one of these elements, $a$, is such that $Rxa$ holds for every other element $x$. Omitting $a$ from the subclass, we have $R''(bc \cdots m)$ for some permutation of the remaining elements. But $Rxa$ holds and $Rax$ fails. Hence $R''(bc \cdots ma)$.

In view of (1) $\sim Rxx$ for every $x$ in $K$. Accordingly, when $R''(abc \cdots n)$ holds we have also $\sim Raa \,.\, \sim Rbb \,.\, \sim Rcc., \cdots, . \sim Rnn$, and, by (9), $a \varepsilon K \,.\, b \varepsilon K \,.\, c \varepsilon K .. \cdots, . n \varepsilon K$. We may denote $R''(abc \cdots n) \,.\, \sim Raa \,.\, \sim Rbb \,. \sim Rcc., \cdots, . \sim Rnn \,.\, a \varepsilon K \,.\, b \varepsilon K \,.\, c \varepsilon K., \cdots, n \varepsilon K$ by $R^0(abc \cdots n)$. Then for some permutation of every $n$ elemets in $K$, $R^0$ holds. We may say that such a set of elements has the form $R^0(xyz \cdots n)$. Note that $R^0$ is significant for a single element. $R^0(a)$ means $\sim Raa \,.\, a \varepsilon K$. Now every $m$ elements not in $K$ are such that $R$ fails for every ordered dyad

formed of them. Let $S(de \cdots m)$ mean $\sim Rde . \sim Red ., \cdots, . \sim Rdm . \sim Rmd ., \cdots, . \sim Rem . \sim Rme ., \cdots, . \sim Rdd . \sim Ree ., \cdots, . \sim Rmm . \sim d \varepsilon K . \sim e \varepsilon K ., \cdots, . \sim m \varepsilon K$, and $d, e, \cdots m$ are all distinct. Then for every $m$ elements not in $K$, $S$ holds, and we may say that such a set of elements has the form $S(xyz \cdots m)$. $S$ is of course significant for a single element. If $a, b, \cdots, n \varepsilon K$ and $d, e, \cdots, m \varepsilon$ not-$K$, then $R$ fails for every ordered dyad which can be formed so as to consist of one element from each of these sets. Thus $\sim Rad . \sim Rda . \sim Rae$, etc. We may denote the failure of $R$ for these dyads by writing $T(ab \cdots n; de \cdots m)$. As a consequence of (1)–(4) and (9) any $n+m$ elements are of the form $R^0(xy \cdots n) . S(zw \cdots m) . T(xy \cdots n; zw \cdots m)$. When $m$ is zero this becomes $R^0(xy \cdots n)$ and when $n$ is zero, $S(zw \cdots m)$.

We now define what will be called the expanded form of a function $f(x, y, \cdots, n)$. Let $\psi(p, q, r, \cdots, w)$ be an elementary function of the propositions $p, q, r, \cdots, w$. Then $\psi$ is a construct on $p, q, r, \cdots, w$ built up in terms of "and", "or", and "$\sim$". By a well known procedure in the logic of propositions, which will be given presently, $\psi$ can be expressed as a disjunctive function of conjunctive functions of the constituents $p, q, r, \cdots, w$. For example, let $\psi$ be $p . \supset . q \vee r ; p . \supset . q \vee r : \equiv : \sim p \vee q \vee r : \equiv : \sim p . q . r . \vee . \sim p . \sim q . r . \vee . \sim p . q . \sim r . \vee . \sim p . \sim q . \sim r . \vee . p . q . r . \vee . p . \sim q . r . \vee . p . q . \sim r$. This is a disjunctive function of conjunctive functions of $p, q, r$. It excludes one possible alternative, *viz*, $p . \sim q . \sim r$. Note that every conjunctive function involves all of the propositional constituents $p, q, r$.

In an elementary function $f(x, y, \cdots, n)$ on $K$, $R_2$ the only elementary propositional constituents which occur are constructs of $x, y, \cdots, n$ on $K, R$, and "$=$" — such as $Rxy$ or $\sim Rxy$, and $x \varepsilon K$ or $\sim x \varepsilon K$, and $x = y$ or $\sim x = y$. These constituents of $f(x, y, \cdots, n)$ correspond to $p, q, r, \cdots, w, \sim p, \sim q, \sim r, \cdots, \sim w$. It may happen that $f(x, y, \cdots, n)$ is a function on $R$ but not explicitly on $K$. For example, $(x \neq y) . \supset . Rxy \vee Ryx$. This function is a case of the function $p . \supset . q \vee r$ in the example just cited. Let $(x \neq y) = p$ and $(Rxy) = q$ and $(Ryx) = r$. Then $x \neq y . \supset . Rxy \vee Ryx : \equiv : x = y . Rxy . Ryx . \vee . x = y . \sim Rxy . Ryx . \vee . x = y . Rxy . \sim Ryx . \vee . x = y . \sim Rxy . \sim Ryx . \vee . x \neq y . \sim Rxy . Ryx . \vee . x \neq y . Rxy . \sim Ryx . \vee . x \neq y . Rxy . Ryx$. In this case some of the alternatives, for example $x = y . \sim Rxy . Ryx$, are impossible and may be dropped from the disjunctive function. Note that in each alternative of the function any construct of $x, y$ on $R$ and "$=$" occurs or else its contradictory occurs. The function is in expanded form with respect to $R$ and "$=$".

A function on $R$ in expanded form involves the condition $x = y$ or $x \neq y$ for any two variables in any constituent conjunctive function. Any

$R$-function can be put in expanded form with respect to $R$ and "$=$" by the following procedure, which also, of course, establishes the existence of an equivalent function in expanded form. Let $f(x, y, \cdots, n)$ be any function which is formulated on $R$ alone. In general $f$ is a function which involves "and", "or", negation, and identity ("$=$"). Some of these may, of course, be lacking. The constituent constructs connected by "and" and "or" and affected by $\sim$ will be such as $Rxy$ or $\sim Rxy$ and $x = y$ or $\sim x = y$ $(x \neq y)$. Since $\sim(p \mathbf{V} q) . \equiv . \sim p . \sim q$ and $\sim(p . q) . \equiv . \sim p \mathbf{V} \sim q$, the sign of negation can be removed from before any conjunctive or disjunctive constituent function, so that $\sim$ will appear before constituents of the form $Rxy$ and $x = y$ only. To put $f(x, y, \cdots, n)$ in expanded form first remove $\sim$ from before every conjunctive or disjunctive constituent function in $f$. Now it may happen that some disjunctive function $p \mathbf{V} q$ is subordinate to a conjunctive function, as in $p \mathbf{V} q . s$. But $p \mathbf{V} q . s : \equiv : p . s \mathbf{V} . q . s$. Hence the function $f(x, y, \cdots, n)$ can be so expressed that no disjunctive function is subordinate to a conjunctive function. This is the second step. We have then a set of conjunctive functions $C_1, C_2, \cdots, C_s$ and $f$ is a disjunctive function $C_1 . \mathbf{V} . C_2 . \mathbf{V}, \ldots, \mathbf{V} . C_s$ of these conjunctive functions. For example, we may have $\sim(\sim p . q : \sim \bar{q} . s) . \mathbf{V} . s . t . u . \mathbf{V} . p . q . r : v . w$, which is equivalent to $p . q . \mathbf{V} . \bar{q} . s . \mathbf{V} . s . t . u . \mathbf{V} . p . q . r : v . w$, which is equivalent to $p . q . v . w . \mathbf{V} . \bar{q} . s . v . w . \mathbf{V} . s . t . u . v . w . \mathbf{V} . p . q . r . v . w$.

Let $f(x, y, \cdots, n)$ be expressed as a disjunctive function of conjunctive functions of its propositional constituents, $Rxy$, $x = y$, and the like. Now it happens in general that for some of the conjunctive functions in such a set of disjunctions some propositional constituent is such that neither it nor its contradictory appears in these conjunctive functions. In the example just given there are three conjunctive functions in which neither $r$ nor $\bar{r}$ appears. But $p . \mathbf{V} . q : \equiv : (p \mathbf{V} q) (r \mathbf{V} \bar{r}) : \equiv : p . (r \mathbf{V} \bar{r}) . \mathbf{V} . q . (r \mathbf{V} \bar{r}) : \equiv : p . r . \mathbf{V} . p . \bar{r} . \mathbf{V} . q . r . \mathbf{V} . q . \bar{r}$, so that the function may be multiplied by $(r \mathbf{V} \bar{r})$. We get $p . q . v . w . r . \mathbf{V} . p . q . v . w . \bar{r} . \mathbf{V} . \bar{q} . s . v . w . r . \mathbf{V} . \bar{q} . s . v . w . \bar{r} . \mathbf{V} . s . t . u . v . w . r . \mathbf{V} . s . t . u . v . w . \bar{r} . \mathbf{V} . p . q . r . v . w . r . \mathbf{V} . p . q . r . v . \bar{r}$. Those conjunctive functions in which both $r$ and $\bar{r}$ occur are impossible and may be dropped, so that we have $p . q . v . w . r . \mathbf{V} . p . q . v . w . \bar{r} . \mathbf{V} . \bar{q} . s . v . w . r . \mathbf{V} . \bar{q} . s . v . w . \bar{r} . \mathbf{V} . s . t . u . v . w . r . \mathbf{V} . s . t . u . v . w . \bar{r}$, and in this function either $r$ or $\bar{r}$ appears in every conjunctive constituent. Accordingly, in $f(x, y, \cdots, n)$, if some propositional constituent $p$ does not appear in every conjunctive function, introduce $p$ by multiplying by $(p \mathbf{V} \bar{p})$.

With $f(x, y, \cdots, n)$ in this form it may still happen that some function of $x, y, \cdots, n$ on $R$ or "$=$" does not occur in $f$ at all. Consider, for example, the function $Rxy . \mathbf{V} . Ryx$. Here neither $x = y$ nor $x \neq y$ occurs.

$Rxy \,.\, \vee \,.\, Ryx :\equiv: Rxy \,.\sim Ryx \,.\, \vee \,.\, Rxy \,.\, Ryx \,.\, \vee \,.\sim Rxy \,.\, Ryx.$ In this function we may introduce $x = y$ and $x \neq y$ by multiplying by $(x = y \vee x \neq y)$. This gives $Rxy \,.\sim Ryx \,.\, x = y \,.\, \vee \,. Rxy \,.\sim Ryx \,.\, x \neq y \,.\, \vee \,. Rxy \,. Ryx \,.\, x = y \,.\, \vee \,. Rxy \,. Ryx \,.\, x \neq y \,.\, \vee \,.\sim Rxy \,. Ryx \,.\, x = y \,.\, \vee \,.\sim Rxy \,. Ryx \,.\, x \neq y.$ In this form the function is expressed in such a way that in each conjunctive function any propositional construct of $x, y$ on $R$, "$=$" occurs or else its contradictory occurs.*

It has been shown that every general proposition can be so expressed that all of the variables have interdependent scopes. This means that any general proposition can be expressed by the use of a single complex quantifier applied to a function $f(x, \cdots, r)$. Thus every general proposition has either the form

$$(x, \cdots, l)\,(\exists y, \cdots, m)\,(z, \cdots, n), .f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n;\ \cdots)$$

or the form

$$(\exists x, \cdots, l)\,(y, \cdots, m)\,(\exists z, \cdots, n), .f(x, \cdots, l;\ y, \cdots, m;\ \ z, \cdots, n;\ \ \cdots).$$

In general $f(x, \cdots r)$ in these functions is, of course, not a construct on $R$ alone, but a construct on $R$ and $K$. Such a construct will involve propositional constituents $Rxy$ or $\sim Rxy$ and $x = y$ or $x \neq y$ and in addition $x \varepsilon K$ or $\sim x \varepsilon K$. Any function can be put in expanded form with respect to $K$, $R$, and "$=$" by the procedure which has been used to expand functions in terms of $R$ and "$=$". For the argument there does not depend on what the elementary constituents are. As an example, consider $Rxy \,.\, \vee \,. Ryx$ which has been expanded into $Rxy \,.\sim Ryx \,.\, x = y$

* And not both. If a constituent and its contradictory both occur in the same alternative, then clearly that alternative may be dropped without altering the truth-value of the function; and we shall consider a function in expanded form to be one from which all such alternatives have been discarded. This gives rise to a limiting case which we take account of here and dismiss from further consideration. It may happen that when a function is put in expanded form every alternative vanishes. (To express such a function positively we should have to write a function involving conjunctively the denial of every possible alternative.) Now any function from which every alternative vanishes and which demands existence (*e.g.*, $(\exists x) \,.\, x \varepsilon K \,.\sim x \varepsilon K$) is selfcontradictory. Its truth-value is determined; and since we are only concerned with showing determination of truth-value, these functions may be dismissed from further consideration. Any function from which every alternative vanishes and which does not demand existence (*e.g.*, $(x) \,.\, x \varepsilon K \,.\sim x \varepsilon K$) fails if there is at least one element within the range of significance of its variables. But (1)–(10) imply that there is at least one element in the system and therefore that every such function fails. We may, then, confine attention in what follows to functions which when put in expanded form involve at least one alternative.

Another limiting case is that in which a function involves every possible alternative. All functions of this kind which do not demand existence hold. Any function of this kind which does demand existence holds if there is at least one element within the range of the variables. Hence all such functions hold in view of (1)–(10).

$.\vee.\, Rxy\,.\sim Ryx\,.\,x \neq y\,.\vee.\, Rxy\,.\,Ryx\,.\,x = y\,.\vee.\, Rxy\,.\,Ryx\,.\,x \neq y$ $.\vee.\sim Rxy\,.\,Ryx\,.\,x = y\,.\vee.\sim Rxy\,.\,Ryx\,.\,x \neq y$. This is in terms of $R$ and "$=$", and it is sufficient to multiply the function by $(x \varepsilon K \vee \sim x \varepsilon K)$ and by $(y \varepsilon K \vee \sim y \varepsilon K)$. In a function so expanded any conjunctive constituent involves, for any two variables $x$, $y$, $Rxy$ or it involves $\sim Rxy$, and it involves $Ryx$ or else $\sim Ryx$, and $x \varepsilon K$ or $\sim x \varepsilon K$, and $y \varepsilon K$ or $\sim y \varepsilon K$, and $x = y$ or $x \neq y$. Postulate (1), $(x)\,.\sim Rxx$, is a function of one variable on $R$ alone. $\sim Rxx$ is equivalent to $x \varepsilon K\,.\sim Rxx\,.\vee.$ $\sim x \varepsilon K\,.\sim Rxx$, so that the postulate becomes $(x) : x \varepsilon K\,.\sim Rxx\,.\vee.$ $\sim x \varepsilon K\,.\sim Rxx$. It may, of course, be expanded as a function of two variables, since $(x)\,.\sim Rxx : \equiv : (x, y) : x = y\,.\supset.\sim Rxy$, or as a function of three variables, since $(x)\,.\sim Rxx : \equiv : (x, y, z) : x = y\,.\,y = z\,.\,x = z$ $.\supset.\sim Rxy\,.\sim Ryz\,.\sim Rxz$, or as a function of $n$ variables.

It is easily seen that any singly quantified function in two variables has its truth-value determined by (1)–(10). These functions are of two forms, $(x, y)\,.f(x, y)$ and $(\exists x, y)\,.f(x, y)$. Put any function of the form $(x, y)$ $.f(x, y)$ in expanded form with respect to $K$, $R$, and "$=$". If this function is to hold $f(x, y)$ must involve the alternatives $R^0(xy)$, $R^0(yx)$, $R^0(x)$, $R^0(x)\,.\,S(y)\,.\,T(x; y)$, $R^0(y)\,.\,S(x)\,.\,T(y; x)$, $S(xy)$, and $S(x)$. The condition is necessary and sufficient. With regard to functions of the form $(\exists x, y)\,.f(x, y)$, it is necessary and sufficient that a function of this form involve at least one of these alternatives. Functions of one variable occur as special cases of these two types of function, so that the argument includes them.

A similar analysis is applicable to doubly quantified functions in two variables. Such functions are of the form $(x) : (\exists y)\,.f(x, y)$ or the form $(\exists y) : (x)\,.f(x, y)$.

Consider the first case first. Let $f(x, y)$ be put in expanded form. Such a function might be $(x) : (\exists y)\,.\,R^0(xy) \vee S(xy)$, for example. Consider the following alternatives which may be involved in $f(x, y)$ : (1) $R^0(xy)$, $R^0(yx)$, $R^0(x)$, $R^0(x)\,.\,S(y)\,.\,T(x; y)$, and (2) $S(x)\,.\,R^0(y)\,.\,T(y; x)$, $S(xy)$, $S(x)$. Alternatives (1) have to do with values of $x$ which belong to $K$ and alternatives (2) have to do with values of $x$ not in $K$. There are at least $n$ elements in $K$ and at least $n$ elements not in $K$. From postulates (1), (5), (6), and (9) it follows that any function $(x) : (\exists y)\,.f(x, y)$ which involves at least one of the alternatives (1) and at least one of the alternatives (2) is true. Conversely, any function which does not involve at least one alternative from (1) and at least one from (2) is false. The condition is equivalent to the function.

In the case of functions $(\exists y) : (x)\,.f(x, y)$, for some one and the same $y$ every $x$ is such that $f(x, y)$, let $f(x, y)$ be expressed in expanded form.

Consider the possible alternatives (1) $R^0(xy)$, $R^0(yx)$, $R^0(y)$, $S(x) . R^0(y) . T(y; x)$, and (2) $R^0(x) . S(y) . T(x; y)$, $S(xy)$, $S(y)$. From postulates (1), (2), (3), and (9), if either all of the alternatives in (1) or all of the alternatives in (2) are involved in $f(x, y)$, then $(\exists y) : (x) . f(x, y)$ is true and if this is not the case the function is false. Since the truth-value of every singly quantified function in two variables is determined by (1)–(10), it follows that *every function in two variables is determined by the set.*

Let $F(x, \cdots, n)$ be any multiply quantified propositional function so expressed as to involve a single complex quantifier affecting the function $f(x, \cdots, n)$. Let $f$ be put in expanded form. Then $f(x, \cdots, n)$ is a disjunctive function $f_1(x, \cdots, n) . \vee . f_2(x, \cdots, n) . \vee . \cdots$ etc, where $f_r(x, \cdots, n)$ is a conjunctive function of $x, \cdots, n$ on $K$, $R$, "$=$" and such that it involves the assertion or denial of every propositional construct of $x, \cdots, n$ on $K$, $R$, "$=$". Thus $f_r(x, \cdots, n)$ involves $Rxn$ or it involves $\sim Rxn$; it involves $x \varepsilon K$ or else $\sim x \varepsilon K$; it involves $x = n$ or else $x \neq n$. Denote by $f'(x, \cdots, n)$ a function in expanded form which involves every possible alternative which can be formulated of the variables $x, \cdots, n$. As pointed out above, $F'(x, \cdots, n)$ is true whatever complex quantifier $F'$ may involve. In general $f$ differs from $f'$ in that certain alternatives belonging to $f'$ are excluded from $f$.

Suppose that $F(x. \cdots, n)$ is a function constructed by excluding certain alternatives from $F'(x, \cdots, n)$ in the following way. If $x$ is any variable in $F'$ to which the applicative "some" attaches, then every alternative in $f'$ which involves the constituent $\sim x \varepsilon K$ is excluded in forming $f$. Also, all other alternatives in $f'$ for which at least one variable $z$ is such that $\sim z \varepsilon K$ is asserted are to be retained in $f$. Any remaining alternatives, i. e., those all of whose elements are assigned to $K$, may either be retained or excluded. Such propositions $F(x, \cdots, n)$ place no restriction on elements not in $K$ and, in this sense, may be said to be about $K$-elements only.

There are two theorems which are important in connection with propositions involving a single or higher quantification.

THEOREM IV. $(a, b, c, \cdots, n) :: R^0(abc \cdots n) :. \supset :. (m) :. m \varepsilon K . m \neq a, b, c, \cdots, n : \supset : R^0(mabc \cdots n) . \vee . R^0(ambc \cdots n) . \vee . R^0(abmc \cdots n) . \vee ., \cdots, . \vee . R^0(abc \cdots mn) . \vee . R^0(abc \cdots nm)$.

If $R^0(abc \cdots n)$ holds, then every $m$ in $K$ other than $a, b, c, \cdots n$ is such that some permutation of it with the elements $a, b, c, \cdots n$ is such that $R^0$ holds. This follows immediately from Theorem III.

THEOREM V. $(a, b, c \cdots, n) :. R^0(abc \cdots n) : \supset : (\exists x) . R^0(xabc \cdots n) : (\exists y) . R^0(aybc \cdots n) : (\exists z) . R^0(abzc \cdots n) :, \cdots, : (\exists w) . R^0(abc \cdots wn) : (\exists t) . R^0(abc \cdots nt)$.

If $R^0(abc\cdots n)$ holds, then there is at least one element $x$ such that $R^0(xabc\cdots n)$ and at least one element $y$ such that $R^0(aybc\cdots n)$, and so on. The theorem follows from (1), (5), (6), and (7).

We may determine in a particular case of doubly a quantified function, say $(x, y) :. (\exists z, w) : x, y \varepsilon K \cdot x \neq y \,.\supset. R^0(xyzw) \vee R^0(yxzw)$, whether its truth-value is determined by (1)–(10). When put in expanded form this function may be written $(x, y) : (\exists z, w) \,.\, R^0(xyzw) \vee R^0(yxzw) \vee P(x, y, z, w)$, where $P(x, y, z, w)$ is a set of alternatives involving (*a*) all possible alternatives in which $z \varepsilon K$ and $w \varepsilon K$ occur and in which $\sim x \varepsilon K$ or $\sim y \varepsilon K$ occurs, and (*b*) all possible alternatives in which $x, y, z, w$ are all assigned to $K$ and in which $x = y$. No limitation is placed on these cases by the function. [1]: $(x, y) : (\exists z, w) \,.\, R^0(xyzw) \vee R^0(yxzw) \vee P(x, y, z, w)$ implies [2]: $(x, y) : (\exists z) \,.\, R^0(xyz) \vee R^0(yxz) \vee P(x, y, z)$. Here we simply drop the variable of narrowest scope $w$ and consider those alternatives in the function which involve permutations of the remaining variables. By Theorem V, $R^0(xyz) \,.\supset. (\exists w) \,.\, R^0(xyzw)$ and $R^0(yxz) \,.\supset. (\exists w) \,.\, R^0(yxzw)$ and $P(x, y, z)$, of course, implies $(\exists w) \,.\, P(x, y, z, w)$. Hence [2] implies [1], so that [2] is equivalent to [1]. We have then a function in three variables equivalent to the original function in four variables. [2] implies [3]: $(x, y) \,.\, R^0(xy) \vee R^0(yx) \vee P(x, y)$, and [3] implies [2], by Theorem V. Hence [1] is equivalent to [3], and we have a function in two variables equivalent to the original function in four variables. But [3] is a singly quantified function and its truth-value is determined by (1)–(10). The truth-value of the original function is therefore determined.

The general procedure of the proofs which follow may be outlined: Every multiply quantified function in $n$ variables is shown to be equivalent, in view of (1)–(10), to some function in two variables. This equivalence is, of course, material and not strict since it depends on (1)–(10). The function in two variables to which a function in $n$ variables is shown to be materially equivalent is therefore in no sense another form of the same function. The functions have same truth-value solely in view of (1)–(10).

We consider any multiply quantified propositional function $F(x, \cdots, l, n)$ in expanded form, and the elementary function $f(x, \cdots, l, n)$ in $F$. As we have seen, $f(x, \cdots, l, n) = f_1(x, \cdots, l, n) \vee f_2(x, \cdots, l, n) \vee, \cdots, \vee f_t(x, \cdots, l, n)$, where each of these alternatives is a conjunctive function on $K$, $R$, and "$=$". Any alternative in $f$ which does not have one of the forms $R^0(a\cdots j)$, $S(a\cdots j)$, or $R^0(a\cdots g) \,.\, S(h\cdots j) \,.\, T(a\cdots g; h\cdots j)$ cannot be satisfied by any set of $n$ elements. Every alternative of this kind is to be discarded. Then, by dropping the variable of narrowest scope in $F(x, \cdots, l, n)$, say $n$, from $f(x, \cdots, l, n)$ we obtain a function $f'(x, \cdots, l), = f'_1(x, \cdots, l) \vee f'_2(x, \cdots, l) \vee, \cdots, \vee f'_p(x, \cdots, l)$, which is, of course, implied by $f(x, \cdots, l, n)$.

Let $f_r(x, \cdots, l, n)$ be any alternative in $f(x, \cdots, l, n)$ and let $f'_r(x, \cdots, l)$ be any alternative in $f'(x, \cdots, l)$. Every alternative $f'_r$ is involved in at least one alternative $f_r$ in $f(x, \cdots, l, n)$, and in general $f'_r$ will occur in more than one alternative $f_r$. $f'_r$ is implied by any alternative $f_r$ in which it occurs since $f_r$ involves the assertion of $f'_r$. But $f'_r$ may be true and yet every alternative $f_r$ in which it occurs be false. Now the variable $n$ which is dropped in forming $f'$ may be affected by the applicative "some" or it may be affected by the applicative "every". Suppose first that "some" attaches to $n$. Then it will be shown, by the aid of Theorem V and some supplementary theorems to be given presently, that if $f_r$ gives rise to $f'_r$ when $n$ is dropped from $f_r$, then $f'_r(x, \cdots, l) . \supset . (\exists n) . f_r(x, \cdots, l, n)$. It follows that $f'(x, \cdots, l) \supset . (\exists n) . f(x, \cdots, l, n)$ and therefore that $F'(x, \cdots, l) . \equiv . F(x, \cdots, l, n)$.

Suppose that "every" attaches to $n$. Then, by the aid of Theorems IV and V together with certain supplementary theorems, it is always possible to show that the disjunction $f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$, formed of all those alternatives in which $f'_r$ is involved, is such that $f'_r . \supset . (n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$, or such that $f'_r . \supset . \sim (n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$; $(n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$ is never independent of $f'_r$. But if $f'_r . \supset . \sim (n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$, then $(n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m} . \supset . \sim f'_r$. But it is also true that $(n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m} . \supset . f'_r$. So that when $f'_r \supset . \sim (n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$, $(n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$ is impossible. These alternatives may therefore be dropped from $f(x, \cdots, l, n)$ without altering the truth-value of $F(x, \cdots, l, n)$. When we discard those alternatives shown to be impossible there results an elementary function $f_0(x, \cdots, l, n)$ such that $(n) . f_0(x, \cdots, l, n) . \equiv . (n) . f(x, \cdots, l, n)$ and thus a function $F_0(x, \cdots, l, n)$ such that $F_0(x, \cdots, l, n) . \equiv . F(x, \cdots, l, n)$. We may now drop the variable of narrowest scope $n$ from $f_0(x, \cdots, l, n)$ and obtain an elementary function $f''(x, \cdots, l)$. $(n) . f_0(x, \cdots, l, n)$ implies $f''(x, \cdots, l)$. But also $f''(x, \cdots, l)$ implies $(n) . f_0(x, \cdots, l, n)$, so that $F''(x, \cdots, l) . \equiv . F_0(x, \cdots, l, n) . \equiv . F(x, \cdots, l, n)$, and we have a function in $n-1$ variables with the same truth-value as $F(x, \cdots, l, n)$.

The following theorems are supplementary to IV and V.

THEOREM VI. $S(y \cdots n) . \supset . (\exists a) . S(y \cdots n a)$.

THEOREM VII. $S(y \cdots n) : \supset : (a) : \sim a \varepsilon K . a \neq y, \cdots, n . \supset . S(y \cdots n a)$.

THEOREM VIII. $\Phi(x, \cdots, z, \cdots, n) . \supset . (\exists w) . w = z . \Phi(x, \cdots, w, \cdots, n)$.

THEOREM IX. $\Phi(x, \cdots, z, \cdots, n) . \supset . (w) . w = z \supset \Phi(x, \cdots, w, \cdots, n)$.

In particular, $R_0(x \cdots n) . \supset . (m) . m = n \supset R_0(x \cdots m)$, and $S(y \cdots n) . \supset . (m) . m = n \supset S(x \cdots m)$.

Consider any singly quantified function of the form $(\exists x, \cdots, l, n) . f(x, \cdots, l, n)$. Let $f$ be put in expanded form. The alternatives in $f$ fall into three classes: those in which every element is assigned to the class $K$, those in which every element is assigned to the class not-$K$, and those in which

at least one element is assigned to $K$ and at least one element is assigned to not-$K$. Each of these classes may again be subdivided into those alternatives in which all elements are asserted to be distinct and those which involve an identity condition for at least one pair of elements.

Any alternative in $f$ in which every element is assigned to $K$ and which does not have the form $R^0(x \cdots n)$, where $x, \cdots, n$ are the distinct elements in the alternative, cannot be satisfied by any set of $n$ elements, since every $n$ elements in $K$ must have this form. Note that this applies to alternatives involving identical elements, as well as to those in which all elements are distinct, since identity conditions simply reduce the number of distinct elements. Any such alternative may be dropped from $f$ since the function so obtained will be equivalent to the original one. Likewise, any alternative in which every element is assigned to not-$K$ and which does not have the form $S(y \cdots n)$, where $y, \cdots, n$ are the distinct elements involved, will be false for every $n$ elements and may be dropped from $f$. Any alternative involving $x, \cdots, m$ assigned to $K$, where $x, \cdots, m$ are distinct, and $y, \cdots, n$ assigned to not-$K$, where $y, \cdots, n$ are distinct, must have the form $R^0(x \cdots m) . S(y \cdots n) . T(x \cdots m; y \cdots n)$. Alternatives not having this form are to be discarded.

$(\exists x, \cdots, l, n) . f(x, \cdots, l, n) . \equiv . (\exists x, \cdots, l)\ (\exists n) . f(x, \cdots, l, n)$. Form the function $f'(x, \cdots, l)$ from $f$ by dropping the variable of narrowest scope $n$ and considering those conjunctive functions of $x, \cdots, l$ which remain. Consider the function $f'$, and first any alternative in $f'$ in which $x, \cdots, l$ are all assigned to $K$. Let $a, \cdots, g$ denote $x, \cdots, l$ in any such alternative such that $R^0(a \cdots g)$ holds. Consider those alternatives in $f$ in which $R^0(a \cdots g)$ occurs. Let $P_1, P_2, \cdots, P_S$ be these alternatives. Each of the alternatives is a function of $a, \cdots, g, n$ and asserts $R^0(a \cdots g)$. There are three classes of alternatives in the set $P_1, P_2, \cdots, P_S$: those alternatives in which $n$ is assigned to $K$ and is not identified with any element $a, \cdots, g$, those alternatives in which $n$ is identified with one of the elements $a, \cdots, g$, and those alternatives in which $n$ is assigned to not-$K$.

Take the first case first, $n$ is assigned to $K$ and is not identical with any one of the set $a, \cdots, g$. Let $P_x$ be any such alternative. Then $R^0(a \cdots g)$ implies that for some $n$ in $K$, $P_x$ holds, by Theorem V.

Suppose that $n$ is identified with one of the set $a, \cdots, g$. If there is any such alternative $P_y$ in $f$, $P_y$ must assert simply $R^0(a \cdots g)$, by Theorem VIII. Any other possibility has already been dropped from $f$.

If $P_z$ be any alternative in which $n$ is assigned to not-$K$, then $P_z$ must be $R^0(a \cdots g) . S(n) . T(a \cdots g; n)$. Alternatives of the kind not having this structure have already been dropped from $f$.

We may next take account of those alternatives in $f'$ in which every element $x, \cdots, l$ is assigned to not-$K$. Let $P_x$ be any alternative in $f$ in which $x, \cdots l$ are assigned to not-$K$ and $n$ is assigned to $K$. Then $P_x$ must be $R^0(n) \,.\, S(x \cdots l) \,.\, T(n; x, \cdots, l)$. Alternatives in which this is not the case have already been excluded. When $n$ is assigned to not-$K$ and is distinct from each of the elements $x, \cdots, l$, we must have $S(x \ldots l\, n)$. When $n$ is identical with some one of the elements $x, \cdots, l$, we must have simply $S(x \cdots l)$ in $f$.

Finally, there are those alternatives in $f'$ in which at least one element is assigned to $K$ and at least one element is assigned to not-$K$. Let $a, \cdots, g, h, \cdots, j$ be distinct elements in any such alternative and such that $a, \cdots, g$ are assigned to $K$ and $h, \cdots, j$ to not-$K$ and $R^0(a \cdots g)$ holds. Then we have $R^0(a \cdots g) \,.\, S(h \cdots j) \,.\, T(a \cdots g; h \cdots j)$ since other possibilities have been excluded. Here there are three possibilities. (1) $n$ is identified with one of the set $a, \cdots, g, h, \cdots, j$. Any such alternative in $f$ must be identical with $R(a \cdots g) \,.\, S(h \cdots j)\, T(a \cdots g;\ h \ldots j)$. (2) $n$ is assigned to $K$ and is distinct from $a, \cdots, g$. In this case, for any $P_x$ in $f$, Theorem V implies $P_x$. (3) $n$ is assigned to not-$K$ and is distinct from $h, \cdots, j$. In this case any alternative $P_y$ in $f$ must have the form $R^0(a \cdots g)$. $S(h \cdots j\, n) \,.\, T(a \cdots g;\ h \cdots j\, n)$, and this will in fact be the case.

It follows that, after eliminating from $f(x, \cdots, l, n)$ those alternatives the form of which it is impossible that any $n$ elements should have, every alternative in $f$ is entailed, for some $n$, by the corresponding alternative in $f'$, and since $f$ implies $f'$, we have $(\exists x, \cdots, l, n) \,.\, f(x, \cdots, l, n) \,.\equiv.\, (\exists x, \cdots, l) \,.\, f'(x, \cdots, l)$. Here we have a function in $n-1$ variables equivalent to the original function in $n$ variables. In precisely the same way we are able to find a function in $n-2$ variables which is equivalent to the function in $n-1$ variables and consequently to the function in $n$ variables. This process may be continued until a function in two variables (whose truth-value is determined) is obtained. It follows that every function of the form $(\exists x, \cdots, l) \,.\, f(x, \cdots, l)$ has its truth-value determined by (1)–(10).

Let $F(x, \cdots, n)$ be any function of the form $(x, \cdots, l, n) \,.\, f(x, \cdots, l, n)$. It will be shown that $F(x, \cdots, n)$ follows from (1)–(10) or else $\sim F(x, \cdots, n)$ follows. As before, put $f$ in expanded form and discard all those alternatives which do not have one of the forms $R^0(a \cdots g)$, $S(h \cdots j)$, or $R^0(a \cdots g) \cdot S(h \cdots j) \cdot T(a \cdots g; h \cdots j)$. Then form the function $f'$ by dropping from $f$ the variable of narrowest scope $n$.

Let $a, \cdots, g$ denote the elements in any alternative in $f'$ in which all elements are assigned to $K$ and such that $R^0(a \cdots g)$ holds. Consider the set of alternatives $P_1, P_2, \cdots, P_s$ in $f$ each of which involves the assertion $R^0(a \cdots g)$. Let $P_a, \cdots, P_d$ be those alternatives among $P_1, P_2, \cdots, P_s$

in which $n$ is assigned to $K$ and is distinct from every element $a, \cdots, g$. Then $R^0(a \cdots g)$ may imply, by Theorem IV, that $P_a \vee, \cdots, \vee P_d$ holds. But if this is not the case, then Theorem V implies that $P_a \vee, \cdots, \vee P_d$ is false for some value of $n$. But if $P_a \vee, \cdots, \vee P_d$ is false, then all the alternatives $P_1, P_2, \cdots, P_s$ may be dropped from $f$. For the function $F(x, \cdots, n)$ demands $P_1 \vee P_2 \vee, \cdots, \vee P_s$ for every value of $n$ when $R^0(a \cdots g)$ is satisfied. But $P_a \vee, \cdots, \vee P_d$ fails for some value of $n$ in $K$ and different from $a, \cdots, g$. And none of the other alternatives can be satisfied by this value of $n$, so that $P_1 \vee P_2 \vee, \cdots, \vee P_s$ fails for this value. Hence $P_1 \vee P_2 \vee, \cdots, \vee P_s$ cannot be true for every value of $n$ when $R^0(a \cdots g)$ holds, and it, of course, cannot be true for at least one value when $R^0(a \cdots g)$ fails. We omit for the present consideration of the case in which Theorem IV implies $P_a \vee, \cdots, \vee P_d$.

When $n$ is identified with one of the elements $a, \cdots, g$ we must have $R^0(a \cdots g)$ as an alternative in $f$. If there is no such alternative, $P_1 \vee P_2 \vee, \cdots, \vee P_s$ cannot be true for every value of $n$, as demanded by $F(x, \cdots, n)$, and may accordingly be discarded.

When $n$ is assigned to not-$K$, we must have $R^0(a \cdots g) . S(n) . T(a \cdots g; n)$ among $P_1, P_2, \cdots, P_s$. If no such alternative occurs, then $P_1 \vee P_2 \vee, \cdots, \vee P_s$ cannot hold for every value of $n$.

On the other hand, if, by Theorem IV, $R^0(a \cdots g)$ entails $P_a \vee, \cdots, \vee P_d$, and if there is an alternative among $P_1, P_2, \cdots, P_s$ which asserts simply $R^0(a \cdots g)$, and if there is an alternative of the form $R^0(a \cdots g) . S(n) . T(a \cdots g; n)$, then $R^0(a \cdots g) . \supset . (n) . P_1 \vee P_2 \vee, \cdots, \vee P_s$, and these alternatives are to be retained in $f$.

Let $h, \cdots, j$ denote the elements in any alternative in $f'$ in which all of the elements are assigned to not-$K$. Then we have $S(h \cdots j)$ in $f'$. For every $n$ not in $K$ and distinct from $h, \cdots, j$, $S(h \cdots j)$ entails $S(h \cdots jn)$, so that we must have at least one alternative $S(h \cdots jn)$ in $f$. When $n$ is identified with one of the elements $h, \cdots, j$, we must have an alternative $S(h \cdots j)$ in $f$. For the case in which $n$ is assigned to $K$, we must have an alternative $R^0(n) . S(h \cdots j) . T(n; h \cdots j)$ in $f$. If all of these demands are satisfied, then all alternatives in $f$ which involve $h, \cdots, j$, where $h, \cdots, j$ are assigned to not-$K$, are to be retained. If one of the demands fails, then every alternative in $f$ involving $h, \cdots, j$ is to be dropped.

Let $a, \cdots, g, h, \cdots, j$ denote the elements in any alternative in $f'$ in which $a, \cdots, g$ are assigned to $K$ and such that $R^0(a \cdots g)$ holds and $h, \cdots, j$ are assigned to not-$K$. For any such alternative in $f'$, let $P_1, P_2, \cdots, P_s$ be the set of alternatives in $f$ which contain it. (1) When $n$ is identified with one of the elements $a, \cdots, g; h, \cdots, j$, we must have

simply $R^0(a \cdots g) . S(h \cdots j) . T(a \cdots g; h \cdots j)$ as an alternative in $f$. If no such alternative occurs, then all of the set $P_1, P_2, \cdots, P_s$ may be dropped, for $F$ asserts $P_1 \vee P_2 \vee, \cdots, \vee P_s$ for every value of $n$. (2) Let $P_a, \cdots, P_d$ be those alternatives in which $n$ is assigned to $K$ and is distinct from $a, \cdots, g$. Then it may follow from Theorem IV that $P_a \vee, \cdots, \vee P_d$ holds for every such $n$. If not it follows from Theorem V that the function $P_a \vee, \cdots, \vee P_d$ fails for at least one value of $n$. (3) When $n$ is assigned to the class not-$K$ we must have an alternative $R^0(a \cdots g) . S(h \cdots jn) . T(a \cdots g; h \cdots jn)$ in $f$. If no such alternative occurs, then $P_1, P_2, \cdots, P_s$ may all be discarded. But if conditions (1), (2), and (3) are satisfied, $P_1, P_2, \cdots, P_s$ are to be retained in $f$.

Consider the function $F(x, \cdots, n)$ as reconstituted by the elimination of the alternatives indicated. $(x, \cdots, l, n) . f(x, \cdots, l, n) . \supset . (x, \cdots, l) f''(x \cdots, l)$, where the latter function is obtained from $F$ by dropping the variable of narrowest scope $n$. But also, in view of the foregoing eliminations, $(x, \cdots, l) . f''(x, \cdots, l) . \supset . (x, \cdots, l, n) . f(x, \cdots, l, n)$, so that we have a function in $n - 1$ variables which has the same truth-value as the original function in $n$ variables. In like manner a function in $n - 2$ variables can be found which has the same truth-value as the function in $n - 1$ variables, and so on.

It may happen at some stage of the process of passing from a function $F_r(x, \cdots, m)$ in $m$ variables to a function $F_r''(x, \cdots, m-1)$ in $m-1$ variables that every alternative in $f_r(x, \cdots, m)$ vanishes. But, whatever complex quantifier $F_r$ may involve, $F_r(x, \cdots, m) . \supset . (\exists x, \cdots, m) . f_r(x, \cdots, m)$ And any function of the form $(\exists x, \cdots, m) . f_r(x, \cdots, m)$ must, if it is to be true, involve at least one alternative. So that $F_r(x, \cdots, m)$ must be false, and consequently the original function $F(x, \cdots, n)$, to which it is equivalent, must be false, and the truth-value of $F(x, \cdots, n)$ is therefore determined. But even here it is still possible to proceed to a function in two variables $F_w(x, y)$. For $F_r''(x, \cdots, m-1)$ is obtained from $F_r(x, \cdots, m)$ by dropping the variable of narrowest scope, and $F_r''(x, \cdots, m-1)$ will, therefore, involve no alternatives—i. e., it will involve the denial of every possible alternative. And finally, we obtain $F_w(x, y)$, in two variables, which involves no alternatives.

In any case therefore we reach a function in two variables whose truth-value is determined. It follows that *every singly quantified first-order function on the base $K$, $R_2$ has its truth-value determined by conditions* (1)–(10).

Strictly it is only necessary to carry out the proof of the theorem just given for at least one of the functions $(\exists x, \cdots, l, n) . f(x, \cdots, l, n)$ or $(x, \cdots, l, n) . f(x, \cdots, l, n)$. For suppose that the theorem has been established for functions of the first kind. The contradictory of any function

of the second kind is of the form $(\exists x, \cdots, l, n) . \sim f(x, \cdots, l, n)$, which is a function of the first kind and its truth-value is, by hypothesis, determined. But if the truth-value of the contradictory of a function is determined, the truth-value of the function is, of course, determined. Similarly, the contradictory of any function of the first kind is a function of the second kind.

Let $F(x, \cdots, l, n)$ be any multiply quantified propositional function on the base $K$, $R_2$, and let $f(x, \cdots, l, n)$ be the elementary function in $F$. Put $f$ in expanded form. Discard from $f$ every alternative which does not have one of the forms $R^0(a \cdots j)$, $S(a \cdots j)$, or $R^0(a \cdots g) . S(h \cdots j) . T(a \cdots g; h \cdots j)$. This on the ground that the resulting function is equivalent to the original one since other forms of alternative are impossible.

Form the function $f'$ by dropping the variable of narrowest scope $n$ from $f$. Suppose that the applicative "some" attaches to $n$. Then, by an argument identical with that employed in the first part of the theorem just given, there exists a function $F''(x, \cdots, l)$ — which in this case is identical with $F'(x, \cdots, l)$—such that $F''(x, \cdots, l) . \equiv . F(x, \cdots, l, n)$. Suppose that the applicative "every" attaches to $n$. Then, by an argument identical with that of the second part of the theorem just given, there exists a function $F''(x, \cdots, l)$ such that $F''(x, \cdots, l) . \equiv . F(x, \cdots, l, n)$. In either case then there exists a function in $n-1$ variables having the same truth-value as the original function in $n$ variables, and by a repetition of the argument, there exists such a function in $n-2$ variables, and finally, a function in two variables having the same truth-value as the original function. But every function in two variables has its truth-value determined. It follows that *every first-order function on the base $K$, $R_2$ has its truth-value determined by postulates* (1)–(10).

**Dense Series with a First Element.** For a dense series having a first but no last element we require an alteration in one postulate of the set (1)–(10). (6) is to be replaced by $(6') : (\exists x) :. (y) : x \in K : y \in K . x \neq y . \supset . Rxy$. In connection with the set (1)–(6′)–(10) we wish to prove theorems analogous to those proved for the set (1)–(10). All of the results not dependent on (6) carry over unchanged. As a consequence of (1)–(4), every $n$ elements in $K$ are such that for some permutation, $x, \cdots, n$, $R^0(x \cdots n)$ holds, and from (1)–(4) together with (5) and (8) some $n$ elements in $K$ are such that $R^0(x \cdots n)$ holds. There are at least $n$ elements not in $K$, from (10), and for every $n$ elements not in $K$, $S(y \cdots n)$ holds. Every singly quantified first-order function in two variables on the base $K$, $R_2$ has its truth-value determined by (1)–(6′)–(10). None of these facts depends on (6), so that all of them carry over from previous proofs.

From (6′) it follows that $(\exists x)\ (y, \cdots n) : x \neq y, \cdots, n . R^0(y \cdots n) . \supset . R^0(xy \cdots n)$. The element $x$ here is clearly unique; there can be only

one first element. We may denote the element $x$, which is such that for every other $K$-element $y$, $Rxy$ holds, by "$a$". Then $(y):y\varepsilon K.y\neq a.\supset.Ray$. This function of "$a$" may be denoted by $g(a)$. For any other $K$-element $b$, $(\exists y).y\varepsilon K.y\neq b.\sim Rby$ holds. This function of $b$ is, of course, $\sim g(b)$. Accordingly, $g(a)$ holds and $(z):z\neq a.\supset.\sim g(z)$. Note that this latter function refers to values of $z$ not in $K$ as well — $\sim g(z)$ for every $z$ other than $a$. Since there is only one element $a$ such that $g(a)$ holds, the class $K$ can be separated into two non-overlapping subclasses $K_1$ and $K_2$ such that $K_1$ has in it the single element $a$ and $K_2$ has every other $K$-element.

If $R^0(bc\cdots e)$ holds and $b, c, \cdots, e\varepsilon K_2$, then also $\sim g(b).\sim g(c).,\cdots,.\sim g(e)$ is true. But if $R^0(ab\cdots e)$ holds, where $a\varepsilon K_1$, then $g(a).\sim g(b).,\ldots,.\sim g(e)$. Let us denote $R^0(bc\ldots e).\sim g(b).\sim g(c).,\cdots,\sim g(e)$ by $R_0(bc\cdots e)$. But $R_0(ab\cdots e)$ is to mean $R^0(ab\cdots e).g(a).\sim g(b).,\cdots,.\sim g(e)$: when the element $a$ occurs in $R_0$ it is to be understood to assert $g(a)$. For any other element $z$ it asserts $\sim g(z)$. Similarly, if $S(yz\cdots n)$ holds, then $\sim g(y).\sim g(z).,\cdots,.\sim g(n)$. We may denote $S(yz\cdots n).\sim g(y).\sim g(z).,\cdots,.\sim g(n)$ by $S_0(yz\cdots n)$.

Denote by $F(x, y, \cdots, n)$ any propositional function on $K$, $R_2$ and let $f(x, y, \cdots, n)$ be the elementary function in $F$. Put $f(x, y, \cdots, n)$ in expanded form with respect to $K$, $R$, and "$=$". On the variables $x, y, \cdots, n$ form the functions $g(x)\vee\sim g(x); g(y)\vee\sim g(y);\cdots; g(n)\vee\sim g(n)$. Multiply $f(x, y, \cdots, n)$ by each of these functions. This puts $f(x, y, \cdots, n)$ in expanded form with respect $K$, $R$, "$=$", and $g$. Every alternative in $f$ involves, for any variable $z$, $g(z)$ or else it involves $\sim g(z)$.

The following theorems are of importance.

THEOREM X. $R_0(ab\cdots n):\supset:(z):z\neq a, b, \cdots, n.z\varepsilon K.\supset.R_0(azb\cdots n)\vee R_0(abz\cdots n)\vee,\cdots,\vee R_0(ab\cdots zn)\vee R_0(ab\cdots nz)$.

The only possibility excluded here is $R_0(zab\cdots n)$.

THEOREM XI. $R_0(ab\cdots n):\supset:(\exists z).R_0(azb\cdots n):(\exists t).R_0(abt\cdots n):,\cdots,:(\exists u).R_0(ab\cdots un):(\exists w).R_0(ab\cdots nw)$.

The only possibility omitted is $(\exists s).R_0(sab\cdots n)$. In these theorems $a\varepsilon K_1$. For elements in $K_2$ theorems analogous to IV and V are clearly entailed.

THEOREM XII. $S_0(xy\cdots n):\supset:(w):\sim w\varepsilon K.w\neq x, y, \cdots, n.\supset.S_0(xy\cdots nw)$.

THEOREM XIII. $S_0(xy\cdots n):\supset:(\exists w).S_0(xy\cdots nw)$.

It will be shown that every doubly quantified first order function in two variables has its truth-value determined by (1)–(6′)–(10). Any such function has one of the forms $(x):(\exists y).\varphi(x, y)$ or $(\exists x):(y).\varphi(x, y)$. Consider any function of the form $(x):(\exists y).\varphi(x, y)$. Put $\varphi(x, y)$ in expanded form with respect to $K$, $R$, "$=$", and $g$. Then in accordance with (1)–(6′)–(10), if the function is to be true, $\varphi(x, y)$ must contain at

least one alternative from each of the following three sets: (1) When $x$ takes the value $a$ in $K_1$ we must have $R_0(a)$, or $R_0(ay)$, or $R_0(a) . S_0(y) . T(a;y)$. (2) when $x$ takes values in $K_2$ we must have $R_0(ax)$, or $R_0(xy)$, or $R_0(yx)$, or $R_0(x)$, or $R_0(x) . S_0(y) . T(x; y)$. (3) when $x$ takes values in not-$K$, $R_0(a) . S_0(x) . T(a: x)$, or $R_0(y) . S_0(x) . T(y; x)$, or $S_0(xy)$, or $S_0(x)$. For a function $(x) : (\exists y) . \varphi(x, y)$ to be true it is clearly necessary and sufficient that it contain at least one alternative from each of these sets. With regard to functions of the form $(\exists x) : (y) . \varphi(x, y)$, note that $(\exists x) : (y) . \varphi(x, y) :\equiv: \sim (x) : (\exists y) . \sim \varphi(x, y)$, and that $(x) : (\exists y) . \sim \varphi(x, y)$ has its truth-value determined.

Since every singly quantified function in two variables is determined by (1)–(6′)–(10), it follows that every function in two variables has its truth-value determined by the set.

Let $F(x, \cdots, l, n)$ be any multiply quantified function on $K, R_2$ such that $f(x, \cdots, l, n)$ is in expanded form with respect to $K, R$, "$=$", and $g$. It is clear that any alternative in $f$, if it is ever to be satisfied, must have one of the forms $R_0(a)$, $R_0(b \cdots e)$, $S_0(h \cdots j)$, $R_0(ab \cdots e)$, $R_0(a) . S_0(h \cdots j) . T(a; h \cdots j)$, $R_0(b \cdots e) . S_0(h \cdots j) . T(b \cdots e; h \cdots j)$, or $R_0(ab \cdots e) . S_0(h \cdots j) . T(ab \cdots e; h \cdots j)$. Every alternative not having one or the other of these forms is to be dropped from $f$. Any alternative which involves the element $a$ and elements belonging to $K_2$, if it is to be possible, must be such that $a$ is asserted to have the relation $R$ to every element assigned to $K_2$. Also, no alternative can hold which involves $g(x) . g(y) . x \neq y$; there cannot be more than one element $x$ such that $g(x)$ holds in any alternative.

Form $f'(x, \cdots, l)$ by dropping from $f$ the variable of narrowest scope $n$. The alternatives in $f'$ fall into seven classes corresponding to the seven types of alternative given above. One or more of these classes may, of course, be null.

Consider $f'$ in relation to $f$, and suppose first that $n$ is a variable to which the applicative "some" attaches. Now any alternative in $f'$ must have one of the seven forms given above, as must any alternative in $f$. (1) Suppose that $R_0(a)$ appears in $f'$. Then it must have been derived from one or more of the functions $R_0(a)$, $R_0(an)$, $R_0(a) . S_0(n) . T(a; n)$. But $R_0(a) : \supset : (\exists n) . n = a . R_0(a) : (\exists n) . R_0(an) : (\exists n) . R_0(a) . S_0(n) . T(a; n)$. So that whatever alternatives in $f$ may have given rise to $R_0(a)$, $R_0(a)$ implies any one of them and therefore their disjunction. (2) Suppose that $R_0(b \cdots e)$ occurs in $f'$. Then it may have arisen from $R_0(b \cdots e)$, or $R_0(ab \cdots e)$, where $n = a$, or $R_0(nb \cdots e)$, where $n \neq a$, or $. R_0(b \cdots n \cdots e)$, or $R_0(b \cdots en)$, or $R_0(b \cdots e) . S_0(n) . T(b \cdots e; n)$, or from any combination of these. But $R_0(b \cdots e)$ implies, for some $n$, any one of these alternatives. (3) If $S_0(h \cdots j)$ occurs in $f'$, then it must have arisen from some of the

set $S_0(h \cdots j)$, $S_0(h \cdots jn)$, $R_0(a) . S_0(h \cdots j) . T(a; h \cdots j)$, where $n = a$, or $R_0(n) . S_0(h \cdots j) . T(n; h \cdots j)$, where $n \neq a$. But $S_0(h \cdots j)$ implies any one of these for some $n$. (4) If $R_0(ab \cdots e)$ occurs in $f'$, then it must have arisen from $R_0(ab \cdots e)$, or $R_0(anb \cdots e)$, or $R_0(ab \cdots n \cdots e)$, or $R_0(ab \cdots en)$, or $R_0(ab \cdots e) . S_0(n) . T(ab \cdots e; n)$. (5) If $R_0(a) . S_0(h \cdots j) . T(a; h \cdots j)$ occurs in $f'$, then it must have arisen from $R_0(a) . S_0(h \cdots j) . T(a; h \cdots j)$, or $R_0(an) . S_0(h \cdots j) . T(an; h \cdots j)$, or $R_0(a) . S_0(h \cdots jn) . T(a; h \cdots jn)$. (6) If an alternative $R_0(b \cdots e) . S_0(h \cdots j) . T(b \cdots e; h \cdots j)$ occurs in $f'$, then it must have arisen from $R_0(b \cdots e) . S_0(h \cdots j) . T(b \cdots e; h \cdots j)$, or $R_0(ab \cdots e) . S_0(h \cdots j) . T(ab \cdots e; h \cdots j)$, where $a = n$, or $R_0(nb \cdots e) . S_0(h \cdots j) . T(nb \cdots e; h \cdots j)$, or $R_0(b \cdots n \cdots e) . S_0(h \cdots j) . T(b \cdots n \cdots e; h \cdots j)$, or $R_0(b \cdots en) . S_0(h \cdots j) . T(b \cdots en; h \cdots j)$, or $R_0(b \cdots e) . S_0(h \cdots jn) . T(b \cdots e; h \cdots jn)$. (7) If $R_0(ab \cdots e) . S_0(h \cdots j) . T(ab \cdots e; h \cdots j)$ occurs, then it arises from $R_0(ab \cdots e) . S_0(h \cdots j) . T(ab \cdots e; h \cdots j)$, or $R_0(anb \cdots e) . S_0(h \cdots j) . T(anb \cdots e; h \cdots j)$, or $R_0(ab \cdots n \cdots e) . S_0(h \cdots j) . T(ab \cdots n \cdots e; h \cdots j)$, or $R_0(ab \cdots en) . S_0(h \cdots j) . T(ab \cdots en; h \cdots j)$, or $R_0(ab \cdots e) . S_0(h \cdots jn) . T(ab \cdots e; h \cdots jn)$.

It follows that $F'(x, \cdots, l) . \supset . F(x, \cdots, l, n)$, and since $F(x, \cdots, l, n) . \supset . F'(x, \cdots, l)$, that $F'(x, \cdots, l) . \equiv . F(x, \cdots, l, n)$.

Suppose that the variable of narrowest scope $n$ in $f$ is such that the applicative "every" attaches to it, that is, that $n$ is quantified universally. As before there are seven classes of alternatives in $f'$, any one of which may be null. We have pointed out in the first part of this theorem that to each alternative in $f'$, there corresponds a set of alternatives in $f$, and we have specified under seven headings precisely what the alternatives are from which a given alternative in $f'$ may arise. In view of Theorems VIII, IX, X, XI, XII and XIII, if a given alternative in $f'$ is satisfied, a disjunctive function involving all of the corresponding alternatives in $f$ is implied, and moreover any disjunctive function lacking at least one of these alternatives is implied to be false. For example, $R_0(a) : \supset : (n) : n = a . R_0(a) . \vee . R_0(an) . \vee . R_0(a) . S_0(n) . T(a; n)$. But $R_0(a) : \supset : (\exists n) . R_0(a) . n = a : (\exists n) . R_0(an) : (\exists n) . R_0(a) . S_0(n) . T(a; n)$. So that if any alternative is lacking in the first implication, it is false that the implication holds for every $n$. But if this is so, the alternatives in $f$ can never by satisfied, since they cannot be satisfied when $R_0(a)$ is false. Accordingly, we may discard every set of alternatives which does not follow, for every $n$, from the corresponding alternative in $f'$, and retain all those sets which do follow.

Consider $F(x, \cdots, l, n)$ thus reconstituted. Form $F''(x, \cdots, l)$ by dropping $n$ from $F$. $F(x, \cdots . l, n) . \equiv . F''(x, \cdots, l)$. Whether "some"

applies to $n$ or "every" applies to $n$, we have a function in $n-1$ variables equivalent to $F(x, \cdots, l, n)$. It follows that *every first-order function on* $K, R_2$ *has its truth-value determined by* (1)–(6′)–(10).

**Dense Series with Both Extreme Elements.** The properties of dense series having both extreme elements differ from those of the set (1)–(10) in that (5) is replaced by $(5'): (\exists x) :. (y) : x \varepsilon K : y \neq x \,.\, y \varepsilon K \,.\supset.\, Rxy$, and in that (6) is replaced by $(6'): (\exists x) :. (y) : x \varepsilon K : y \neq x \,.\, y \varepsilon K \,.\supset.\, Ryx$. It is also necessary to replace (8) by $(8'): (\exists x, y) \,.\, x \neq y \,.\, x \varepsilon K \,.\, y \varepsilon K$. All of the remaining properties are the same. In view of (7) and (8′) there are at least $n$ elements in $K$. That every $n$ elements in $K$ are such that for some permutation, $x, y, \cdots, n$, $R^0(x, y, \cdots, n)$ holds carries over from the properties of the set (1)–(10). We may denote the set obtained by altering these three properties of (1)–(10) by $(1'), \cdots, (10')$. Then the truth-value of every singly quantified function in two variable on $K, R_2$ is determined by the set (1′)–(10′). It follows immediately that

$$(\exists x, y) :. (z) : z \varepsilon K \,.\, z \neq x \,.\, z \neq y \,.\supset.\, R^0(xzy)$$

and that

$$(\exists x, y) :. (z, \cdots, n) : x \neq z, \cdots, n \,.\, y \neq z, \cdots, n \,.\, R^0(z \cdots n) \,.\supset.\, R^0(xz \cdots ny).$$

$x$ and $y$ are clearly unique and distinct. And since this is the case, the class $K$ can be divided into three non-overlapping subclasses, $K_1$, $K_2$ and $K_3$, where $K_1$ comprises the single element $x$ and $K_3$ the single element $y$ and $K_2$ comprises all other elements which belong to $K$. Denote the element in $K_1$ by "$a$" and the element in $K_3$ by "$c$". Then $(x) : x \varepsilon K \,.\, x \neq a \,.\supset Rax$, and "$a$" is the only element which has this property. As before, denote this function by $g$, so that $g(a)$ holds and $(z) : z \neq a \,.\supset.\, \sim g(z)$. $(x) : x \varepsilon K \,.\, x \neq c \,.\supset.\, Rxc$, and "$c$" is the only element which has this property. Denote this function by $q$, so that $q(c)$ holds and $(z) : z \neq c \,.\supset.\, \sim q(z)$.

Let $F(x, \cdots, l, n)$ be any function on $K, R_2$. Then $f(x, \cdots, l, n)$ can be expressed in expanded form with respect to $K$, $R$, "$=$", $g$, and $q$. $R_0$ and $S_0$ will be used in this section in senses very much like their previous ones. $R_0(b \cdots e)$ is to mean $R^0(b \cdots e) \,.\sim g(b) \,.,\cdots,.\sim g(e) \,.\sim q(b) \,.,\cdots,.\sim q(e)$. But $R_0(ab \cdots e)$ is to mean $R^0(ab \cdots e) \,.\, g(a) \,.\sim g(b) \,.,\cdots,.\sim g(e) \,.\sim q(a) \,.\sim q(b) \,.,\cdots,.\sim q(e)$, and $R_0(b \cdots ec)$ is to mean $R^0(b \cdots ec) \,.\sim g(b) \,.,\cdots,.\sim g(e) \,.\sim g(c) \,.\sim q(b) \,.,\cdots,.\sim q(e) \,.\, q(c)$, while $R_0(ab \cdots ec)$ is to mean $R^0(ab \cdots ec) \,.\, g(a) \,.\sim g(b) \,.,\cdots,.\sim g(e) \,.\sim g(c) \,.\sim q(a) \,.\sim q(b) \,.,\cdots,.\sim q(e) \,.\, q(c)$. $S_0(h \cdots j)$ is to mean $S(h \cdots j) \,.\sim g(h) \,.,\cdots,.\sim g(j) \,.\sim q(h) \,.,\cdots,.\sim q(j)$.

THEOREM XIV. $R_0(ab \cdots nc) :\supset: (z) \,.\, z \varepsilon K \,.\, z \neq a, b, \cdots, n, c \,.\supset.\, R_0(azb \cdots nc) \vee R_0(abz \cdots nc) \vee, \cdots, \vee R_0(ab \cdots znc) \vee R_0(ab \cdots nzc)$.

The only possibilities excluded are $R_0(zab \cdots nc)$ and $R_0(ab \cdots ncz)$.

THEOREM XV. $R_0(ab\cdots nc):\supset:(\exists z).R_0(azb\cdots nc):(\exists t).R_0(abt\cdots nc):,$ $\cdots,:(\exists u).R_0(ab\cdots unc):(\exists w).R_0(ab\cdots nwc)$.

This omits the possibilities $(\exists r).R_0(rab\cdots nc)$ and $(\exists s).R_0(ab\cdots ncs)$. They would be incompatible with XIV.

THEOREM XVI. $S_0(xy\cdots n):\supset:(z):\sim z\varepsilon K.z\neq x,y,\cdots,n.\supset.S_0(xy\cdots nz)$.

THEOREM XVII. $S_0(xy\cdots n).\supset.(\exists z).S_0(xy\cdots nz)$.

It will be shown that every doubly quantified first-order function in two variables has its truth-value determined by (1′)–(10′). If every function of the form $(x):(\exists y).\varphi(x,y)$ is dependent on (1′)–(10′), then every function $(\exists x):(y).\varphi(x,y)$ is dependent, since $(\exists x):(y).\varphi(x,y):\equiv:\sim(x):(\exists y).\sim\varphi(x,y)$. Consider any function $(x):(\exists y).\varphi(x,y)$ in which $\varphi(x,y)$ is in expanded form with respect to $K$, $R$, "$=$", $g$, and $q$. Then when $x$ takes the value $a$ in $K_1$, if the function is to hold, we must have at least one of the following alternatives in $\varphi(x,y)$. (1) $R_0(a)$, where $y=a$, or $R_0(ay)$, or $R_0(ac)$, where $c\varepsilon K_3$, or $R_0(a).S_0(y).T(a;y)$. When $x$ takes values in $K_2$ we must have (2) $R_0(x)$, where $y=x$, or $R_0(xy)$, or $R_0(yx)$, or $R_0(ax)$, where $y=a$, or $R_0(xc)$, where $y=c$, or $R_0(x).S_0(y).T(x;y)$. When $x$ takes the value $c$ we must have (3) $R_0(c)$, where $y=c$, or $R_0(ac)$, or $R_0(yc)$, or $R_0(c).S_0(y).T(c;y)$. For values of $x$ in not-$K$ we must have (4) $S_0(x)$, where $y=x$, or $S_0(xy)$, or $R_0(a).S_0(x).T(a;x)$ or $R_0(y).S_0(x).T(y;x)$, or $R_0(c).S_0(x).T(c;x)$. To contain at least one alternative from each of these sets is necessary and sufficient for the truth of $(x):(\exists y).\varphi(x,y)$. Since every singly quantified function is dependent on (1′)–(10′), it follows that every first-order function in two variables on $K$, $R_2$ is dependent on conditions (1′)–(10′).

To return to functions of $n$ variables; consider $F(x,\cdots,l,n)$ where $f(x,\cdots,l,n)$ is in expanded form with respect to $K$, $R$, "$=$", $g$, and $q$. Every alternative in $f$ must have one of the forms $R_0(a)$, $R_0(c)$, $R_0(b\cdots e)$, $R_0(ab\cdots e)$, $R_0(b\cdots ec)$, $R_0(ab\cdots ec)$, $R_0(ac)$, $S_0(h\cdots j)$, $R_0(a).S_0(h\cdots j).T(a;h\cdots j)$, $R_0(c).S_0(h\cdots j).T(c;h\cdots j)$, $R_0(b\cdots e).S_0(h\cdots j).T(b\cdots e;h\cdots j)$, $R_0(ab\cdots e).S_0(h\cdots j).T(ab\cdots e;h\cdots j)$, $R_0(b\cdots ec).S_0(h\cdots j).T(b\cdots ec;h\cdots j)$, $R_0(ab\cdots ec).S_0(h\cdots j).T(ab\cdots ec;h\cdots j)$, or $R_0(ac).S_0(h\cdots j).T(ac;h\cdots j)$. Any alternative in $f$ not having one of these forms is to be discarded. The resulting function will have the same truth-value as the original one. Form $f'(x,\cdots,l)$ from $f(x,\cdots,l,n)$ by dropping the variable of narrowest scope $n$. Let $P(x,\cdots,l,n)$ represent any alternative in $f$, and let $P'(x,\cdots,l)$ be the alternative obtained from $P$ by dropping $n$. In view of theorems VIII, XV, and XVII, it is clear that $P'(x,\cdots,l).\supset.(\exists n).P(x,\cdots,l,n)$, and, of course, $(\exists n).P(x,\cdots,l,n).\supset.P'(x,\cdots,l)$. It follows that $F'(x,\cdots,l).\equiv.F(x,\cdots,l,n)$ if the applicative "some" attaches to $n$.

Suppose that the applicative "every" attaches to $n$, and consider the following cases. (1) Let $R_0(a)$ be an alternative in $f'$, and let $P_1, P_2, \cdots, P_s$ be the corresponding alternatives in $f$. Then for $n = a$ we must have $R_0(a)$ among these alternatives. For $n \varepsilon K_2$ we must have $R_0(an)$, and for $n \varepsilon K_3$ we must have $R_0(ac)$, where $n = c$. For $n \varepsilon$ not-$K$ we must have $R_0(a) . S_0(n) . T(a; n)$. If all of these alternatives occur among $P_1, P_2, \cdots, P_s$, $P_1 \vee P_2 \vee, \cdots, \vee P_s$ holds for every $n$ when $R_0(a)$ holds. But if one of these alternatives is lacking, $P_1 \vee P_2 \vee, \cdots, \vee P_s$ fails for some $n$ when $R_0(a)$ holds; and, of course, it fails when $R_0(a)$ fails, so that in this case these alternatives may be discarded. Similarly, (2) let $R_0(b \cdots e)$ be an alternative in $f'$ and let $P_1, P_2, \cdots, P_s$ be the corresponding alternatives in $f$. When $n$ takes the value $a$ in $K_1$ we must have $R_0(ab \cdots e)$ among these alternatives. When $n \varepsilon K_3$ we must have $R_0(b \cdots ec)$. When $n \varepsilon K_2$ and $n \neq b, \cdots, e$ we must have the set of alternatives entailed by Theorem XIV. When $n$ is identified with one of the set $b, \cdots, e$ we must have $R_0(b \cdots e)$. When $n \varepsilon$ not-$K$ we must have $R_0(b \cdots e) . S_0(n) . T(b \cdots e; n)$. (3) Let $R_0(c)$ be an alternative in $f'$. Then this implies $R_0(ac)$ in $f$ when $n = a$. When $n \varepsilon K_2$ it implies $R_0(nc)$, and when $n = c$, $R_0(c)$. When $n \varepsilon$ not-$K$ we require $R_0(c) . S_0(n) . T(c; n)$. (4) Let $S_0(h \cdots j)$ be an alternative in $f'$. When $n$ is identified with one of the terms $h, \cdots, j$ this alternative requires $S_0(h \cdots j)$ in $f$. When $n \varepsilon$ not-$K$ and $n \neq h, \cdots, j$ it requires $S_0(h \cdots jn)$. When $n = a$ it requires $R_0(a) . S(h \cdots j) . T(a; h \cdots j)$, and when $n \varepsilon K_2$, $R_0(n) . S_0(h \cdots j) . T(n; h \cdots j)$, and when $n = c$, $R_0(c) . S_0(h \cdots j) . T(c; h \cdots j)$. Every other case is a simple combination of these four cases.

Accordingly, $f''(x, \cdots, l)$ exists such that $F''(x, \cdots, l) .\equiv. F(x, \cdots, l, n)$. It follows that *every first-order function on the base* $K, R_2$ *has its truth-value determined by conditions* (1′)–(10′).

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# NOTE ON CERTAIN ASSOCIATED SYSTEMS OF LINEAR EQUALITIES AND INEQUALITIES.*

By L. L. Dines.

1. Certain associated systems of linear equalities and inequalities relative to a matrix $\|a_{ij}\|$ of real coefficients have been considered by the author in an earlier paper.† One of the principal results may be stated in the following form:

THEOREM I. *A necessary and sufficient condition that the system of linear inequalities*

$$\sum_{j=1}^{m} a_{ij} x_j > 0, \qquad (i = 1, 2, \cdots, n) \tag{1}$$

*admit no solution* $(x_1, x_2, \cdots, x_m)$ *is that the associated system of linear equalities*

$$\sum_{i=1}^{n} y_i a_{ij} = 0, \qquad (j = 1, 2, \cdots, m) \tag{2}$$

*admit a solution* $(y_1, y_2, \cdots, y_n)$ *in which not every* $y_i$ *is zero, and the non-zero ones are positive.*

The present note presents a theorem closely related to this, which is apparently of equal significance. It results from the replacement of the symbol $>$ in (1) by a symbol $\geqq$, which is to be interpreted "is not less than, and for at least one value of the range of indices is definitely greater than". The new theorem can then be stated as follows:

THEOREM II. *A necessary and sufficient condition that the system*

$$\sum_{j=1}^{m} a_{ij} x_j \geqq 0, \qquad (i = 1, 2, \cdots, n) \tag{3}$$

*admit no solution* $(x_1, x_2, \cdots, x_m)$ *is that the associated system*

$$\sum_{i=1}^{n} y_i a_{ij} = 0, \qquad (j = 1, 2, \cdots, m) \tag{4}$$

*shall admit a solution* $(y_1, y_2, \cdots, y_n)$ *in which every* $y_i$ *is positive.*

As the proof requires only slight and rather obvious alterations in the procedure used to prove the earlier theorem it will not be given. Instead,

* Presented to the Americal Mathematical Society, June 12, 1926.

† Annals of Math., vol. 27, (1925), pp. 57–64.

we present in the next section a possible property of the matrix of coefficients $\|a_{ij}\|$ sufficient to assure the conditions of Theorem II. It is a very restrictive property, by no means necessary, but it has the merit of being easily detected when present.

2. Let us denote by $\sigma_{ij}$ the sign of the element $a_{ij}$. A matrix $\|a_{ij}\|$ of $m$ columns and $2^m$ rows, in which all elements are different from zero, and in which the $2^m$ rows of signs

$$\sigma_{i1}, \sigma_{i2}, \cdots, \sigma_{im} \qquad (i = 1, 2, \cdots, 2^m)$$

include all the $2^m$ possible arrangements of signs (+ and —) will be said to be a *completely signed matrix.*

THEOREM III. *If the matrix* $\|a_{ij}\|$ *is completely signed, the conditions of Theorem* II *are satisfied, that is the system* (3) *admits no solution, and the system* (4) *admits an everywhere positive solution.*

Suppose, for a completely signed matrix, the system (3) did have a solution $(x_1, x_2, \cdots, x_m)$. Let the elements of the solution be divided into three groups: those which are positive

$$x_{\alpha_1}, x_{\alpha_2}, \cdots, x_{\alpha_p},$$

those which are negative

$$x_{\beta_1}, x_{\beta_2}, \cdots, x_{\beta_N},$$

and those which are zero, for which we shall need no notation. Some row of the completely signed matrix, say the $q$th, will have the character

$$\sigma_{q\alpha} = -, \qquad (\alpha = \alpha_1, \alpha_2, \cdots, \alpha_p),$$
$$\sigma_{q\beta} = +, \qquad (\beta = \beta_1, \beta_2, \cdots, \beta_N).$$

Then upon substitution of the supposed solution, the $q$th statement of (3) may be written

$$\sum_{\alpha} a_{q\alpha} x_\alpha + \sum_{\beta} a_{p\beta} x_\beta \gtrless 0,$$

which is obviously untrue. The contradiction proves the theorem.

COROLLARY: *If any subset of rows of the matrix* $\|a_{ij}\|$ *forms a completely signed matrix, the system* (3) *admits no solution, and the associated system* (4) *admits an everywhere positive solution.*

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# SOME FURTHER THEOREMS CONCERNING THE FORMATION OF CHAINS.

By A. Arwin.

In this article I shall sketch some further points of view on the formation of chains. I shall try to prove what was inferred in a previous paper* from the results of Mr. Thue, namely, that if $\omega_n = (\alpha_n \mu^2 + \beta_n \mu + \gamma_n)/R_n$, then $R_n \to \infty$. More precisely, I shall prove that $\omega_n$ is bounded, and I shall deduce some of the consequences of this fact, giving finally the chain formation in the case of complex roots.

Referring to my former investigations, I quote the following formulas. For $\mu$ a real root of $f(x) = x^3 + A_2 x + A_3 = 0$, $\omega_n = (\alpha_n \mu^2 + \beta_n \mu + \gamma_n)/R_n$ and $\mu = a_0 + \frac{1}{\omega_1}, \cdots, \omega_{n-1} = a_{n-1} + \omega'_{n-1} = a_{n-1} + \frac{1}{\omega_n}$, we found

$$\mu = \frac{\beta_n \omega_n + \beta_{n-1}}{\alpha_n \omega_n + \alpha_{n-1}}, \qquad \beta_n \alpha_{n-1} - \beta_{n-1} \alpha_n = (-1)^n,$$

$$(1) \qquad \mu - \frac{\beta_n}{\alpha_n} = \frac{(-1)^{n-1}}{\alpha_n^2 (\omega_n + \alpha_{n-1}/\alpha_n)}, \qquad \omega_n + \frac{\alpha_{n-1}}{\alpha_n} \to \frac{\alpha_n}{R_n} f'(\mu),$$

$$\beta_{n+1} = a_n \beta_n + \beta_{n-1}, \qquad \alpha_{n+1} = a_n \alpha_n + \alpha_{n-1};$$

$$R_{n-1} \alpha_n = \alpha_{n-1}^2 A_2 + \beta_{n-1}^2 - \alpha_{n-1} \gamma_{n-1},$$

$$R_n \alpha_{n-1} = \alpha_n^2 A_2 + \beta_n^2 - \alpha_n \gamma_n,$$

$$(2) \qquad R_{n-1} \beta_n = -\alpha_{n-1}^2 A_3 - \beta_{n-1} \gamma'_{n-1},$$

$$R_n \beta_{n-1} = -\alpha_n^2 A_3 - \beta_n \gamma_n,$$

$$R_{n-1} \beta_n = (\gamma'_{n-1} - A_2 \alpha_{n-1})^2 + \beta_{n-1} (\beta_{n-1} A_2 + \alpha_{n-1} A_3),$$

$$R_n \gamma'_{n-1} = (\gamma_n - A_2 \alpha_n)^2 + \beta_n (\beta_n A_2 + \alpha_n A_3),$$

where $\omega_n$ satisfies the set of equations

$$\psi_n(x) = \psi_{n-1}(a_{n-1}) x^3 + \psi'_{n-1}(a_{n-1}) x^2 + \frac{1}{2!} \psi''_{n-1}(a_{n-1}) x + \frac{1}{3!} \psi'''_{n-1}(a_{n-1}) = 0$$

for $\frac{1}{3!} \psi'''_{n-1}(a_{n-1}) = R_{n-1}$, $-\frac{1}{2!} \psi''_{n-1}(a_{n-1}) = 3\gamma'_{n-1} - 2A_2 \alpha_{n-1}$, $\psi'_{n-1}(a_{n-1}) = 3\gamma_n - 2A_2 \alpha_n$, $-\psi_{n-1}(a_{n-1}) = R_n$, determined from the relations

* A. Arwin, On continued fractions in the theory of binary forms, *Ann. of Math.*, vol. 26, pp. 247–272.

$$\begin{aligned} \omega_n + \bar{\omega}_n + \bar{\bar{\omega}}_n &= (3\gamma_n - 2A_2\,\alpha_n)/R_n, \\ \omega_n\,\bar{\omega}_n + \omega_n\,\bar{\bar{\omega}}_n + \bar{\omega}_n\,\bar{\bar{\omega}}_n &= (3\gamma'_{n-1} - 2A_2\,\alpha_{n-1})/R_n, \\ \omega_n\,\bar{\omega}_n\,\bar{\bar{\omega}}_n &= R_{n-1}/R_n; \end{aligned} \tag{3}$$

for instance, in the case of the last expression, $\beta_n - \alpha_n\mu = (-1)^n/(\omega_n\alpha_n + \alpha_{n-1})$ and therefore

$$\prod_{\mu}(\beta_n - \alpha_n\mu) = (-1)^n R_n = \frac{(-1)^n}{\prod\limits_{\omega_n}(\omega_n\alpha_n + \alpha_{n-1})}$$

$$= \frac{(-1)^n}{\prod\left[\alpha_n\left(a_n + \dfrac{1}{\omega_{n+1}}\right) + \alpha_{n-1}\right]} = \frac{(-1)^n \prod \omega_{n+1}}{\prod(\omega_{n+1}\alpha_{n+1} + \alpha_n)},$$

whence $\prod \omega_{n+1} = R_n/R_{n+1}$.

THEOREM. *In* $\omega_n$ *from* (1) *we have* $R_n \to \infty$.

To show this we shall investigate the law of formation of the following chain, which we shall say is of the third type. We form

$$\mu = b_0 \pm \frac{1}{\varrho_1}, \cdots, \varrho_{n-1} = b_{n-1} \pm \varrho'_{n-1} = b_{n-1} \pm \frac{1}{\varrho_n},$$

where $1 > \varrho'_{n-1} > \frac{1}{2}$ and therefore $1 = \varrho'_{n-1}\,\varrho_n > \frac{1}{2}\varrho_n$, $\varrho_n < 2$ for all $n$, whence the coefficients $b_i$ are 1 or 2, so that for $b_i = 1$ the sign is $+$ and for $b_i = 2$ it is $-$. We shall say that fractions $\varrho_\tau = 2 - 1/\varrho_{\tau+1}$ in a given negative set belong to the same "minus period". If this set is broken off with $\varrho_i = 1 + 1/\bar{\varrho}_{\tau+1}$, say, we have always $\bar{\varrho}_{\tau+1} > 1$ and an equivalence

$$\mu = \frac{\beta'_{\tau+1}\,\bar{\varrho}_{\tau+1} + \beta'_\tau}{\alpha'_{\tau+1}\,\varrho'_{\tau+1} + \alpha'_\tau}, \qquad \beta'_{\tau+1}\,\alpha'_\tau - \beta'_\tau\,\alpha'_{\tau+1} = \pm 1. \tag{4}$$

Determining $r > 0$ from $\beta'_{\tau+1} > \beta'_\tau - r\,\beta'_{\tau+1} > 0$, we have also $\alpha'_{\tau+1} > \alpha'_\tau - r\,\alpha'_{\tau+1}$ and from the reduced equivalence

$$\mu = \frac{\beta'_{\tau+1}(\bar{\varrho}_{\tau+1} + r) + (\beta'_\tau - r\,\beta'_{\tau+1})}{\alpha'_{\tau+1}(\bar{\varrho}_{\tau+1} + r) + (\alpha'_\tau - r\,\alpha'_{\tau+1})} \tag{4'}$$

and, since $\bar{\varrho}_{\tau+1} > 1$, we infer that $\bar{\varrho}_{\tau+1} + r$ must coincide with a fraction in the regular chain of type 1. Moreover, if the formation is broken off in a minus period, we are led to a fraction in the regular chain, and we conclude from

$$\begin{aligned} \frac{\beta'_{\tau+1}}{\alpha'_{\tau+1}} &= b_0 \pm \frac{1}{|b_1} \pm \cdots \pm \frac{1}{|b_i} \pm \frac{1}{2} - \frac{1}{|2} - \frac{1}{|2} - \cdots - \frac{1}{|2} - \frac{1}{|1} \\ &= b_0 \pm \frac{1}{|b_1} \pm \cdots \pm \frac{1}{|b_\nu} \pm \frac{1}{1} \end{aligned}$$

that $\beta'_{\tau+1}/\alpha'_{\tau+1}$ will be the same wherever the minus period is broken off, which will evidently not happen for the fraction $\beta'_\tau/\alpha'_\tau$. This is illustrated by the following example.

*Example.* $x^3-3x+1=0$, $\mu\sim 1.53\ldots$.

The regular chain gives

$$\mu = 1+\frac{\mu-1}{1} \qquad = 1+\cfrac{1}{\cfrac{\mu^2+\mu-2}{1}}$$

$$\frac{\mu^2+\mu-2}{1} = 1+\frac{\mu^2+\mu-3}{1} \qquad = 1+\cfrac{1}{\cfrac{\mu^2+2\mu-2}{3}}$$

$$\frac{\mu^2+2\mu-2}{3} = 1+\frac{\mu^2+2\mu-5}{3} \qquad = 1+\cfrac{1}{\cfrac{2\mu^2+3\mu-2}{1}}$$

$$\frac{2\mu^2+3\mu-2}{1} = 7+\frac{2\mu^2+3\mu-9}{1} = 7+\cfrac{1}{\cfrac{15\mu^2+23\mu-12}{17}}$$

. . . . . .

and the chain of the third kind

$$\mu = 1+\frac{\mu-1}{1} \qquad = 1+\cfrac{1}{\cfrac{\mu^2+\mu-2}{1}}$$

$$\frac{\mu^2+\mu-2}{1} = 1+\frac{\mu^2+\mu-3}{1} \qquad = 1+\cfrac{1}{\cfrac{\mu^2+2\mu-2}{3}}$$

$$\frac{\mu^2+2\mu-2}{3} = 2+\frac{\mu^2+2\mu-8}{3} \qquad = 2-\cfrac{1}{\cfrac{3\mu^2+5\mu+5}{17}}$$

$$\frac{3\mu^2+5\mu+5}{17} = 2+\frac{3\mu^2+5\mu-29}{17} \qquad = 2-\cfrac{1}{\cfrac{5\mu^2+8\mu+20}{37}}$$

$$\frac{5\mu^2+8\mu+20}{37} = 2+\frac{5\mu^2+8\mu-54}{37} \qquad = 2-\cfrac{1}{\cfrac{7\mu^2+11\mu+37}{57}}$$

$$\frac{7\mu^2+11\mu+37}{57} = 2+\frac{7\mu^2+11\mu-77}{57} \qquad = 2-\cfrac{1}{\cfrac{9\mu^2+14\mu+50}{71}}$$

$$\frac{9\mu^2+14\mu+50}{71} = 2+\frac{9\mu^2+14\mu-92}{71} \qquad = 2-\cfrac{1}{\cfrac{11\mu^2+17\mu+53}{73}}$$

$$\frac{11\mu^2+17\mu+53}{73} = 2+\frac{11\mu^2+17\mu-93}{73} = 2-\cfrac{1}{\cfrac{13\mu^2+20\mu+40}{57}}$$

$$\frac{13\mu^2+20\mu+40}{57} = 1+\frac{13\mu^2+20\mu-17}{57} = 1+\cfrac{1}{\cfrac{2\mu^2+3\mu-8}{1}}$$

$$\frac{2\mu^2+3\mu-8}{1} = 2+\frac{2\mu^2+3\mu-10}{1} = 2-\cfrac{1}{\cfrac{17\mu^2+26\mu-5}{53}}$$

. . . . . .

where, for example,

$$\frac{3\mu^2+5\mu+5}{17} = 1+\frac{3\mu^2+5\mu-12}{17} = 1+\cfrac{1}{\cfrac{2\mu^2+3\mu-3}{1}}$$

and

$$\frac{5\mu^2+8\mu+20}{37} = 1+\frac{5\mu^2+8\mu-17}{37} = 1+\cfrac{1}{\cfrac{2\mu^2+3\mu-4}{1}}$$

as above.

Let us now say that $\beta'_{\tau+1}$, $\alpha'_{\tau+1}$ in (4′) belong to the regular chain as $\beta_n$, $\alpha_n$. Then

$$(4'') \qquad \mu = \frac{\beta_n\omega_n+\beta_{n-1}}{\alpha_n\omega_n+\alpha_{n-1}};$$

that is, $\beta'_\tau = \beta_{n-1}+r\beta_n$, expressed in terms from the regular chain. The length of the minus period being $a_n-1$ we can give $r$ all values $1, 2, \cdots, a_n-1$, since we can break off in $\varrho_\tau = 1+1/\varrho'_{\tau+1}$ at any place in the minus period. As the period ends with $r = a_n-1$, we obtain $\varrho_\tau = 1+1/\varrho_{\tau+1}$ and this $\varrho_{\tau+1}$ remains in our chain of the third kind. Let us in general put $\beta'_{\tau+r} = \beta_{n-1}+r\beta_n$ $(r = 1, 2, \cdots, a_n-1)$, since in a minus period $\beta'_{\tau+r+1} = 2\beta'_{\tau+r}-\beta'_{\tau+r-1}$, for $r = a_n-1$, (4′) gives

$$(4'') \qquad \mu = \frac{\beta'_{\tau+a_n}(\varrho_{\tau+a_n}+a_n-1)+(\beta'_{\tau+a_{n-1}}-(a_n-1)\beta'_{\tau+a_n})}{\alpha'_{\tau+a_n}(\varrho_{\tau+a_n}+a_n-1)+(\alpha'_{\tau+a_{n-1}}-(a_n-1)\alpha'_{\tau+a_n})}.$$

Hence $\beta'_{\tau+a_n} = \beta_n$, $\varrho_{\tau+a_n}+a_n-1 = \omega_n$ and, since $2 > \varrho_{\tau+a_n} > 1$, then $\varrho_{\tau+a_n}-1 < 1$ and $a_n$ is the greatest integer in $\omega_n = a_n+1/\omega_{n+1}$. Hence we have determined the composition of these chains of the third kind; they are minus periods of length $a_i-1$, where for $a_i > 1$ the $a_i$ are the integers from $\omega_i = a_i+1/\omega_{i+1}$. To the fraction $\omega_i = (\alpha_i\mu^2+\beta_i\mu+\gamma_i)/R_i$ of the regular chain corresponds the fraction $\varrho_{\tau+a_i} = [\alpha_i\mu^2+\beta_i\mu+(\gamma_i-(a_i-1)R_i)]/R_i$

in the third type, differing only in the last element $\gamma_i$; the numerators and denominators of the fractions in the minus period are given by $\beta'_{\tau+r} = \beta_{n-1} + r\beta_n$, $\alpha'_{\tau+r} = \alpha_{n-1} + r\alpha_n$ $(r = 1, 2, \cdots, a_n - 1)$, $\beta'_{\tau+a_n} = \beta_n$, $\alpha'_{\tau+a_n} = \alpha_n$, and it is well known that the inserted fractions form all the branch fractions in the theory of continued fractions. Also in this third type of chain our formulas (1), (2) and (3) are true, and we have, for example,

$$\beta'^3_\nu + A_2\beta'_\nu\alpha'^2_\nu + A_3\alpha'^3_\nu = (-1)^n R'_\nu,$$

$$\varrho_\nu \mp \frac{\alpha'_{\nu-1}}{\alpha'_\nu} \to \frac{\alpha'_\nu}{R'_\nu} f'(\mu), \qquad \mu - \frac{\beta'_\nu}{\alpha'_\nu} \to \frac{(-1)^{n-1} R'_\nu}{\alpha'^3_\nu f'(\mu)}, \tag{5}$$

where the upper sign and $(-1)^n$ belong to the minus period. Let us take $\varrho_i = 1 + 1/\varrho_{i+1}$, $\varrho_{i+1} = 2 - 1/\varrho_{i+2}$; hence $\alpha'_{i+2} = 2\alpha'_{i+1} + \alpha'_i$, and further $\alpha'_{i+3} = 2\alpha'_{i+2} - \alpha'_{i+1}$ in the next minus period and hence in the minus period through $\alpha'_{\tau+1} > \alpha'_\tau$ without exception; but beyond $\varrho_i = 2 - 1/\varrho_{i+1}$, $\varrho_{i+1} = 1 + 1/\varrho_{i+2}$ it follows that $\alpha'_{i+2} = \alpha'_{i+1} - \alpha'_i$, that is, $\alpha'_{\tau+1} < \alpha'_\tau$. Hence, since in (5) $2 > \varrho_\nu > 1$ and $\alpha'_{\nu-1}/\alpha'_\nu < 1$ i. e., $0 < \alpha'_\nu f'(\mu)/R'_\nu < 2$, we see that $\beta'_\nu/\alpha'_\nu$ approximates $\mu$ as $c/\alpha'^2_\nu$, where $c > 1/2$. This is in agreement with the general theory, since for $c < 1/2$ the fraction itself should appear in the regular chain.

Let us now suppose that in $\omega_n = (\alpha_n\mu^2 + \beta_n\mu + \gamma_n)/R_n$ belonging to the regular chain, $R_n$ is limited as $n \to \infty$. Then from

$$\omega_n + \frac{\alpha_{n-1}}{\alpha_n} \to \frac{\alpha_n f'(\mu)}{R_n},$$

since in this chain $\omega_n = a_n + 1/\omega_{n+1}$, $\alpha_{n+1} = a_n\alpha_n + \alpha_{n-1}$, $0 < \alpha_{n-1}/\alpha_n < 1$, and $\alpha_n \to \infty$, it follows that $\omega_n \to \infty$; hence $a_n = O(\alpha_n)$ and $\alpha_{n-1}/\alpha_n \to 0$. It was shown above that $\varrho_{\tau+a_n} = [\alpha_n\mu^2 + \beta_n\mu + \gamma_n - R_n(a_n - 1)]/R_\nu = \varrho_\nu$. Hence, since $\omega_n - a_n = 1/\omega_{n+1} \to 0$, we have $\varrho_\nu \to 1$; and further from the formation of the chain $\varrho_{\nu-2} = 2 - \varrho'_{\nu-2} = 2 - 1/\varrho_{\nu-1}$, $\varrho_{\nu-1} = 1 + \varrho'_{\nu-1} = 1 + 1/\varrho_\nu$; that is, $\varrho'_{\nu-1} \to 1$, $\varrho_{\nu-1} \to 2$, $\varrho'_{\nu-2} \to 1/2$, $\varrho_{\nu-2} \to 3/2$. We have also proved that

$$\varrho_{\nu-2} - \frac{\alpha'_{\nu-3}}{\alpha'_{\nu-2}} \to \frac{\alpha'_{\nu-2}}{R'_{\nu-2}} f'(\mu), \qquad \varrho_{\nu-1} - \frac{\alpha'_{\nu-2}}{\alpha'_{\nu-1}} \to \frac{\alpha'_{\nu-1}}{R'_{\nu-1}} f'(\mu)$$

with $\alpha'_{\nu-3}/\alpha'_{\nu-2} = (\alpha_{n-1} + (a_n - 3)\alpha_n)/(\alpha_{n-1} + (a_n - 2)\alpha_n) = 1 - O(1/a_n) \to 1$, Similarly, $\alpha'_{\nu-2}/\alpha'_{\nu-1} = 1 - O(1/a_n) \to 1$. Hence

$$\frac{\alpha'_{\nu-2} f'(\mu)}{R'_{\nu-2}} \to \frac{1}{2}, \qquad \frac{\alpha'_{\nu-1} f'(\mu)}{R'_{\nu-1}} \to 1, \qquad R'_{\nu-2} \to \infty, \qquad R'_{\nu-1} \to \infty,$$

and $(\alpha'_{\nu-1}\mu^2 + \beta'_{\nu-1}\mu + \gamma'_{\nu-1})/R'_{\nu-1} = \varrho_{\nu-1} \to 2$ implies $\gamma'_{\nu-1}/R'_{\nu-1} \to 2 - 2\alpha'_{\nu-1}\mu^2/R'_{\nu-1}$ since $\beta'_{\nu-1}/\alpha'_{\nu-1} \to \mu$, and $(\alpha'_{\nu-2}\mu^2 + \beta'_{\nu-2}\mu + \gamma'_{\nu-2})/R'_{\nu-2}$

$= \varrho'_{\nu-2} \to 1/2$ so that $\gamma''_{\nu-2}/R'_{\nu-2} \to 1/2 - 2\alpha'_{\nu-2}\mu^2/R'_{\nu-2}$. Combining these results we get

$$\frac{3\gamma'_{\nu-1} - 2A_2\alpha'_{\nu-1}}{R'_{\nu-1}} \to 6 - 2\frac{\alpha'_{\nu-1}}{R'_{\nu-1}}(3\mu^2 + A_2) \to 6 - 2 = 4,$$

$$\frac{3\gamma''_{\nu-2} - 2A_2\alpha'_{\nu-2}}{R'_{\nu-2}} \to \frac{3}{2} - 2\frac{\alpha'_{\nu-2}}{R'_{\nu-2}}(3\mu^2 + A_2) \to \frac{3}{2} - 1 = \frac{1}{2},$$

and $R'_{\nu-2}/R'_{\nu-1} \to 2$. Further, as in (3) we have

$$\psi_{\nu-1}(x) = R'_{\nu-1}x^3 - (3\gamma'_{\nu-1} - 2A_2\alpha'_{\nu-1})x^2 + (3\gamma''_{\nu-2} - 2A_2\alpha'_{\nu-2})x - R'_{\nu-2} = 0$$

and, since $\varrho_{\nu-1} = 1 + 1/\varrho_\nu$, also $-\psi_{\nu-1}(1) = R'_\nu = R_n$. From the above evaluations it follows that

$$-\frac{R_n}{R'_{\nu-1}} = \frac{\psi(1)}{R'_{\nu-1}} \to \left(1 - 4 + \frac{R'_{\nu-2}}{R'_{\nu-1}} \cdot \frac{1}{2} - \frac{R'_{\nu-2}}{R'_{\nu-1}}\right) \to -4;$$

whence $R_n \to \infty$, since $R'_{\nu-1} \to \infty$, and the theorem is proved for the assumption all $R_n$ limited. Moreover, for $\omega_n \to \infty$ with $R_n \to \infty$ the preceding is true since $a_n \to \infty$ remains valid and hence also $\alpha_{n-1}/\alpha_n \to 0$. Hence again $R'_{\nu-1}/R_n \to 1/4$, and from $\alpha'_{\nu-1}f'(\mu)/R'_{\nu-1} \to 1$ therefore also $\alpha'_{\nu-1}f'(\mu)/R_n \to (\alpha_{n-1} + (a_n - 1)\alpha_n)f'(\mu)/R_n \to 1/4$ in contradiction to $\alpha_n f'(\mu)/R_n \to \infty$.

We have then finally to consider the possibility $\omega_{n_1} \to \infty$ but $\omega_{n_1+1}$ limited. We find as before $\varrho_\nu = 1 + 1/\omega_{n_1+1} \neq 0$, further, since $\alpha_{n_1} \to \infty$,

$$\varrho_{\nu-1} - \frac{\alpha'_{\nu-2}}{\alpha'_{\nu-1}} \to \varrho_{\nu-1} - 1 = \frac{1}{\varrho_\nu} = c_1 \to \frac{\alpha'_{\nu-1}}{R'_{\nu-1}}f'(\mu);$$

$$\varrho_{\nu-2} - \frac{\alpha'_{\nu-3}}{\alpha'_{\nu-2}} \to \varrho_{\nu-2} - 1 = 1 - \frac{1}{1 + \frac{1}{\varrho_\nu}} = \frac{1}{\varrho_\nu + 1} = c_2 \to \frac{\alpha'_{\nu-2}}{R'_{\nu-2}}f'(\mu).$$

Hence

$$\frac{\alpha'_{\nu-1}\mu^2 + \beta'_{\nu-1}\mu + \gamma'_{\nu-1}}{R'_{\nu-1}} = \varrho_{\nu-1} \to 1 + c_1, \quad \frac{\gamma'_{\nu-1}}{R'_{\nu-1}} \to 1 + c_1 - \frac{\alpha'_{\nu-1}}{R'_{\nu-1}}2\mu^2,$$

$$\frac{3\gamma'_{\nu-1} - 2A_2\alpha'_{\nu-1}}{R'_{\nu-1}} \to 3 + 3c_1 - 2\frac{\alpha'_{\nu-1}}{R'_{\nu-1}}(3\mu^2 + A_2) \to 3 + 3c_1 - 2c_1 = 3 + c_1$$

and

$$\varrho'_{\nu-2} = \frac{1}{\varrho_{\nu-1}} = \frac{c_2}{c_1} = \frac{\alpha'_{\nu-2}\mu^2 + \beta'_{\nu-2}\mu + \gamma''_{\nu-2}}{R'_{\nu-2}}, \quad \frac{\gamma''_{\nu-2}}{R'_{\nu-2}} \to \frac{c_2}{c_1} - \frac{\alpha'_{\nu-2}}{R'_{\nu-2}}2\mu^2,$$

$$\frac{3\gamma''_{\nu-2} - 2A_2\alpha'_{\nu-2}}{R'_{\nu-2}} \to 3\frac{c_2}{c_1} - 2\frac{\alpha'_{\nu-2}}{R'_{\nu-2}}(3\mu^2 + A_2) \to 3\frac{c_2}{c_1} - 2c_2,$$

$$\frac{R'_{\nu-2}}{R'_{\nu-1}} \to \frac{c_1}{c_2} = 1 + c_1.$$

Hence

$$\frac{R_n}{R'_{\nu-1}} \to \left[-1+3+c_1-\left(3\frac{c_2}{c_1}-2c_2\right)\frac{c_1}{c_2}+\frac{c_1}{c_2}\right] = 4c_1,$$

and we are led to the same contradiction as before. Hence the important theorem:

*In the case of a cubic irrationality the set $\omega_n$ is limited.*

Probably the theorem is true for any algebraic irrationality, similarly proved. Its consequences are obviously of considerable importance. It gives us a simple criterion to decide whether any expression in the form of a regular chain can appertain to an algebraic number or must give a transcendental number, for example $e$ since, as known,

$$\frac{e-1}{2} = \frac{1|}{|1}+\frac{1|}{|6}+\frac{1|}{|10}+\frac{1|}{|14}+\cdots$$

Further, from the fact that $\omega_n$ is limited, and the general identity

$$\mu-\frac{\beta_n}{\alpha_n} = \frac{(-1)^{n-1}}{\alpha_n^2\left(\left(\omega_n+\frac{\alpha_{n-1}}{\alpha_n}\right)\right.},$$

we infer that the inequality

$$\left|\mu-\frac{\beta_n}{\alpha_n}\right| < \frac{c}{\alpha_n^{2+\varepsilon}},$$

for $c$ an arbitrary constant and $\varepsilon > 0$, can have only a finite number of solutions, i. e., the best approximation of $\mu$ with any integers $(\beta_n, \alpha_n)$ is of order $O(1/\alpha_n^2)$. This indeed is proved above for cubic irrationalities. And we observe incidentally that the curve $x^{1+\varepsilon}(y-\mu x) = 0$, $\mu$ as above, having the $y$-axis and the line $y-\mu x = 0$ as asymptotes, is for $\varepsilon = 0$ a hyperbola. In this case the area between the branch of the hyperbola and the asymptote $y-\mu x = 0$ becomes infinite as $\log x$ with $x$ while in every case $\varepsilon > 0$ for increasing $x$ the area converges to 0 as $x^{-\varepsilon}$. In the first case there can exist, as is well known, an infinite number of "lattice points" in this area, while in the second case we have just shown that the number is finite.

Having found that the $\omega_n$ in (1) are limited we have also shown the existence of an upper finite limit $\chi$, giving $\alpha_n/R_n < 1/\chi$, $R_n > \alpha_n\chi$ for $n$ and hence an upper limit for solutions of $y^3+A_2yx^2+A_3x^3 = \pm R$, since for each $\alpha_i$ such that $\chi\alpha_i > R$ we have, for $n > i$, $R_n > \chi\alpha_n > \chi\alpha_i > R$. As a less precise limit we might, for example, use $\alpha_n^{1/2}$, since $\omega_n+\frac{\alpha_{n-1}}{\alpha_n} \to$

$\frac{\alpha_n}{R_n} f'(\mu)$, and we have just shown that $\alpha_n/R_n$ is limited. For, after a sufficiently great index $n>N$ affording a good approximation to the preceding relation we shall have $f'(\mu)\,\alpha_n/R_n < \alpha_n^{1/2}$, that is, $R_n > \alpha_n^{1/2} f'(\mu)$ and, since we must always reach an $\alpha_i$ for which $\alpha_i^{1/2} f'(\mu) > R$, we get $R_n > R$ in all following fractions, so that after this no further solutions appear in the chain. A sharper limit, however, can certainly be obtained in other ways. For example, when $\gamma_n < 0$ for all $n > N$, it is

$$R_n\,\alpha_{n-1} = -A_2\,\alpha_n^2 + \beta_n^2 + \alpha_n|\gamma_n|$$

or

$$1 > \frac{\alpha_{n-1}}{\alpha_n} > \frac{(\mu^2 - A_2)\,\alpha_n}{R_n}, \qquad R_n > \alpha_n(\mu^2 - A_2),$$

and from

$$\omega_n + \frac{\alpha_{n-1}}{\alpha_n} = a_n + \frac{1}{\omega_{n+1}} + \frac{\alpha_{n-1}}{\alpha_n} \to \frac{\alpha_n}{R_n} f'(\mu)$$

we find the inequality expressed in terms of $\mu$,

$$a_n < \frac{\alpha_n}{R_n} f'(\mu) < \frac{3\mu^2 - A_2}{\mu^2 - A_2} = k(\mu).$$

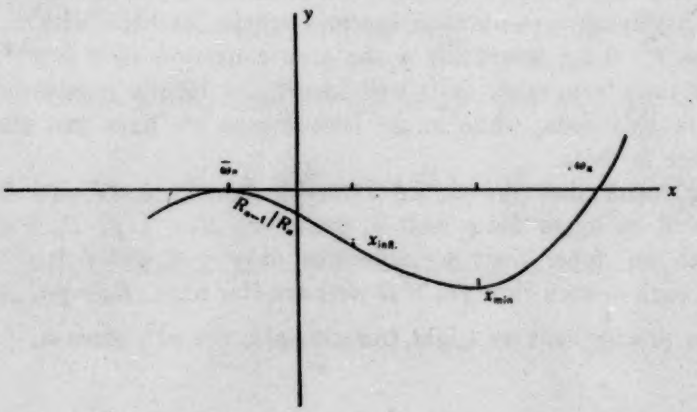
It is easy to draw the curves $R_n\,y = \psi_n(x)$ for $n \to \infty$ since we have

$$\omega_n \to \frac{\alpha_n}{R_n} f'(\mu) - \frac{\alpha_{n-1}}{\alpha_n},$$

$$\bar{\omega}_n \to \bar{\bar{\omega}}_n \to \quad - \frac{\alpha_{n-1}}{\alpha_n},$$

$$x_{\text{infl.}} \to \frac{\alpha_n}{3R_n} f'(\mu) - \frac{\alpha_{n-1}}{\alpha_n},$$

$$x_{\min} \to \frac{2\,\alpha_n}{3R_n} f'(\mu) - \frac{\alpha_{n-1}}{\alpha_n}.$$



Let us now consider the formation of a chain with complex coefficients and assume $\omega = \mu + i\nu$ a complex root of the equation $f_m(x) = 0$ of degree $m$. Determining the integers $a$ and $b$ from $|\mu - a_0| < \frac{1}{2}$, $|\nu - b_0| < \frac{1}{2}$ we form $\varrho_0 = a_0 + ib_0$, getting $|\omega - \varrho_0| < \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}}$, and hence from $\omega = \varrho_0 + 1/\omega_1$ the inequality $|\omega_1| > \sqrt{2}$. Forming in a similar way

$$\omega_\tau = \varrho_\tau + \frac{1}{\omega_{\tau+1}} \qquad (\tau = 1, 2, \cdots) \tag{8}$$

we obtain

$$\omega = \frac{\beta_n \omega_n + \beta_{n-1}}{\alpha_n \omega_n + \alpha_{n-1}}, \qquad \beta_n \alpha_{n-1} - \beta_{n-1} \alpha_n = (-1)^n. \tag{8'}$$

Let us suppose that $|\beta_n|$ and $|\alpha_n|$ in (8′) are limited. Then, since the new relations (1), (2) and (3) remain true in this case, also if $\omega$ is a higher irrationality than the third, we should find $|R_n|$ and $|\gamma_n|$ likewise limited. Hence the chain should be periodic and $\omega$ quadratic in $K(i)$. We must therefore have either $|\alpha_n| \to \infty$ or $|\beta_n| \to \infty$, or both. Assuming $|\alpha_n| \to \infty$ and $|\beta_n|$ limited we get as before

$$\omega - \frac{\beta_n}{\alpha_n} = \frac{(-1)^{n-1}}{\alpha_n^2 \left(\omega_n + \frac{\alpha_{n-1}}{\alpha_n}\right)}. \tag{8''}$$

Since $|\alpha_n| \to \infty$, there must exist a sequence $n_1$ in $n$ such that without exception $|\alpha_{n_1}| > |\alpha_{n_1-1}|$; hence from $|\omega_{n_1}| > \sqrt{2}$, $|\alpha_{n_1-1}/\alpha_{n_1}| < 1$ we infer $\omega_{n_1} + \alpha_{n_1-1}/\alpha_{n_1} \nrightarrow 0$ and consequently, with $|\beta_n|$ limited, $\omega \nrightarrow 0$. From this contradiction we conclude $|\alpha_n| \to \infty$, $|\beta_n| \to \infty$ and

$$\omega - \frac{\beta_{n_1}}{\alpha_{n_1}} = \frac{(-1)^{n_1-1}}{\alpha_{n_1}^2 (\omega_{n_1} + \alpha_{n_1-1}/\alpha_{n_1})} \to 0;$$

but, since in general $\beta_{n_1} \alpha_{n_1-1} - \beta_{n_1-1} \alpha_{n_1} = (-1)^{n_1}$, we have for all $n$

$$\omega - \frac{\beta_n}{\alpha_n} \nrightarrow 0,$$

and we find once more from

$$\left| \omega - \frac{\beta_n}{\alpha_n} \right| \to \left| \frac{R_n}{\alpha_n^3 f'(\mu)} \right|$$

the order of infinity of the approximation.

*Example.* $x^3+2x+5=0$, $\omega\sim 0.7+i1.75$.

$$\omega = 1+2i+\cfrac{1}{\cfrac{\omega^2+(1+2i)\omega+(-1+4i)}{4-2i}}$$

$$\frac{\omega^2+(1+2i)\omega+(-1+4i)}{4-2i} = -2+i+\cfrac{1}{\cfrac{(-2+i)\omega^2-3(1+i)\omega-(1+2i)}{2+7i}}$$

$$\frac{(-2+i)\omega^2-3(1+i)\omega-(1+2i)}{2+7i} = -2-i+\cfrac{1}{\cfrac{6\omega^2+(4+11i)\omega+(3+15i)}{20+11i}}.$$

. . . . . .

The next coefficient will be determined to within $2i$; we compute

$$-[3(1+i)]^3-2\cdot 3(1+i)(-2+i)^2+5(-2+i)^3 = 2+7i,$$
$$(4+11i)^3+2(4+11i)6^2+5\cdot 6^3 = -20-11i,$$

as in the case of real quantities. Further we have

$$1+2i+\frac{1}{|-2+i}+\frac{1}{|-2-i}+\frac{1}{|2i} = \frac{5(5-i)}{2-13i} = 0.664+1.820i.$$

Comparing with the regular formation from $\mu\sim 1.3$ and $x^3+2x-5=0$ we have

$$\mu = 1+\cfrac{1}{\cfrac{\mu^2+\mu+3}{2}}$$

$$\frac{\mu^2+\mu+3}{2} = 3+\cfrac{1}{\cfrac{3\mu^2+4\mu+11}{1}}$$

$$\frac{3\mu^2+4\mu+11}{1} = 21+\cfrac{1}{\cfrac{64\mu^2+85\mu+228}{275}}$$

where $1+\frac{1}{|3}+\frac{1}{|21} = \frac{85}{64}\sim 1.3281$, and from the preceding chain we take $2\times 0.664 = 1.328$.

MALMÖ, SWEDEN,
November, 1925.

# NOTE ON THE EXTENSIONS OF GROUPS TO OBTAIN $n$-TH ROOTS.

By M. H. Ingraham.

1. Among the most important results in algebra during the last century was its development starting from a postulational point of view. The "field" became the basic notion, and much that was true for the rational, real, and complex number systems was found to be true for the general field.

In particular, Steinitz* proved that for every field $F$ there exists another field $F'$ which contains a subfield simply isomorphic with $F$ and such that every polynomial in $F'$ is reducible into linear factors. Moreover, Steinitz showed that there was a minimal such extension in the sense that there existed an $F_1$ effective as $F'$ and such that every field $F_2$ which was effective as $F'$ contained a subset simply isomorphic with $F_1$. It should be noted that in the proof of the above, Steinitz uses the Zermelo Principle and assumes that every field can be well ordered (wohlgeordnet).

The group stands as one of the generalizations of a field. In the field there are two processes of composition, addition $(+)$ and muliplication $(\cdot)$, and in a definition of a polynomial both of these processes enter in. In the case of the general group we have but one process of composition which we shall call multiplication. Questions as to roots of algebraic equations are therefore limited to equations of the type

$$x^n = g.$$

It is the purpose of this paper to prove:

For every group $G_0$ there exists a group $K$ containing a subgroup $K'$ simply isomorphic with $G_0$ and such that for every integer $n$ and element $k_1$ of $K$ there exists an element $k_2$ of $K$ such that $(k_2)^n = k_1$.

It is, of course, not true that in general the number of elements effective as $k_2$ is equal to $n$. It is not true that there exists a minimal group effective as $G_1$. Whether under certain restrictions as to the type of extension, especially as to the commutativity of multiplication between newly introduced roots of elements which are themselves commutative, such a minimal group could be found, I do not know. It seems probable, however that the proof of such a theorem, if it exists, would depend on

* Journ. für Math., 1910, vol. 137, p. 167 ff.

the use of the Zermelo principle. In the proof of the general theorem stated above no use of this principle is made. It is true, however, that whenever $G_0$ is commutative, $K$ may also be chosen commutative and if, besides, $G_0$ can be well ordered a minimal such $K$ exists.

I should remark that this paper was suggested by certain postulates for an arithmetic and geometric mean in a paper when presented to the Mathematics Club of the University of Wisconsin by Professor Huntington of Harvard. The postulates which were of the form of four functional equations would apply to any group and the question was brought up as to the type of groups in which solutions of these functional equations exist, and the type of group in which such a solution is unique. In answering this question it becomes clear that certain theorems hold in all groups for which every element has an $n$-th root; and these postulates become categorical for groups for which every element has a unique $n$-th root. This paper was presented by Professor Huntington to the American Mathematical Society at Kansas City, December 29, 1925, under the title "Sets of Independent Postulates for Four Types of Means."

2. THEOREM 1*. *For every group $G_0$ there exists a group $G_1$ containing a subgroup $G_1'$ simply isomorphic with $G_0$ and such that for every integer $n$ and element $g_{11}'$ of $G_1'$ there exists an element $g_{12}$ of $G_1$ such that* $(g_{12})^n = g_{11}'$.

It should be noted that this is a weaker theorem than the one quoted in the introduction, which appears as Theorem 2 since in Theorem 2 $g_{11}$ is not restricted to be in $G_1'$.

In the proof that follows we construct elements that correspond to products of rational powers of the elements of $G_0$ and give the rule of multiplication of such elements. An infinitude of different elements may correspond to an element $g_0$ of $G_0$ or to a rational power of $g_0$, just as it is possible to factor a real number in an infinite number of ways. We classify these representative elements into mutually exclusive classes in such a way that elements which would be equivalent by the ordinary rules of exponents belong to the same class. It is found that this correspondence of elements is maintained under multiplication, and we may readily build these classes of corresponding elements into a group effective as $G_1$.

Proof: Consider $I =$ [zero and the set of all natural integers]. Consider $H$ the set of all functions $h$ on $I$ to $G_0 I$ with the following restrictions applying to every $h$ of $H$.

* In proofs of the theorems certain almost standard logical notations for equivalence, implication, inclusion, etc., are used; for fuller explanation see E. H. Moore's *Introduction to a form of general analysis*, Yale University Press, 1910, p. 150, or the writer's paper *A general theory of linear sets*, Trans. Am. Math. Soc., vol. 27, 2, p. 163 ff.

1. For every $n$,
   *a)* $h(3n)$ belongs to $G_0$,
   *b)* $h(3n+1)$ belongs to $I$,
   *c)* $h(3n+2)$ belongs to $I$ and $\neq 0$.

2. $\exists n_h : s$:
   *A)* $n \geqq n_h$ .). *a)* $h(3n) = u$ where $u$ is the unit of $G_0$,
   *b)* $h(3n+1) = 1 = h(3n+2)$,
   *B)* $n_h$ is the least $n$ for which $A$ holds.

That is $H$ is the totality of sequences whose first and every subsequent third term belongs to $G_0$ and the others to the set of natural integers and such that beyond a certain term there is a $u$, 1, 1 repetend, where $u$ is the unit of $G_0$.

If for such a function $h$ we consider for all $n < n_h$

$$h(3n) \equiv g_{0n}, \qquad h(3n+1) \equiv p_n, \qquad h(3n+2) \equiv q_n,$$

we may think of $h$ as corresponding to the product

$$\prod_{n}^{0\,n_h-1} (g_{0n})^{\frac{p_n}{q_n}}.$$

We define multiplication between elements of $H$ as follows:

$$h_1 h_2 = h_3 \;:\equiv:\; n < 3n_{h_1} .). h_3(n) = h_1(n) : n \geqq 3n_{h_1} .). h_3(n) = h_2(n - 3n_{h_1}).$$

It is readily seen that this process of multiplication is associative and closed and moreover, there exists a unit, namely, the unique element $h_0$ for which $n_{h_0} = 0$. However multiplication is not commutative and the inverse does not exist.

We now proceed to divide $H$ into a set $G_1$ of classes and define a multiplication process between these classes in such a way that there is a subset of these classes simply isomorphic with $G_0$ and such that $G_1$ forms a group effective as $G_1$ of our theorem.

Consider the following correspondence between elements of $H$. We say that $h_1$ corresponds to $h_2$, $(h_1 \sim h_2)$ if any of the following conditions hold.

1. $h_1 = h_2$.
2. $\exists m . s$.

(Index Law)

a) $n < 3m$ .). $h_1(n) = h_2(n)$,

b) $h_1(3m) = h_2(3m) = h_2(3m+3)$,
   $h_1(3m+1) = h_2(3m+1) + h_2(3m+4)$,
   $h_1(3m+2) = h_2(3m+2) = h_2(3m+5)$,

c) $n \geqq 3m+3$ .). $h_1(n) = h_2(n+3)$,

3. $\exists k, m \,.\, s.$

a) $n < 3m \,.).\, h_1(n) = h_2(n),$

b) $h_1(3m) = h_2(3m),$

$h_1(3m+1) = k h_1(3m+1),$

$h_2(3m+2) = k h_2(3m+2),$

c) $n \geqq 3m+3 \,.).\, h_1(n) = h_2(n).$

This corresponds to $a^{\frac{kp}{kq}} = a^{\frac{p}{q}}$ in ordinary algebra.

4. $\exists m : s:$

a) $n < 3m \,.).\, h_1(n) = h_2(n),$

b) $h_1(3m) = (h_2(3m))^{h_2(3m+1)},$

$h_1(3m+1) = 1,$

$h_1(3m+2) = h_2(3m+2),$

c) $n \geqq 3m+3 \,.).\, h_1(n) = h_2(n).$

This corresponds to $(a)^{\frac{m}{q}} = (a^m)^{\frac{1}{q}}$ in ordinary algebra.

5. $\exists m : s:$

a) $n < 3m \,.).\, h_1(n) = h_2(n),$

b) $h_1(3m) = h_2(3m)\, h_2(3m+3),$

$h_1(3m+1) = h_2(3m+1) = h_2(3m+4) = 1,$

$h_1(3m+2) = h_2(3m+2) = h_2(3m+5) = 1,$

c) $n \geqq 3m+3 \,.).\, h_1(n) = h_2(n+3).$

This corresponds to $(a_1)(a_2) = (a_1 a_2)$ in ordinary algebra.

6. $h_1 \sim h_2 \,.).\, h_2 \sim h_1.$

7. $h_1 \sim h_2 \,.\, h_3 \sim h_2 \,:).\, h_1 \sim h_3.$

Concerning this correspondence we note the following two facts:

1. $h_1 \sim h_2$

$n_{h_1} = n_{h_2} = 1 \,.\, h_1(1) = h_1(2) = h_2(1) = h_2(2) = 1 \,.).\, h_1(0) = h_2(0).$

This corresponds to the invariance of the sum of exponents in representations of $a$ as a product of powers of $a$.

2. $h_1 \sim h_2 \,.\, h_3 \sim h_4 \,.).\, (h_1\, h_3) \sim (h_2\, h_4).$

This is readily seen from the fact that any two corresponding $h$'s are connected by a finite set $h_1 \cdots h_n$ such that $h_k \sim h_{k+1}$ or $h_{k+1} \sim h_k$ under one of the rules 1, 2, 3, 4, 5.

Consider $G = [g]$ the totality of sets of corresponding $h$'s. Let $g = \{h_1\}$ be all members of $H$ which correspond to $h_1$ etc. We define the product of two members $\{h_1\}$ and $\{h_2\}$ of $G$ as the set $\{h_1 h_2\}$. We will now prove that $G$ is a group effective as $G_1$ of our theorem.

As multiplication between the elements of $H$ is associative and closed it is also true of the elements of $G$. Moreover, if $n_{h_0} = 0$ then $h_1 h_0 = h_0 h_1 = h_1$ for every $h_1$ and $\{h_0\}$ acts as the unit of $G$. Obviously any element $h \neq h_0$ of $H$ is the product of elements $h_1 \cdots h_n$ such that $k \leqq n$ :): $n_{h_k} = 1 . h_k(1) \leqq h_k(2)$. For every $h_i (i \leqq n)$, consider $h_i'$ where $n_{h_i'} = 2, h_i'(0) = h_i(0) . h_i'(1) = h_i(2) - h_i(1) . h_i'(2) = h_i(2) . h_i'(3) = [h_i(0)]^{-1} . h_i'(4) = 1 = h_i'(5)$.

Hence

$$\{h_i\}\{h_i'\} = \{h_i h_i'\}$$

and by 2, 3, 5

$$= \{h_0\}.$$

Hence reciprocals of $\{h_1\} \cdots \{h_n\}$ exist and therefore the reciprocal of their product. Hence reciprocals of all elements of $G$ exist. Moreover, if $n_{h_1} = 1$ and $h(1) = h(2) = 1$, then if $h_2$ is such that $n_{h_2} = 1$ and $h_2(0) = h_1(0)$, while $h_2(1) = 1$, $h_2(2) = n$ then $\{h_2\}^n = \{h_2^n\}$ (and by 2, 3) $= \{h_1\}$. Hence if to every $g_0$ of $G_0$ we let correspond the element $h$ of $G_1$ where $n_h = 1$, $h(0) = g_0$, $h(1) = h(2) = 1$, as $G_0'$ the totality of these corresponding elements and as $G_1$ the set $G$ with multiplication defined as above we see that Theorem 1 is true.

3. THEOREM 2. *For every group $G_0$ there exists a group $K$ containing a subgroup $K'$ simply isomorphic with $G_0$ and such that for every integer $n$ and element $k_1$ of $K$ there exists an element $k_2$ of $K$ such that $(k_2)^n = k_1$.*

Proof: Consider $G_0$ as $G_0$ of Theorem 1. We then arrive at a $G_1$ containing (in the sense of isomorphism $G_0$). Let $G_1$ be the $G_0$ of Theorem 1 and we arrive at $G_2$ effective as $G_1$ of Theorem 1. In this way we arrive at a series of groups $G_0, G_1, G_2 \cdots$ etc., each containing in an isomorphic sense the preceding group and each such that every element corresponding to an element of the preceding group has roots of all orders. Corresponding to any element $g_i$ of $G_i$ there is in each $G_k (k \geqq i)$ an element $g_k$. If we consider the groups preceding $G_i$ we will find a first group $G_e (e \leqq i)$ in which there is an element $g_e$ corresponding to $g_i$. Let $\{g_i\}$ be the class of all elements corresponding to $g_i$. Let $K = [k]$ be the totality of all such classes. Define multiplications between two elements of $K$ by the following $\{g_{i1}\}\{g_{i2}\} = \{g_{i1} g_{i2}\}$, for every $i$. This rule is applicable since for any two sets in $K$ there exist elements in every group of the series $G_0, G_1, \cdots$ etc. beyond a certain group.

Moreover, $K$ forms a group, since multiplication is associative due to the associativity in the groups $G_0, G_1, \cdots$, $\{u\}$ acts as the unit of $K$ where $u$ is the unit of $G_0$, and $\{g_{i1}^{-1}\}$ is the reciprocal of $\{g_{i1}\}$.

It is also readily seen that roots of all orders exist in $K$ since by the formation of the series $G_0, G_1 \cdots$ for any integer $n$ and element $g_{i1}$ in $G_i$ there exists a corresponding element $g_{i+1,1}$ in $G_{i+1}$ and an element $g_{i+1,2}$ in $G_{i+1}$ such that $(g_{i+1,2})^n = (g_{i+1,1})$. Hence $\{g_{i+1,2}\}^n = \{g_{i+1,1}\} = \{g_{i,1}\}$. Note also that the sets in $K$ corresponding to elements of $G_0$ form a group $K'$ isomorphic with $G_0$.

Hence our theorem holds.

In order to see that there is no minimal group effective as $K$, consider $G_0$ the multiplication group of the rational number system. If we construct $K$ as above; any square root of 2 will be noncommutative with every square root of 3. But the real number system, less 0, is effective as $K$ and in this multiplication is commutative.

In case $G_0$ is commutative we may construct a commutative group effective as $K$ of Theorem 2 and commutative group effective as $G_1$ of Theorem 1, by the same process as above by merely adding to our definition of corresponding elements of $h$ the condition

$$h_1, h_2 .). h_1 h_2 \sim h_2 h_1,$$

with

$$h_1 \sim h_2 . h_3 \sim h_4 :): h_1 h_2 \sim h_3 h_4.$$

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# PERIODIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS.

By William Benjamin Fite.

If the coefficients in the equation

$$p_0 y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = 0 \tag{1}$$

have the period $l$ and $y_1, y_2, \cdots, y_n$ form a fundamental system of solutions, $y_i(x+l)$ is also a solution. Hence

$$\begin{aligned} y_1(x+l) &= \alpha_{11} y_1(x) + \alpha_{12} y_2(x) + \cdots + \alpha_{1n} y_n(x) \\ y_2(x+l) &= \alpha_{21} y_1(x) + \alpha_{22} y_2(x) + \cdots + \alpha_{2n} y_n(x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n(x+l) &= \alpha_{n1} y_1(x) + \alpha_{n2} y_2(x) + \cdots + \alpha_{nn} y_n(x). \end{aligned} \tag{2}$$

If now (1) has a solution of period $l$, the fundamental equation of this substitution, namely,

$$\begin{vmatrix} \alpha_{11} - \omega & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} - \omega & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} - \omega \end{vmatrix} = 0, \tag{3}$$

has at least one root equal to 1, and conversely. Moreover, the number of independent solutions of (1) of period $l$ is equal to the number of elementary divisors of the left member of (3) that contain the factor $\omega - 1$. If a root of (3) is a $k$th root of unity, there is a solution of (1) of period $kl$.

Consider now the non-homogeneous equation

$$p_0 y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = f, \tag{4}$$

where $f$ also has the period $l$. Let $\varphi$ be any solution of (4). Then

$$\varphi(x+l) = \beta_1 y_1(x) + \beta_2 y_2(x) + \cdots + \beta_n y_n(x) + \varphi(x).$$

If the $\beta$'s are all zero, $\varphi$ is a solution of (4) with the period $l$. If they are not all zero, consider the general solution of (4)

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + \varphi.$$

In order for this to be of period $l$, we must have

$$(5)\qquad \begin{aligned} &c_1 y_1(x+l)+\cdots+c_n y_n(x+l)+\varphi(x+l)\\ &\quad = c_1 y_1(x)+\cdots+c_n y_n(x)+\varphi(x) \end{aligned}$$

or,

$$\begin{aligned} c_1(\alpha_{11}y_1(x)+\cdots+\alpha_{1n}y_n(x))+\cdots+c_n(\alpha_{n1}y_1(x)+\cdots+\alpha_{nn}y_n(x))&\\ +\beta_1 y_1(x)+\cdots+\beta_n y_n(x)+\varphi(x)&\\ = c_1 y_1(x)+c_2 y_2(x)+\cdots+c_n y_n(x)+\varphi(x).& \end{aligned}$$

Hence

$$(6)\qquad \begin{aligned} (\alpha_{11}-1)c_1+\alpha_{21}c_2+\cdots+\alpha_{n1}c_n &= -\beta_1\\ \alpha_{12}c_1+(\alpha_{22}-1)c_2+\cdots+\alpha_{n2}c_n &= -\beta_2\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\\ \alpha_{1n}c_1+\alpha_{2n}c_2+\cdots+(\alpha_{nn}-1)c_n &= -\beta_n. \end{aligned}$$

Since the $\beta$'s are not all zero, there will be a set of $c$'s, not all zero, that satisfy these equations in case the determinant of the coefficients in the left members is not zero. That is, if 1 is not a root of (3), then (4) has a solution of period $l$.

THEOREM I. *If* (1) *has no solution of period* $l$, (4) *has one, while if* (1) *has such a solution,* (4) *may or may not have one.**

If the left member of (1) is an exact derivative,† we have

$$(7)\qquad p_0 y^{(n-1)}+\varphi_2 y^{(n-2)}+\cdots+\varphi_n y = c,$$

where $\varphi_2, \varphi_3, \cdots, \varphi_n$ are polynomials in $p_0, p_1, \cdots, p_n$ and their derivatives, and therefore have the period $l$; and $c$ is an arbitrary constant. From Theorem I we know that (7) has a solution of period $l$ either when $c = 0$ or when $c \neq 0$. But every solution of (7) is a solution of (1). Hence, in this case, (1) has a solution of period $l$.

In order that (4) have a solution of period $l$, under the present circumstances, it is necessary that $\int_x^{x+l} f(x)\,dx = 0$, since when a function of period $l$ is put for $y$ in the left member of (7), this expression has the same value for $x$ and for $x+l$. Conversely, if $\int_x^{x+l} f(x)\,dx = 0$, (4) has, in general, a solution of period $l$. This may be seen as follows:

* See Theorem II.

† The necessary and sufficient condition for this is $p_0^n - p_1^{(n-1)} + \cdots + (-1)^n p_n \equiv 0$. If this condition is satisfied $(-1)^{n-1} p_{n-1} \equiv -p_0^{(n-1)} + p_1^{(n-2)} - \cdots + (-1)^{n-1}\int_a^x p_n\,dx + c$. Hence $p_n$ must change sign, since otherwise $p_{n-1}$ would not be periodic.

For any solution $\varphi$ of (4) which is not of period $l$ we have

$$(8) \qquad p_0\varphi^{(n-1)} + \varphi_2\varphi^{(n-2)} + \cdots + \varphi_n\varphi\Big|_x^{x+l} = \int_x^{x+l} f(x)\,dx = 0.$$

Now (5) and the $n-1$ equations resulting from it by $n-1$ successive differentiations give

$$\begin{aligned}
&c_1[y_1(x)-y_1(x+l)] + c_2[y_2(x)-y_2(x+l)] + \cdots + c_n[y_n(x)-y_n(x+l)] \\
&\qquad = \varphi(x+l)-\varphi(x) \\
&c_1[y_1'(x)-y_1'(x+l)] + c_2[y_2'(x)-y_2'(x+l)] + \cdots + c_n[y_n'(x)-y_n'(x+l)] \\
&\qquad = \varphi'(x+l)-\varphi'(x) \qquad (9) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&c_1[y_1^{(n-1)}(x)-y_1^{(n-1)}(x+l)] + c_2[y_2^{(n-1)}(x)-y_2^{(n-1)}(x+l)] + \cdots \\
&\qquad \cdots + c_n[y_n^{(n-1)}(x)-y_n^{(n-1)}(x+l)] = \varphi^{(n-1)}(x+l)-\varphi^{(n-1)}(x).
\end{aligned}$$

If we multiply these equations through by $\varphi_n$, $\varphi_{n-1}$, $\cdots$, $\varphi_2$, and $p_0$ respectively and add, the sum of the resulting left members will be zero by virtue of (7) and the fact that these multipliers have the period $l$, while the sum of the right members will be zero by virtue of (8) and the same periodicity. For a given value of $x$, say $x = x_1$, we have therefore in (9) a set of not more than $n-1$ independent linear equations in $n$ unknowns, $c_1, c_2, \cdots, c_n$; and the equations are satisfied by a set of $c$'s not all zero, except possibly when all the coefficients in the left members of (9) are zero. For these values of the $c$'s

$$z \equiv c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + \varphi$$

is a solution of (4) such that $z(x_1) = z(x_1+l)$ and $z^{(i)}(x_1) = z^{(i)}(x_1+l)$ for all positive integral values of $i$. Hence $z(x) = z(x+l)$ for all values of $x$ within a sufficiently small interval around $x_1$, and hence for all values of $x$; that is, $z$ has the period $l$. In the exceptional case every solution of (1) has the period $l$.

If the left member of (1) is not an exact derivative, it can be made so by multiplying through by a solution $z$ of the adjoint equation. If $z$ has the period $l$ the preceding discussion applies to the resulting equation. We have therefore the following

THEOREM II. *If a linear homogeneous differential equation whose coefficients have the period $l$ has a solution with the period $l$, the adjoint equation has a solution with the same period. In this case if the corresponding non homogeneous equation has a right member $f(x)$ of period $l$, the necessary and sufficient condition that there be a solution of period $l$ is that*

$\int_x^{x+l} zf\,dx = 0$, *where* $z$ *is a solution of the adjoint equation of period* $l$, *except that when every solution of the homogeneous equation is of period* $l$ *the non-homogeneous equation may not have a periodic solution even if* $\int_x^{x+l} zf\,dx = 0$.*

The exception is one of fact, as may be seen from the equation

$$y'' + y = 2\cos x.$$

Every solution of the corresponding self-adjoint homogeneous equation has the period $2\pi$. But the non-periodic function $y = x\sin x$ is a solution of the given equation, and hence the latter has no periodic solution, although the condition $\int_x^{x+2\pi} 2\sin x \cdot \cos x\,dx = 0$ is satisfied and $\cos x$ has the period $2\pi$, $\sin x$ being a solution of the homogeneous equation. On the other hand, all the conditions described are satisfied in the case of the equation

$$y'' + y = 3\cos 2x,$$

and yet this equation has a solution of period $\pi$, namely $y = -\cos 2x$. As a matter of fact, if $y$ is a solution of (1) with the period $l$, $y+1$ is a solution of the non-homogeneous equation

$$p_0 y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = p_n$$

and has the same period. Hence corresponding to every equation of the form (1) with a solution of period $l$ there are non-homogeneous equations with solutions of the same period.

If we integrate the left member of (1) by parts between the limits $x$ and $x+l$, we get

$$(10)\quad \begin{aligned} &p_0 y^{(n-1)} + (p_1 - p_0')y^{(n-2)} + \cdots + [p_{n-1} - p_{n-2}' + \cdots + (-1)^{n-1} p_0^{(n-1)}]y \Big|_x^{x+l} \\ &+ \int_x^{x+l} y[p_n - p_{n-1}' + \cdots + (-1)^n p_0^{(n)}]\,dx = 0. \end{aligned}$$

If now $y$ has the period $l$, the part of the left member of (10) that is free from the integral sign vanishes, and hence the integral vanishes.

Suppose now conversely that $y$ is a solution of (1) such that

$$(11)\quad \int_x^{x+l} y[p_n - p_{n-1}' + \cdots + (-1)^n p_0^n]\,dx = 0.$$

* Cf. Mason, Trans. Amer. Math. Soc., vol. 6 (1905), p. 164.

If $y$ is not of period $l$, that is, if $y(x+l)-y(x) \not\equiv 0$, put $y(x+l)-y(x) \equiv z$. Obviously $z$ is a non-identically vanishing solution of (1). That is,

$$p_0 z^{(n)} + p_1 z^{(n-1)} + \cdots + p_n z = 0. \tag{12}$$

But we have from (10)

$$p_0 z^{(n-1)} + (p_1 - p_0') z^{(n-2)} + \cdots + [p_{n-1} - p_{n-2}' + \cdots + (-1)^{n-1} p_0^{(n-1)}] z = 0. \tag{13}$$

If we differentiate the left member of (13) and compare the result with (12), we see that

$$(p_n - p_{n-1}' + \cdots + (-1)^n p_0^{(n)}) z = 0. \tag{14}$$

But this is impossible unless the coefficient of $z$ in (14) vanishes identically. This completes the proof of

THEOREM III. *If* $p_n - p_{n-1}' + \cdots + (-1)^n p_0^{(n)} \not\equiv 0$, *the necessary and sufficient condition that a solution* $y$ *of* (1) *be of period* $l$ *is that*

$$\int_x^{x+l} y[p_n - p_{n-1}' + \cdots + (-1)^n p_0^{(n)}]\, dx = 0.$$

Now $p_n - p_{n-1}' + \cdots + (-1)^n p_0^{(n)}$ is the coefficient of the dependent variable in the equation adjoint to (1), and therefore, if it vanishes identically, any constant is a solution of the adjoint equation. This means that the left member of (1) is an exact derivative, and therefore that (1) has at least one solution of period $l$.

We proceed to apply the foregoing considerations to equations of the second order. Consider the equation

$$p_0 y'' + p_1 y' + p_2 y = 0. \tag{15}$$

If $y$ is a solution of (15) of period $l$, multiply the left member of this equation by $y$ and integrate between the limits $x$ and $x+l$. We get, in view of the assumed periodicity of $p_0$, $p_1$, $p_2$, and $y$,

$$\int_x^{x+l} \left[ \left( p_2 - \frac{p_1'}{2} + \frac{p_0''}{2} \right) y^2 - p_0 y'^2 \right] dx = 0. \tag{16}$$

But this is impossible in case $p_0$ and $2p_2 - p_1' + p_0''$ have opposite signs for all values of $x$. Hence under these circumstances equation (15) has no solution of period $l$. The same conclusion holds in case $p_0$ and $p_2$ have opposite signs for all values of $x$, since then no solution could have a positive maximum or a negative minimum.

If the equation

$$p_0 y'' + p_1 y' + p_2 y = f, \tag{17}$$

where $p_0$ is everywhere positive and $f$ everywhere negative, has a solution $\varphi$ that is of period $l$ and everywhere positive, then (15) has no solution of period $l$. For if in (15) we make the transformation $y = \varphi z$, we get

$$p_0 \varphi z'' + (2p_0 \varphi' + p_1 \varphi) z' + (p_0 \varphi'' + p_1 \varphi' + p_2 \varphi) z = 0, \tag{18}$$

where the coefficient of $z''$ is everywhere positive and the coefficient of $z$ everywhere negative. It follows from what has just been said that $z$ cannot be of period $l$. Hence $y$ cannot have this period.

Moreover (15) cannot have a solution of period $l$ in case $p_0$ and $p_0'' - p_1' + p_2$ are everywhere of opposite signs, since the adjoint equation has no solution of this period.

# ON THE INVERSION OF THE ORDER OF INTEGRATION OF A TWO FOLD ITERATED INTEGRAL.

By H. J. Ettlinger.

W. H. Young* has proved that if $f(x, y)$ is bounded and integrable in each variable for *all* values of the other, then both of the iterated integrals exist. Lichtenstein† has shown that under the same conditions these integrals are equal. Gillespie‡ has recently shown that whenever the two integrals exist, they are equal.

It is the object of this note to establish necessary and sufficient conditions for the inversion of the order of integration of a two fold iterated integral.

The following establishes sufficient conditions.

Hypothesis:

1. *$f(x, y)$ is a real bounded function of $(x, y)$ on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$, i. e., there exists a constant $K$ independent of $x$ and $y$, such that $|f(x, y)| < K$ for all points $(x, y)$ of the square.*
2. *For each $x$ on $(a, b)$, save a discrete§ set, $C_0^{(x)}$, $f(x, y)$ is integrable in $y$ on $(a, b)$.*
3. *For each $y$ on $(a, b)$, save a discrete set, $C_0^{(y)}$, $f(x, y)$ is integrable in $x$ on $(a, b)$.*

Conclusion: $\int_a^b dy \int_a^b f(x, y)\, dx$ *exists.*

Proof. *Let* $g_n(x) = (b-a)/n \sum_{i=1}^{n} f(x, y_{in})$, *$x$ not in $C_0^{(x)}$, such that*

$$\lim_{n \to \infty} g_n(x) = \int_a^b f(x, y)\, dy.$$

It follows from hypothesis 1. that $|g_n(x)| \leqq K(b-a)$.

We shall now apply the following generalization of a theorem due to Lebesgue. *Let $g_n(x)$ be an infinite sequence of summable functions on $(a, b)$, which converges for each value of $x$ in $(a, b)$ save for a null set to a limit*

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* *On parametric integration*, Monat. für Math. u. Physik, vol. 2 (1910), p. 127.

† *Über die Integration eines bestimmten Integrals in bezug auf einen Parameter*, Gött. Nach., 1910, pp. 468–470.

‡ *Repeated integrals*, Annals of Math., (2), vol. 20 (1919), pp. 224–225.

§ A *discrete* set is a set of content zero, i. e., a set which can be enclosed in a *finite* set of intervals, the sum of whose lengths is as small as you please.

*function $g(x)$. Let $g_n(x)$ be bounded for all values of $n$ and for all values of $x$ on $(a, b)$. Then*

$$\lim_{n\to\infty} L\int_a^b g_n(x)\,dx = L\int_a^b g(x)\,dx.$$

Lebesgue has also shown that *any function which is R-integrable on $(a, b)$ is summable on $(a, b)$ and the two integrals have the same value.*

Now $f(x, y_{in})$ is $R$-integrable in $x$ and hence is summable. Hence $g(x)$ is summable by Lebesgue's theorem,

$$L\int_a^b dx\int_a^b f(x, y)\,dy = \lim_{n\to\infty} (b-a)/n \sum_{i=1}^{n} L\int_a^b f(x, y_{in})\,dx$$
$$= \int_a^b dy\int_a^b f(x, y)\,dx,$$

since the limit on the left exists as $n\to\infty$, the limit of the right must also exist as $n\to\infty$. This establishes the above conclusion. By interchanging $x$ and $y$ in the proof we obtain the existence of the integral

$$\int_a^b dx\int_a^b f(x, y)\,dy.$$

By Gillespie's theorem, the two integrals must have the same value and hence we have established sufficient conditions for the inversion of the order of integration of the two fold iterated integrals.

The following establishes necessary conditions.

HYPOTHESIS:

1. *$f(x, y)$ is a real bounded function of $(x, y)$ on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$.*
2. *$\int_a^b dx\int_a^b f(x, y)\,dy$ exists.*

CONCLUSION:

1. *$\int_a^b dx\int_a^b f(x, y)\,dy$ exists.*
2. *$\int_a^b f(x, y)\,dy - \int_a^b f(x, y)\,dy$ is an integrable null function of $x$ on $(a, b)$.*

Proof. Hypothesis 2. carries with it the existence of the integral of conclusion 1. It also follows from this same hypothesis that

$$\int_a^b dx\int_a^b f(x, y)\,dy$$

exists and is equal to the former integral.* A necessary condition that this be true is conclusion 2.

By interchanging $x$ and $y$, we obtain other necessary conditions symmetric to those given above. We shall now prove these conditions sufficient.

HYPOTHESIS:

1. *$f(x, y)$ is a real bounded function of $(x, y)$ on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$.*
2. *$\int_a^b f(x, y)\,dy$ is integrable on $(a, b)$.*
3. *$\int_a^b f(x, y)\,dx$ is integrable on $(a, b)$.*
4. *$\int_a^b f(x, y)\,dy - \int_a^b f(x, y)\,dy$ is an integrable null function on $(a, b)$.*
5. *$\int_a^b f(x, y)\,dx - \int_a^b f(x, y)\,dx$ is an integrable null function on $(a, b)$.*

CONCLUSION:

1. *$\int_a^b dx \int_a^b f(x, y)\,dy$ exists.*
2. *$\int_a^b dy \int_a^b f(x, y)\,dx$ exists.*
3. $\int_a^b dx \int_a^b f(x, y)\,dy = \int_a^b dy \int_a^b f(x, y)\,dx$.

Proof. From hypotheses 2. and 4. it follows that $\int_a^b dx \int_a^b f(x, y)\,dy$ exists, since if two bounded functions differ by an integrable null function and one of them is integrable, the other is also and the two integrals are equal.

Similarly conclusion 2. is obtained from hypotheses 3. and 5. Hence by Gillespie's theorem we have conclusion 3.

Hence we have proved that hypotheses 2., 3., 4., and 5. are necessary and sufficient that the bounded function $f(x, y)$ have its two iterated integrals and that these be equal.

If the sufficient conditions first established are satisfied, it is clearly seen that the latter sufficient conditions will also be satisfied, but not conversely. The second set of sufficient conditions is therefore weaker than the first.

From the first theorem proved above, we derive the following two corollaries.

* Gillespie, l. c. p. 225. See also E. W. Hobson, *The theory of functions of a real variable*, second edition, Cambridge, 1921, p. 480.

*Corollary* 1. $g(x) = \int_a^b f(x, y)\, dy$ is continuous at every point of $(a, b)$ save a null set, $M_0^{(x)}$.

*Corollary* 2. If $x$ is not in $C_0^{(x)}$ and $y$ not in $C_0^{(y)}$,

$$F(x, y) = \int_a^x ds \int_a^y f(s, t)\, dt = \int_a^y dt \int_a^x f(s, t)\, ds$$

has first partial derivatives, $F_x = \int_a^y f(x, t)\, dt$, continuous in $y$ for each $x$ not in $C_0^{(x)}$, and $F_y = \int_a^x f(s, y)\, ds$, continuous in $x$ for each $y$ not in $C_0^{(y)}$, and a second order cross derivative, $F_{xy} = F_{yx} = f(x, y)$.

The foregoing theorems and corollaries have been extended to the case of functions of $m$ variables.*

The first theorem also holds if we replace $R$-integrability in *one* of the variables by *summability* except for a *null* set. The final statements will be changed from $R$-integrals with respect to this variable to $L$-integrals.

* See my paper *On multiple iterated integrals*, Amer. Journ. of Math., vol. 48 (1926), pp. 215–222.

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# THE SUMMABILITY OF SINGLE AND MULTIPLE FOURIER SERIES.

By H. W. Bailey.

## Introduction.

It is the purpose of this paper to prove the summability of Fourier series, both single and multiple, by two general definitions of summability and to exhibit various definitions of summability which have already been applied to Fourier series as equivalent to definitions which are special cases of these general definitions.

The method used is essentially a generalization of the method for convergence developed by Birkhoff* and applied by him to the so-called "Birkhoff series", the tools being a Green's function and contour integration. Birkhoff's method has been used by Camp† to prove the convergence of multiple Fourier series and of multiple Birkhoff series and by Carman‡ to prove the convergence of both single and multiple Fourier series. The generalized method has been used by Hosford§ to prove the summability of the Fourier-Bessel and Dini expansions in Bessel's functions and by Conkwright|| to prove the summability of single and multiple Birkhoff series. The fundamental ideas on which this extension is based were given by Professor R. D. Carmichael in a course of lectures on Linear Differential Equations in 1924–25 at the University of Illinois.

In section I, definitions of summability $(\Phi)$ and summability $(\Psi)$ for single series are given, and the single Fourier series is proved summable by these definitions. The main results are contained in Theorems I and II. In section II, definitions of summability $(\Phi_1, \Phi_2, \cdots, \Phi_p)$ and summability $(\Psi_1, \Psi_2, \cdots, \Psi_p)$ for $p$-fold series are given and the $p$-fold Fourier series is proved summable by these methods. The main results are contained in Theorems III and IV. In section III, it is shown by means of various subsidiary lemmas that the definitions $(\Phi)$ and $(\Psi)$ for single series include as special cases definitions which are equivalent to summability $(C1)$, $(Cr)$ ($r$ an integer greater than unity), two forms equivalent to $(Cr)$, and $(VP)$ which is the notation I shall use for the definition of De la Vallée Poussin. Limitations of the definitions as here set up for summability $(\Phi)$ and $(\Psi)$

* Birkhoff, Trans. Amer. Math. Soc., 9 (1908), 373–395.

† Camp, Trans. Amer. Math. Soc., 25 (1923), 123–134; ibid, 338–342.

‡ Carman, Bull. Amer. Math. Soc., (2), 30 (1923–24), 410–416.

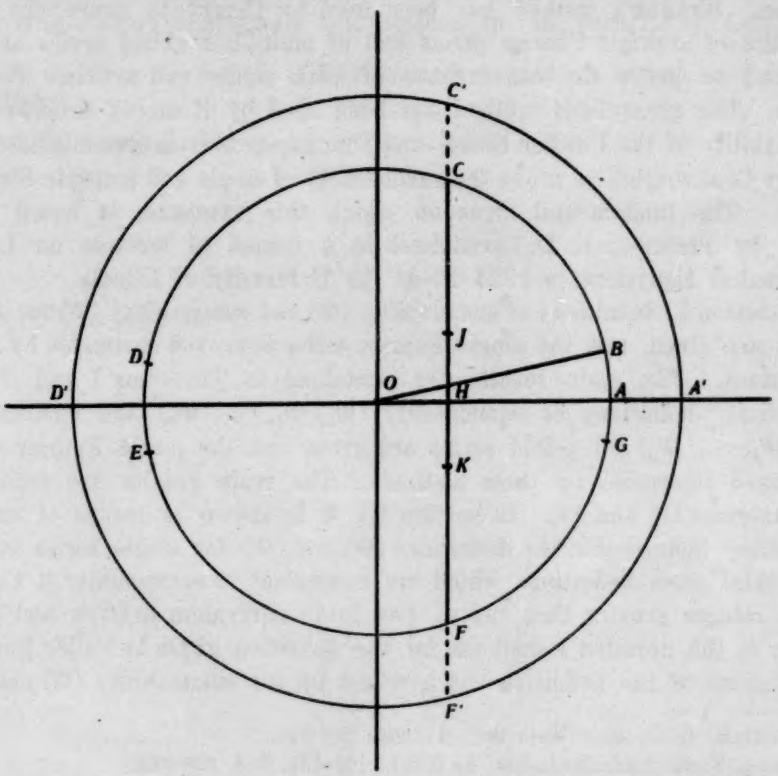
§ Hosford, Dissertation, Illinois, 1926.

|| Conkwright, Dissertation, Illinois, 1926.

are discussed. The main result of section III is to be found in Theorem V. In section IV, various definitions of summability of a $p$-fold Fourier series are discussed as special cases of summability $(\Phi_1, \cdots, \Phi_p)$ and $(\Psi_1, \cdots, \Psi_p)$. All of these definitions are either monomorphic or polymorphic types of definition. The main result is to be found in Theorem VI.

## I. Single series.

1. **Definition of summability** $(\Phi)$. Let $\varrho$ be a complex variable and consider the contour in the $\varrho$-plane made up of the circle $ADG$, of radius $n-\frac{1}{2}$ about the origin, and the line $CF$, which is the portion of the line $R(\varrho)=\mu$ cut off by the circle; $\mu$ is not an integer. Let $\mu$ be taken to be positive; this can be done without loss of generality and the modifications in the argument if $\mu$ be taken negative are evident. Let $m$ be an integer such that $m<\mu<m+1$, $\mu-m\geqq\zeta>0$, and $m+1-\mu\geqq\zeta>0$. We shall consider values of $n$ such that $n>m+1$.




Separate the circle into arcs by the points $B$, $D$, $E$, $G$, where $D$, $E$, and $G$ are the reflections of $B$ in the imaginary axis, origin, and real

axis respectively. Letting $\angle AOB = \theta_n$, we impose the following conditions:

$$(1) \qquad 1)\ \theta_n \to 0, \qquad 2)\ n\theta_n \to \infty, \qquad \text{as } n \to \infty.$$

Separate the line $CF$ into segments by the points $J$ and $K$ such that $HJ = HK = \lambda_n$. We impose the following conditions on $\lambda_n$:

$$(2) \qquad 1)\ \lambda_n \to \infty, \qquad 2)\ n^{-1}\lambda_n^2 \to 0, \qquad \text{as } n \to \infty.$$

Call $ACFA$ the contour $\Gamma_n$ and $CEFC$ the contour $\bar{\Gamma}_n$. On the circular part of the contour we shall use the substitution $\varrho = (n-\frac{1}{2})e^{i\theta}$, and on the line $CF$ the substitution $\varrho = \mu + is$. Now draw the circle $A'C'D'F'A'$ which is of radius $n-\frac{1}{2}+\gamma$, $\gamma > 0$, with center at the origin.

Let us consider two functions $\varphi_n(\varrho)$ and $\bar{\varphi}_n(\varrho)$ which shall have the following properties:

1) $\varphi_n(\varrho)$ is analytic on and within the contour bounded by $A'C'F'A'$;

1a) $\bar{\varphi}_n(\varrho)$ is analytic on and within the contour bounded by $C'D'F'C'$;

2) If $k$ is any positive integer less than $n$, $\bar{\varphi}_n(k) + \bar{\varphi}_n(-k) = 2a_{nk}$ when $k \leqq m$, and $\varphi_n(k) + \bar{\varphi}_n(-k) = 2a_{nk}$ when $k > m$; moreover write $\bar{\varphi}_n(0) = a_{n0}$;

$$(3) \qquad 3)\ \int_G^B \frac{|\varphi_n(\varrho)|}{|\sin\theta|}\,d\theta = o(1),^* \quad \int_B^C \frac{|\varphi_n(\varrho)|}{\sin\theta}\,d\theta < M_1, \quad \int_F^G \frac{|\varphi_n(\varrho)|}{|\sin\theta|}\,d\theta < M_1;$$

$$3a)\ \int_D^E \frac{|\bar{\varphi}_n(\varrho)|}{|\sin\theta|}\,d\theta = o(1), \quad \int_C^D \frac{|\bar{\varphi}_n(\varrho)|}{\sin\theta}\,d\theta < M_1, \quad \int_E^F \frac{|\bar{\varphi}_n(\varrho)|}{|\sin\theta|}\,d\theta < M_1;$$

$$4)\ \int_K^J |\bar{\varphi}_n(\varrho) - \varphi_n(\varrho)|\,ds = o(1), \quad \int_J^C \frac{|\bar{\varphi}_n(\varrho) - \varphi_n(\varrho)|}{s}\,ds < M_1,$$

$$\int_F^K \frac{|\bar{\varphi}_n(\varrho) - \varphi_n(\varrho)|}{|s|}\,ds < M_1,$$

where $M_1$ is a constant.

The series $\sum_{k=0}^{\infty} u_k$ is said to be summable $(\Phi)$ to the sum $t$ if $\underset{n=\infty}{L}\ t_n = t$, where $t_n = \sum_{k=0}^{n-1} a_{nk}\,u_k$. This definition is regular† if

$$(4) \qquad 1)\ \underset{n=\infty}{L}\ a_{nk} = 0 \text{ for every } k,$$

$$2)\ \sum_{j=0}^{n-1} |a_{nj} - a_{nj+1}| < N, \text{ independent of } n.$$

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* $v_n = o(1)$ means that $\underset{n=\infty}{L}\ v_n = 0$.

† Carmichael, Bull. Amer. Math. Soc., (2), 25 (1918–19), 97–131. For the conditions of regularity see pp. 107–111.

We shall be concerned only with regular definitions of summability. Since the $a_{nk}$ are defined in terms of $\varphi_n(\varrho)$ and $\bar{\varphi}_n(\varrho)$, this condition puts added restrictions upon these functions.

**2. Summability ($\Phi$) of Fourier series.** We shall now prove the following theorem:

THEOREM I. *Let $f(x)$ be a function of a real variable, defined in the interval $-\pi \leqq x \leqq \pi$, bounded, and such that $\int_{-\pi}^{\pi} |f(\xi)|\, d\xi$ exists. Then the Fourier series associated with $f(x)$ is summable ($\Phi$) to the sum $\frac{1}{2}[f(x+0)+f(x-0)]$ at all the interior points at which the limits $f(x\pm 0)$ exist. At $x = \pm\pi$, the series is summable ($\Phi$) to the sum $\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$, providing these limits exist. If $f(x)$ is continuous in a subinterval defined by the inequalities $\alpha \leqq x \leqq \beta$, then in the interval $\alpha+\eta \leqq x \leqq \beta-\eta$, $\eta>0$, the series is uniformly summable ($\Phi$).*

We shall start with the differential equation and boundary conditions

$$y''+\varrho^2 y = 0, \qquad y(\pi) = y(-\pi), \qquad y'(\pi) = y'(-\pi), \tag{5}$$

where the accents represent derivatives with respect to $x$. We shall employ the usual Green's function*

$$\varrho\, G(x, \xi, \varrho^2) = \frac{\cos\varrho\,[x-\xi+\pi\,\mathrm{sgn}\,(\xi-x)]}{2\sin\varrho\,\pi}. \tag{6}$$

If $k$ be a positive integer, the residue of $\varphi_n(\varrho)\,\varrho\, G(x, \xi, \varrho^2)$ at $\varrho = k$, $k>m$, is easily found to be $\varphi_n(k)\frac{1}{2\pi}\cos k(x-\xi)$; of $\bar{\varphi}_n(\varrho)\,\varrho\, G(x, \xi, \varrho^2)$ at $\varrho = k$, $k \leqq m$, to be $\bar{\varphi}_n(k)\frac{1}{2\pi}\cos k(x-\xi)$; and of $\bar{\varphi}_n(\varrho)\,\varrho\, G(x, \xi, \varrho^2)$ at $\varrho = -k$, $k<n$, to be $\bar{\varphi}_n(-k)\frac{1}{2\pi}\cos k(x-\xi)$. Moreover the residue of $\bar{\varphi}_n(\varrho)\,\varrho\, G(x, \xi, \varrho^2)$ at $\varrho = 0$ is $\bar{\varphi}_n(0)\frac{1}{2\pi}$. Then using the definitions of $\Gamma_n$ and $\bar{\Gamma}_n$ given above, we may write

$$\frac{1}{2\pi i}\int_{\Gamma_n} \varphi_n(\varrho)\,\varrho\, G(x, \xi, \varrho^2)\, d\varrho = \frac{1}{2\pi}\sum_{k=m+1}^{n-1} \varphi_n(k)\cos k(x-\xi)$$

and

$$\frac{1}{2\pi i}\int_{\bar{\Gamma}_n} \bar{\varphi}_n(\varrho)\,\varrho\, G(x, \xi, \varrho^2)\, d\varrho = \frac{1}{2\pi}\bar{\varphi}_n(0) + \frac{1}{2\pi}\sum_{k=1}^{m} \bar{\varphi}_n(k)\cos k(x-\xi)$$
$$+ \frac{1}{2\pi}\sum_{k=m+1}^{n-1} \bar{\varphi}_n(-k)\cos k(x-\xi) + \frac{1}{2\pi}\sum_{k=1}^{m} \bar{\varphi}_n(-k)\cos k(x-\xi).$$

* Carman, loc. cit., p. 411.

Adding, we have

$$\frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\varrho\,G(x,\xi,\varrho^2)\,d\varrho=\frac{1}{2\pi}a_{n0}+\sum_{k=1}^{n-1}a_{nk}\frac{1}{\pi}\cos k(x-\xi).$$

On multiplying both members by $f(\xi)$ and integrating from $-\pi$ to $\pi$, we get

$$(7)\qquad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{-\pi}^{\pi}\varrho\,G(x,\xi,\varrho^2)f(\xi)\,d\xi\,d\varrho$$
$$=\frac{1}{2}a_{n0}\cdot a_0+\sum_{k=1}^{n-1}a_{nk}(a_k\cos kx+b_k\sin kx),$$

where the $a_k$ and $b_k$ are the ordinary Fourier constants. Thus the integral in the left member of (7) represents what we may call the $n$th partia sum ($\Phi$) of the Fourier series associated with $f(x)$. Hence if we can provel that, for $-\pi<x<\pi$, the limit as $n$ becomes infinite of the integral in the left member of (7) is $\frac{1}{2}[f(x+0)+f(x-0)]$, we shall have proved the theorem for interior points, namely,

$$\lim_{n=\infty}\left[\frac{1}{2}a_{n0}\cdot a_0+\sum_{k=1}^{n-1}a_{nk}(a_k\cos kx+b_k\sin kx)\right]=\frac{1}{2}[f(x+0)+f(x-0)]$$

To do this, let us consider the two integrals

$$(8)\qquad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{-\pi}^{x}\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x-0)]\,d\xi\,d\varrho$$

and

$$(9)\qquad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{x}^{\pi}\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x+0)]\,d\xi\,d\varrho.$$

By breaking the path of integration with respect to $\varrho$ into pieces as indicated in the figure, it is easy to see that (8) may be written in the form

$$\frac{1}{2\pi i}\left\{\left[\int_G^B+\int_B^C+\int_F^G\right]\varphi_n(\varrho)+\left[\int_C^D+\int_D^E+\int_E^F\right]\bar{\varphi}_n(\varrho)\right.$$
$$\left.+\int_F^C[\bar{\varphi}_n(\varrho)-\varphi_n(\varrho)]\right\}\int_{-\pi}^{x}\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x-0)]\,d\xi\,d\varrho.$$

The path of integration with respect to $\xi$ may likewise be broken into two parts, from $-\pi$ to $x-\delta$ and from $x-\delta$ to $x$. Let $I\begin{pmatrix}B & x\\ G & -\pi\end{pmatrix}$ denote the quantity

$$\left|\frac{1}{2\pi i}\int_G^B\int_{-\pi}^{x}\varphi_n(\varrho)\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x-0)]\,d\xi\,d\varrho\right|;$$

we shall use a corresponding notation for other like quantities which arise from (8).

Since $f(\xi)$ is bounded, $|f(\xi)-f(x-0)|<M$ for $-\pi \leqq \xi \leqq \pi$; and for every positive $\varepsilon$, small at will, there exists a $\delta>0$ such that $|f(\xi)-f(x-0)|<\varepsilon$ for $0 \leqq x-\xi<\delta$, and such that $\pi-\delta>x$. For $\xi<x$,

$$\varrho G(x, \xi, \varrho^2) = \frac{\cos\varrho(x-\xi-\pi)}{2\sin\varrho\pi}$$

from (6). Let $\varrho = (n-\frac{1}{2})e^{i\theta}$; then it is easy to show that

$$\left|\int_{-\pi}^{x-\delta} \cos\varrho(x-\xi-\pi)\,d\xi\right| \leqq \frac{1}{(n-\frac{1}{2})\sin\theta}\{\sinh[(n-\tfrac{1}{2})(\pi-\delta)\sin\theta]+\sinh[(n-\tfrac{1}{2})x\sin\theta]\},$$

$$\left|\int_{x-\delta}^{x} \cos\varrho(x-\xi-\pi)\,d\xi\right| \leqq \frac{1}{(n-\frac{1}{2})\sin\theta}\{\sinh[(n-\tfrac{1}{2})\pi\sin\theta]-\sinh[(n-\tfrac{1}{2})(\pi-\delta)\sin\theta]\},$$

and that

$$|\sin\varrho\pi| = \frac{1}{2}\sqrt{2\cosh[2(n-\tfrac{1}{2})\pi\sin\theta]-2\cos[2(n-\tfrac{1}{2})\pi\cos\theta]} \equiv A(n,\theta).$$

Then, using the well known theorem that the absolute value of an integral is not greater than the integral of the absolute value and the inequalities above, we have

$$I\begin{pmatrix} B & x \\ G & -\pi \end{pmatrix} \leqq \frac{M}{2\pi}\int_G^B \frac{|\varphi_n(\varrho)|}{\sin\theta}\,\frac{\sinh[(n-\frac{1}{2})\pi\sin\theta]+\sinh[(n-\frac{1}{2})x\sin\theta]}{A(n,\theta)}\,d\theta.$$

It will be seen without difficulty that this integral exists and that, if both numerator and denominator be divided by $\exp.[(n-\frac{1}{2})\pi\sin\theta]$, the second quotient in the integrand is bounded. Thus we have

$$I\begin{pmatrix} B & x \\ G & -\pi \end{pmatrix} < \bar{M}\int_G^B \frac{|\varphi_n(\varrho)|}{|\sin\theta|}\,d\theta = o(1)$$

by (3), 3).

Now

$$I\begin{pmatrix} C & x-\delta \\ B & -\pi \end{pmatrix} \leqq \frac{M}{2\pi}\int_B^C \frac{|\varphi_n(\varrho)|}{\sin\theta}\,\frac{\sinh[(n-\frac{1}{2})(\pi-\delta)\sin\theta]+\sinh[(n-\frac{1}{2})x\sin\theta]}{A(n,\theta)}\,d\theta.$$

The second quotient in the integrand is dominated by a constant times exp. $[-(n-\frac{1}{2})\delta \sin\theta]$, so that

$$I\begin{pmatrix} C & x-\delta \\ B & -\pi \end{pmatrix} < \bar{M}\int_B^C \frac{|\varphi_n(\varrho)|}{\sin\theta}\,\frac{1}{\exp.\,[(n-\frac{1}{2})\delta\sin\theta]}\,d\theta$$
$$\leqq \bar{M}\,\frac{1}{\exp.\,[(n-\frac{1}{2})\delta\sin\theta_n]}\int_B^C \frac{|\varphi_n(\varrho)|}{\sin\theta}\,d\theta = o(1)$$

by (3), 3). By a similar argument

$$I\begin{pmatrix} G & x-\delta \\ F & -\pi \end{pmatrix} = o(1).$$

Next

$$I\begin{pmatrix} C & x \\ B & x-\delta \end{pmatrix} \leqq \varepsilon\cdot\frac{1}{2\pi}\int_B^C \frac{|\varphi_n(\varrho)|}{\sin\theta}\,\frac{\sinh[(n-\frac{1}{2})\pi\sin\theta]-\sinh[(n-\frac{1}{2})(\pi-\delta)\sin\theta]}{A(n,\theta)}\,d\theta.$$

In this integrand the second quotient is bounded, so that

$$I\begin{pmatrix} C & x \\ B & x-\delta \end{pmatrix} < \varepsilon\cdot\bar{M}\int_B^C \frac{|\varphi_n(\varrho)|}{\sin\theta}\,d\theta < \varepsilon\cdot\bar{M}\cdot M_1 = \varepsilon\cdot\text{constant};$$

hence the contribution of this integral can be made small at will. Similarly we show that

$$I\begin{pmatrix} G & x \\ F & x-\delta \end{pmatrix} < \varepsilon\cdot\text{constant}.$$

By precisely the same type of argument, using (3), 3a), we show that

$$I\begin{pmatrix} E & x \\ D & -\pi \end{pmatrix} = o(1), \quad I\begin{pmatrix} D & x-\delta \\ C & -\pi \end{pmatrix} = o(1), \quad I\begin{pmatrix} D & x \\ C & x-\delta \end{pmatrix} < \varepsilon\cdot\text{constant},$$
$$I\begin{pmatrix} F & x-\delta \\ E & -\pi \end{pmatrix} = o(1), \quad I\begin{pmatrix} F & x \\ E & x-\delta \end{pmatrix} < \varepsilon\cdot\text{constant}.$$

There remains the evaluation of the integral along the line $CF$. Let $\varrho = \mu + is$; then again it is easy to show that

$$\left|\int_{-\pi}^{x-\delta} \cos\varrho(x-\xi-\pi)\,d\xi\right| \leqq \frac{1}{s}[\sinh s(\pi-\delta)+\sinh sx],$$
$$\left|\int_{x-\delta}^{x} \cos\varrho(x-\xi-\pi)\,d\xi\right| \leqq \frac{1}{s}[\sinh s\pi-\sinh s(\pi-\delta)],$$

and that

$$|\sin \varrho\pi| = \frac{1}{2}\sqrt{2\cosh 2s\pi - 2\cos 2\mu\pi} \equiv \Delta(s).$$

Then

$$I\begin{pmatrix} J & x \\ K & -\pi \end{pmatrix} \leqq \frac{M}{2\pi}\int_K^J |\bar{\varphi}_n(\varrho) - \varphi_n(\varrho)| \frac{\sinh s\pi + \sinh sx}{s\Delta(s)}\, ds.$$

The quotient in the integrand is bounded over the range of integration, so that

$$I\begin{pmatrix} J & x \\ K & -\pi \end{pmatrix} < \bar{M}\int_K^J |\bar{\varphi}_n(\varrho) - \varphi_n(\varrho)|\, ds = o(1)$$

by (3), 4).

Now

$$I\begin{pmatrix} C & x-\delta \\ J & -\pi \end{pmatrix} \leqq \frac{M}{2\pi}\int_J^C \frac{|\bar{\varphi}_n(\varrho) - \varphi_n(\varrho)|}{s} \frac{\sinh s(\pi-\delta) + \sinh sx}{\Delta(s)}\, ds.$$

The second quotient in the integrand is dominated by a constant times $e^{-s\delta}$, so that

$$I\begin{pmatrix} C & x-\delta \\ J & -\pi \end{pmatrix} < \bar{M}\int_J^C \frac{|\bar{\varphi}_n(\varrho) - \varphi_n(\varrho)|}{s} \cdot \frac{1}{e^{s\delta}}\, ds \leqq \bar{M}\cdot M_1 \cdot e^{-\delta\lambda_n} = o(1)$$

in view of (3), 4). Similarly we can show that

$$.I\begin{pmatrix} K & x-\delta \\ F & -\pi \end{pmatrix} = o(1).$$

Again

$$I\begin{pmatrix} C & x \\ J & x-\delta \end{pmatrix} \leqq \varepsilon\cdot\frac{1}{2\pi}\int_J^C \frac{|\bar{\varphi}_n(\varrho) - \varphi_n(\varrho)|}{s} \frac{\sinh s\pi - \sinh s(\pi-\delta)}{\Delta(s)}\, ds;$$

the second quotient is bounded, so that

$$I\begin{pmatrix} C & x \\ J & x-\delta \end{pmatrix} < \varepsilon\cdot\bar{M}\cdot M_1 = \varepsilon\cdot\text{constant}.$$

By a similar argument we can show that

$$I\begin{pmatrix} K & x \\ F & x-\delta \end{pmatrix} < \varepsilon\cdot\text{constant}.$$

Therefore, on combining the separate quantities, we show that

$$(10)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho) + \int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{-\pi}^{x}\varrho\, G(x,\xi,\varrho^2)[f(\xi) - f(x-0)]\, d\xi\, d\varrho = o(1).$$

Exactly the same type of argument applied to (9) shows that

$$(11)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_x^{\pi}\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x+0)]\,d\xi\,d\varrho = o(1).$$

Let $f(\xi)=1$, $\varphi_n(\varrho)=\bar{\varphi}_n(\varrho)=1$. The path of integration with respect to $\varrho$ may then be replaced by the circle of radius $n-\frac{1}{2}$ with center at the origin which we shall denote by $\Gamma_n'$. From the paper by Carman*, or by evaluation of the integrals, it follows that

$$\frac{1}{2\pi i}\int_{\Gamma_n'}\int_{-\pi}^{x}\varrho\,G(x,\xi,\varrho^2)\,d\xi\,d\varrho = \frac{1}{2}+o(1),$$

and that

$$\frac{1}{2\pi i}\int_{\Gamma_n'}\int_{x}^{\pi}\varrho\,G(x,\xi,\varrho^2)\,d\xi\,d\varrho = \frac{1}{2}+o(1).$$

Since the definition of summability $(\Phi)$ is regular, it follows that

$$(12)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{-\pi}^{x}\varrho\,G(x,\xi,\varrho^2)\,d\xi\,d\varrho = \frac{1}{2}+o(1),$$

$$(13)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{x}^{\pi}\varrho\,G(x,\xi,\varrho^2)\,d\xi\,d\varrho = \frac{1}{2}+o(1).$$

Combining (10) and (12), we have

$$\frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{-\pi}^{x}\varrho\,G(x,\xi,\varrho^2)f(\xi)\,d\xi\,d\varrho = \frac{1}{2}f(x-0)+o(1);$$

combining (11) and (13), we have

$$\frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{x}^{\pi}\varrho\,G(x,\xi,\varrho^2)f(\xi)\,d\xi\,d\varrho = \frac{1}{2}f(x+0)+o(1).$$

On adding these two integrals and expressing the result with an explicit limit symbol, we have

$$\lim_{n\to\infty}\left[\frac{1}{2\pi i}\left\{\int_{\Gamma_n}\varphi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\varphi}_n(\varrho)\right\}\int_{-\pi}^{\pi}\varrho\,G(x,\xi,\varrho^2)\,f(\xi)\,d\xi\,d\varrho\right]$$
$$=\frac{1}{2}[f(x-0)+f(x+0)],$$

and the theorem is proved for interior points.

* Carman, loc. cit., p. 413.

To prove the theorem for boundary points, let $\chi(x)$ be a function of period $2\pi$ identical with $f(x)$ for the range $-\pi \leqq x < \pi$. By the part of the theorem already proved, the Fourier series associated with $\chi(x)$ is summable $(\Phi)$ at every point of the interval $0 \leqq x \leqq 2\pi$ at which the limits $\chi(x \pm 0)$ exist, with the possible exception of $x = \pi$. Let $g(x) = \chi(x+\pi)$; the Fourier constants for $g(x)$ are identical with those obtained by replacing $x$ by $x+\pi$ in the Fourier series for $\chi(x)$. But $g(x)$ satisfies the conditions of the theorem for interior points; hence it is summable $(\Phi)$ at $x = 0$ to the sum $\frac{1}{2}[g(+0)+g(-0)]$, providing these limits exist. Therefore at $x = \pi$ the Fourier series associated with $f(x)$ is summable $(\Phi)$ to the sum

$$\frac{1}{2}[g(+0)+g(-0)] = \frac{1}{2}[\chi(\pi+0)+\chi(\pi-0)] = \frac{1}{2}[f(-\pi+0)+f(\pi-0)],$$

and to the same sume at $x = -\pi$.

To prove the statement concerning the uniformity of the summability, it should first be noticed that under the hypotheses of the theorem the integrals in (10) and (11) approach their limits uniformly; this is easy to see on considering the separate quantities used in the evaluation of the limit as $n$ becomes infinite of (8) and (9). Then the following theorem, the proof of which is given in the paper by Hosford,* is necessary to complete the proof: If a series is uniformly convergent in an interval, it is uniformly summable to the same sum in that interval by any definition of summability by a method of mean values with finite reference which is regular. Using these facts, the uniformity of the summability follows at once.

This completes the proof of the theorem.

**3. Definition of summability $(\Psi)$.** Let $\varrho$ be a complex variable, and consider the same contour as that associated with Theorem I. Now let $\theta_n$ satisfy the following conditions:

(14) 1) $n\theta_n \to \infty$, 2) $n\theta_n^2 \to 0$, 3) $ne^{-\nu n\theta_n} \to 0$ for every positive $\nu$, as $n \to \infty$.

Let $\lambda_n$ satisfy the following conditions:

(15) 1) $\lambda_n \to \infty$, 2) $n^{-1} \cdot \lambda_n^2 \to 0$, 3) $ne^{-\nu\lambda_n} \to 0$ for every positive $\nu$, as $n \to \infty$.

As in § 1, we shall let $\varrho = \mu + is$ on the straight line $CF$ and let $\varrho = (n-\frac{1}{2})e^{i\theta}$ on the circular part of the contour.

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* Hosford, loc. cit. For the term "method of mean values with finite reference", see Carmichael, loc. cit., p. 101.

Now let us consider two functions $\psi_n(\varrho)$ and $\bar{\psi}_n(\varrho)$ which shall have the following properties:

1) $\psi_n(\varrho)$ is analytic on and within the contour bounded by $A'C'F'A'$;

1a) $\bar{\psi}_n(\varrho)$ is analytic on and within the contour bounded by $C'D'F'C'$;

2) If $k$ is any positive integer less than $n$, $\bar{\psi}_n(k)+\bar{\psi}_n(-k)=2\alpha_{nk}$ when $k \leqq m$, and $\psi_n(k)+\bar{\psi}_n(-k)=2\alpha_{nk}$ when $k>m$; moreover write $\bar{\psi}_n(0)=\alpha_{n0}$;

(16) 3) $n\int_G^B|\psi_n(\varrho)|\,d\theta = o(1)$, $\int_B^C|\psi_n(\varrho)|\,d\theta<M_1$, $\int_F^G|\psi_n(\varrho)|\,d\theta<M_1$;

3a) $n\int_D^E|\bar{\psi}_n(\varrho)|\,d\theta = o(1)$, $\int_C^D|\bar{\psi}_n(\varrho)|\,d\theta<M_1$, $\int_E^F|\bar{\psi}_n(\varrho)|\,d\theta<M_1$;

4) $\int_K^J|\bar{\psi}_n(\varrho)-\psi_n(\varrho)|\,ds=o(1)$, $e^{-\nu\lambda_n}\int_J^C|\bar{\psi}_n(\varrho)-\psi_n(\varrho)|\,ds=o(1)$,

$e^{-\nu\lambda_n}\int_F^K|\bar{\psi}_n(\varrho)-\psi_n(\varrho)|\,ds=o(1)$;

$M_1$ is again a constant, though not the same one as that used in § 2.

The series $\sum_{k=0}^{\infty}u_k$ is said to be summable $(\Psi)$ to the sum $t$ if $\underset{n=\infty}{L}\,t_n=t$, where $t_n=\sum_{k=0}^{n-1}\alpha_{nk}u_k$. Let the conditions of regularity (4) be imposed upon the $\alpha_{nk}$; this, of course, places added conditions on $\psi_n(\varrho)$ and $\bar{\psi}_n(\varrho)$ since the $\alpha_{nk}$ are expressed in terms of these functions.

**4. Summability $(\Psi)$ of the Fourier series.** We shall now prove the following theorem, thus extending the class of functions for which the associated Fourier series is summable. In Theorem I the class of functions considered was bounded and absolutely integrable; here we remove the condition of boundedness.

THEOREM II. *Let $f(x)$ be a function of a real variable, defined in the interval $-\pi \leqq x \leqq \pi$, and such that $\int_{-\pi}^{\pi}|f(\xi)|\,d\xi$ exists. Then the Fourier series associated with $f(x)$ is summable $(\Psi)$ to the sum $\frac{1}{2}[f(x+0)+f(x-0)]$ at all interior points at which the limits $f(x\pm 0)$ exist. At $x=\pm\pi$, the series is summable $(\Psi)$ to the sum $\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$, providing these limits exist. If $f(x)$ is continuous in a subinterval defined by the inequalities $\alpha \leqq x \leqq \beta$, then in the interval $\alpha+\eta \leqq x \leqq \beta-\eta$, $\eta>0$, the series is uniformly summable $(\Psi)$.*

Starting again from the differential equation and boundary conditions (5), by the procedure in Theorem I we can show that

$$\frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_{-\pi}^{\pi}\varrho\,G(x,\xi,\varrho^2)f(\xi)\,d\xi\,d\varrho$$
$$=\frac{1}{2}a_0\cdot\alpha_{n0}+\sum_{k=1}^{n-1}\alpha_{nk}[a_k\cos kx+b_k\sin kx].$$

The theorem is proved for interior points if

$$(17)\quad \lim_{n\to\infty}\left[\frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_{-\pi}^{\pi}\varrho\,G(x,\xi,\varrho^2)f(\xi)\,d\xi\,d\varrho\right]$$
$$=\frac{1}{2}[f(x+0)+f(x-0)].$$

Let us consider the two integrals

$$(18)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_{-\pi}^{x}\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x-0)]\,d\xi\,d\varrho$$

and

$$(19)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_{x}^{\pi}\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x+0)]\,d\xi\,d\varrho.$$

Since $\int_{-\pi}^{\pi}|f(\xi)|\,d\xi$ exists, the integrals $\int_{-\pi}^{x}|f(\xi)-f(x-0)|\,d\xi$ and $\int_{x}^{\pi}|f(\xi)-f(x+0)|\,d\xi$ also exist; let $L$ be an upper bound of all three integrals. For $\xi<x$,

$$\varrho\,G(x,\xi,\varrho^2)=\frac{\cos\varrho(x-\xi-\pi)}{2\sin\varrho\pi}$$

from (6). Letting $\varrho=(n-\frac{1}{2})e^{i\theta}$, it is easy to show that

$$|\cos\varrho(x-\xi-\pi)|\leqq\cosh\left[\left(n-\frac{1}{2}\right)(x-\xi-\pi)\sin\theta\right]$$

and that

$$|\sin\varrho\pi|=A(n,\theta),$$

the latter being the same function as that used in the proof of Theorem I. Since the bound on $|\cos\varrho(x-\xi-\pi)|$ is independent of the sign of $x-\xi-\pi$, we shall take $x-\xi-\pi>0$. Now (18), by breaking up the path of integration with respect to $\varrho$ as indicated in the figure, may be written in the form

$$\frac{1}{2\pi i}\left\{\left[\int_G^B+\int_B^C+\int_F^G\right]\psi_n(\varrho)+\left[\int_C^D+\int_D^E+\int_E^F\right]\bar{\psi}_n(\varrho)\right.$$
$$\left.+\int_F^C[\bar{\psi}_n(\varrho)-\psi_n(\varrho)]\right\}\int_{-\pi}^{x}\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x-0)]\,d\xi\,d\varrho.$$

We shall use the same notation for the absolute values of the separate pieces as that used in Theorem I.

Then we have

$$I\binom{B\quad x}{G\quad -\pi} \leqq \frac{1}{2\pi}\int_{-\pi}^{x} |f(\xi)-f(x-0)| \int_G^B |\psi_n(\varrho)| \frac{\cosh[(n-\frac{1}{2})(x-\xi-\pi)\sin\theta]}{\Delta(n,\theta)}\left(n-\frac{1}{2}\right)d\theta\, d\xi.$$

The quotient in the integrand is bounded for all $n$ and $\theta$ in the range of integration, so that

$$I\binom{B\quad x}{G\quad -\pi} < \bar{M}\int_{-\pi}^{x} |f(\xi)-f(x-0)| \int_G^B |\psi_n(\varrho)| \left(n-\frac{1}{2}\right) d\theta\, d\xi < \bar{M}\cdot L\cdot o(1) = o(1)$$

by (16), 3).

Now

$$I\binom{C\quad x}{B\quad -\pi} \leqq \frac{1}{2\pi}\int_{-\pi}^{x} |f(\xi)-f(x-0)| \int_B^C |\psi_n(\varrho)| \frac{\cosh[(n-\frac{1}{2})(x-\xi-\pi)\sin\theta}{\Delta(n,\theta)}\left(n-\frac{1}{2}\right) d\theta\, d\xi.$$

The quotient in the integrand is dominated by a constant times $\exp.[-(n-\frac{1}{2})(2\pi+\xi-x)\sin\theta]$. Hence, using (14) and (16), 3) we have

$$\begin{aligned} I\binom{C\quad x}{B\quad -\pi} &< \bar{M}\int_{-\pi}^{x} |f(\xi)-f(x-0)| \int_B^C |\psi_n(\varrho)| \frac{n-\frac{1}{2}}{\exp.[(n-\frac{1}{2})(2\pi+\xi-x)\sin\theta]}\, d\theta\, d\xi \\ &< \bar{M}\cdot M_1 \int_{-\pi}^{x} |f(\xi)-f(x-0)| \frac{n-\frac{1}{2}}{\exp.[(n-\frac{1}{2})(2\pi+\xi-x)\sin\theta_n]}\, d\xi \\ &< L\cdot\bar{M}\cdot M_1 \frac{n-\frac{1}{2}}{\exp.[(n-\frac{1}{2})(\pi-x)\sin\theta_n]} = o(1), \end{aligned}$$

since $x$ is an interior point and $\pi-x=\nu>0$.

The same sort of argument shows that

$$I\binom{G\quad x}{F\quad -\pi} = o(1), \quad I\binom{D\quad x}{C\quad -\pi} = o(1), \quad I\binom{E\quad x}{D\quad -\pi} = o(1),$$

$$\text{and } I\binom{F\quad x}{E\quad -\pi} = o(1).$$

Now let $\varrho = \mu + is$, and let us find what comes out of the integration along $CF$. In this case

$$|\cos\varrho(x-\xi-\pi)| \leqq \cosh s(x-\xi-\pi), \qquad |\sin\varrho\pi| \equiv A(s),$$

the latter being the same function used in the proof of Theorem I.

We have

$$I\binom{J\ \ x}{K\ -\pi} \leqq \frac{1}{2\pi}\int_{-\pi}^{x}|f(\xi)-f(x-0)| \int_K^J |\bar{\psi}_n(\varrho)-\psi_n(\varrho)|\frac{\cosh s(x-\xi-\pi)}{A(s)}\,ds\,d\xi.$$

The quotient in the integrand is bounded for all $s$ and $\xi$ in the range of integration, and by (16), 4), the remainder of the integrand is $o(1)$, so that

$$I\binom{J\ \ x}{K\ -\pi} = o(1).$$

Again

$$I\binom{C\ \ x}{J\ -\pi} \leqq \frac{1}{2\pi}\int_{-\pi}^{x}|f(\xi)-f(x-0)| \int_J^C |\bar{\psi}_n(\varrho)-\psi_n(\varrho)|\frac{\cosh s(x-\xi-\pi)}{A(s)}\,ds\,d\xi.$$

The quotient in the integrand is dominated by a constant times $\exp.[-s(2\pi+\xi-x)]$, so that

$$I\binom{C\ \ x}{J\ -\pi} < \overline{M}\int_{-\pi}^{x}\frac{|f(\xi)-f(x-0)|}{\exp.[\lambda_n(2\pi+\xi-x)]}\int_J^C|\bar{\psi}_n(\varrho)-\psi_n(\varrho)|\,ds\,d\xi$$

$$< L\cdot\overline{M}\cdot\exp.[-\lambda_n(\pi-x)]\int_J^C|\bar{\psi}_n(\varrho)-\psi_n(\varrho)|\,ds = o(1)$$

by reason of the fact that $x$ is an interior point so that $\pi-x=\nu>0$ and the condition (16), 4). A similar argument shows that

$$I\binom{K\ \ x}{F\ -\pi} = o(1).$$

By combining the separate pieces, we show that

$$(20)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_{-\pi}^{x}\varrho\,G(x,\xi,\varrho^2)[f(\xi)-f(x-0)]\,d\xi\,d\varrho = o(1),$$

and in like manner we show that

$$(21)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_x^{\pi}\varrho G(x,\xi,\varrho^2)[f(\xi)-f(x+0)]\,d\xi\,d\varrho = o(1).$$

As in the proof of Theorem I, since the definition of summability $(\Psi)$ is regular, we have

$$(22)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_{-\pi}^{x}\varrho G(x,\xi,\varrho^2)\,d\xi\,d\varrho = \frac{1}{2}+o(1),$$

$$(23)\quad \frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_x^{\pi}\varrho G(x,\xi,\varrho^2)\,d\xi\,d\varrho = \frac{1}{2}+o(1).$$

Combining (20), (21), (22), and (23), we can write the result explicitly as a limit in the form

$$\lim_{n=\infty}\left[\frac{1}{2\pi i}\left\{\int_{\Gamma_n}\psi_n(\varrho)+\int_{\bar{\Gamma}_n}\bar{\psi}_n(\varrho)\right\}\int_{-\pi}^{\pi}\varrho G(x,\xi,\varrho^2)f(\xi)\,d\xi\,d\varrho\right]$$
$$=\frac{1}{2}[f(x+0)+f(x-0)];$$

this is precisely (17) and the theorem is proved for interior points.

The proof for the boundary points, $x=\pm\pi$, and for the statement regarding the uniformity of the summability is a repetition of the argument in Theorem I. This completes the proof of the theorem.

## II. Multiple series.

5. **Definition of summability** $(\Phi_1, \Phi_2, \cdots, \Phi_p)$. Consider $p$ independent complex variables $\varrho_j$, $j=1, 2, \cdots, p$, and associate with each a contour like that used in Theorem I, the circles being of radius $n_j-\frac{1}{2}$ and the lines having the equations $R(\varrho_j)=\mu_j$. $\theta_n$ and $\lambda_n$ of (1) and (2) become $\theta_{n_j}$ and $\lambda_{n_j}$ respectively, and the contours become $\Gamma_{n_j}^{(j)}$ and $\bar{\Gamma}_{n_j}^{(j)}$. For $j$ fixed, the functions $\varphi_{n_j}^{(j)}(\varrho_j)$ and $\bar{\varphi}_{n_j}^{(j)}(\varrho_j)$ shall satisfy the same conditions as those for $\varphi_n(\varrho)$ and $\bar{\varphi}_n(\varrho)$ respectively in (3); the condition 2) becomes explicitly

2) If $k_j$ is any positive integer less than $n_j$, $\bar{\varphi}_{n_j}^{(j)}(k_j)+\bar{\varphi}_{n_j}^{(j)}(-k_j)=2a_{n_jk_j}^{(j)}$ when $k_j\leqq m_j$, and $\varphi_{n_j}^{(j)}(k_j)+\bar{\varphi}_{n_j}^{(j)}(-k_j)=2a_{n_jk_j}^{(j)}$ when $k_j>m_j$; moreover write $\bar{\varphi}_{n_j}^{(j)}(0)=a_{n_j0}^{(j)}$.

The $p$-fold series

$$(24)\quad \sum_{k_1,k_2,\cdots,k_p=0}^{\infty} u_{k_1,k_2,\cdots,k_p}$$

is said to be summable $(\Phi_1, \Phi_2, \cdots, \Phi_p)$ to the sum $T$ if

$$\underset{n_1, n_2, \cdots, n_p \doteq \infty}{L} T_{n_1, n_2, \cdots, n_p} = T,$$

where

$$T_{n_1, n_2, \cdots, n_p} = \sum_{k_1, \cdots, k_p = 0}^{n_1 - 1, \cdots, n_p - 1} a^{(1)}_{n_1 k_1} a^{(2)}_{n_2 k_2} \cdots a^{(p)}_{n_p k_p} u_{k_1, k_2, \cdots, k_p}.$$

6. **Summability $(\Phi_1, \cdots, \Phi_p)$ of the $p$-fold Fourier series.** We shall now prove the following theorem:

THEOREM III. *Let $f(x_1, \cdots, x_p)$ be a function of $p$ real variables, defined in the domain $R$, $-\pi \leqq x_j \leqq \pi$, bounded with respect to the several variables together, and such that the $p$-fold integral $\int_R |f(\xi_1, \cdots, \xi_p)|\, d\xi_1, \cdots, d\xi_p$ exists. Then the Fourier series associated with $f(x_1, \cdots, x_p)$ is summable $(\Phi_1, \cdots, \Phi_p)$ to the sum $(\frac{1}{2})^p \sum f(x_1 \pm 0, \cdots, x_p \pm 0)$ at every point where the limits*

$$\underset{\sigma \doteq 0}{L} f(x_1 \pm 0, \cdots, x_{j-1} \pm 0, x_j \pm \sigma, \xi_{j+1}, \cdots, \xi_p), \qquad j = 1, 2, \cdots, p,$$

*exist. The summation means the sum of the $2^p$ terms obtained by taking all possible combinations of the positive and negative signs. For $x_j = \pm\pi$, instead of $x_j - 0$ and $x_j + 0$ read respectively $\pi - 0$ and $-\pi + 0$. If $R'$, $\alpha_j \leqq x_j \leqq \beta_j$, is any subdomain of $R$ in which $f(x_1, \cdots, x_p)$ is a continuous function with respect to each variable, then in the domain $\alpha_j + \eta \leqq x_j \leqq \beta_j - \eta$, $\eta > 0$, the Fourier series associated with $f(x_1, \cdots, x_p)$ is uniformly summable $(\Phi_1, \cdots, \Phi_p)$.*

For simplicity we shall give the proof of this theorem for the case $p = 2$; the method is seen to be applicable to the general case.

We start with the system of differential equations and boundary conditions, analogous to (5)

$$(25) \quad y_j'' + \varrho_j^2 y_j = 0, \quad y_j(\pi) = y_j(-\pi), \quad y_j'(\pi) = y_j'(-\pi), \quad j = 1, 2.$$

We have a set of Green's functions

$$(26) \qquad \varrho_j\, G(x_j, \xi_j, \varrho_j^2) = \frac{\cos \varrho_j [x_j - \xi_j + \pi \operatorname{sgn}(\xi_j - x_j)]}{2 \sin \varrho_j \pi}, \quad j = 1, 2$$

The residue of $\varphi^{(j)}_{n_j}(\varrho_j)\, \varrho_j\, G(x_j, \xi_j, \varrho_j^2)$ at $\varrho_j = k_j$ when $k_j > m_j$ is $\varphi^{(j)}_{n_j}(k_j) \frac{1}{2\pi} \cos k_j (x_j - \xi_j)$; of $\bar{\varphi}^{(j)}_{n_j}(\varrho_j) \varrho_j G(x_j, \xi_j, \varrho_j^2)$ at $\varrho_j = k_j$ when $k_j \leqq m$ is

$\overline{\varphi}_{n_j}^{(j)}(k_j)\frac{1}{2\pi}\cos k_j(x_j-\xi_j)$; and of $\overline{\varphi}_{n_j}^{(j)}(\varrho_j)\varrho_j G(x_j,\xi_j,\varrho_j^2)$ at $\varrho_j=-k_j$ when $k_j<n_j$ is $\overline{\varphi}_{n_j}^{(j)}(-k_j)\frac{1}{2\pi}\cos k_j(x_j-\xi_j)$. Moreover, the residue of $\overline{\varphi}_{n_j}^{(j)}(\varrho_j)\varrho_j G(x_j,\xi_j,\varrho_j^2)$ at $\varrho_j=0$ is $\overline{\varphi}_{n_j}^{(j)}(0)\frac{1}{2\pi}$. Then we may write

$$\frac{1}{2\pi i}\left\{\int_{\Gamma_{n_1}^{(1)}}\varphi_{n_1}^{(1)}(\varrho_1)+\int_{\overline{\Gamma}_{n_1}^{(1)}}\overline{\varphi}_{n_1}^{(1)}(\varrho_1)\right\}\varrho_1 G(x_1,\xi_1,\varrho_1^2)\,d\varrho_1$$
$$=\frac{1}{2\pi}a_{n_1 0}^{(1)}+\sum_{k_1=1}^{n_1-1}a_{n_1k_1}^{(1)}\frac{1}{\pi}\cos k_1(x_1-\xi_1),$$

$$\frac{1}{2\pi i}\left\{\int_{\Gamma_{n_2}^{(2)}}\varphi_{n_2}^{(2)}(\varrho_2)+\int_{\overline{\Gamma}_{n_2}^{(2)}}\overline{\varphi}_{n_2}^{(2)}(\varrho_2)\right\}\varrho_2 G(x_2,\xi_2,\varrho_2^2)\,d\varrho_2$$
$$=\frac{1}{2\pi}a_{n_2 0}^{(2)}+\sum_{k_2=1}^{n_2-1}a_{n_2k_2}^{(2)}\frac{1}{\pi}\cos k_2(x_2-\xi_2).$$

On multiplying these two integrals together, we get

$$\left(\frac{1}{2\pi i}\right)^2\left\{\int_{\Gamma_{n_1}^{(1)}}\varphi_{n_1}^{(1)}(\varrho_1)+\int_{\overline{\Gamma}_{n_1}^{(1)}}\overline{\varphi}_{n_1}^{(1)}(\varrho_1)\right\}$$
$$\left\{\int_{\Gamma_{n_2}^{(2)}}\varphi_{n_2}^{(2)}(\varrho_2)+\int_{\overline{\Gamma}_{n_2}^{(2)}}\overline{\varphi}_{n_2}^{(2)}(\varrho_2)\right\}\varrho_1 G(x_1,\xi_1,\varrho_1^2)\varrho_2 G(x_2,\xi_2,\varrho_2^2)\,d\varrho_1\,d\varrho_2$$

$$=\left[\frac{1}{2\pi}a_{n_1 0}^{(1)}+\sum_{k_1=1}^{n_1-1}a_{n_1k_1}^{(1)}\frac{1}{\pi}\cos k_1(x_1-\xi_1)\right]$$
$$\times\left[\frac{1}{2\pi}a_{n_2 0}^{(2)}+\sum_{k_2=1}^{n_2-1}a_{n_2k_2}^{(2)}\frac{1}{\pi}\cos k_2(x_2-\xi_2)\right]$$
$$=\sum_{k_1,k_2=0}^{n_1-1,n_2-1}\varepsilon_{k_1k_2}a_{n_1k_1}^{(1)}a_{n_2k_2}^{(2)}\frac{1}{\pi^2}\cos k_1(x_1-\xi_1)\cos k_2(x_2-\xi_2), \tag{27}$$

where

$$\varepsilon_{k_1k_2}=\begin{cases}\frac{1}{2}, & k_1=0,\ k_2>0,\ \text{or } k_1>0,\ k_2=0;\\ \frac{1}{4}, & k_1=k_2=0;\\ 1, & k_1>0,\ k_2>0.\end{cases}$$

Now if we multiply both members of (27) by $f(\xi_1,\xi_2)$ and integrate from $-\pi$ to $\pi$ with respect to each variable, we get

$$\left(\frac{1}{2\pi i}\right)^2\left\{\int_{\Gamma_{n_1}^{(1)}}\varphi_{n_1}^{(1)}(\varrho_1)+\int_{\overline{\Gamma}_{n_1}^{(1)}}\overline{\varphi}_{n_1}^{(1)}(\varrho_1)\right\}\left\{\int_{\Gamma_{n_2}^{(2)}}\varphi_{n_2}^{(2)}(\varrho_2)+\int_{\overline{\Gamma}_{n_2}^{(2)}}\overline{\varphi}_{n_2}^{(2)}(\varrho_2)\right\}$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varrho_1 G(x_1,\xi_1,\varrho_1^2)\varrho_2 G(x_2,\xi_2,\varrho_2^2) f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\,d\varrho_1\,d\varrho_2 \tag{28}$$

$$=\sum_{k_1,k_2=0}^{n_1-1,\,n_2-1}\epsilon_{k_1k_2}\,a_{n_1k_1}^{(1)}\,a_{n_2k_2}^{(2)}\left[A_{k_1k_2}\cos k_1x_1\cos k_2x_2+B_{k_1k_2}\cos k_1x_1\sin k_2x_2\right.$$

$$\left.+C_{k_1k_2}\sin k_1x_1\cos k_2x_2+D_{k_1k_2}\sin k_1x_1\sin k_2x_2\right],$$

where $A_{k_1k_2}$, $B_{k_1k_2}$, $C_{k_1k_2}$, and $D_{k_1k_2}$ are the Fourier constants of the double series.

The integral (28) may be evaluated by integrating first with respect to $\xi_1$ and $\varrho_1$, and then integrating the result with respect to $\xi_2$ and $\varrho_2$; the integral (7) is of the form of the separate integrals in this procedure. Then if we take the limits of this result as $n_1$ and $n_2$ become infinite, we get

$$\frac{1}{4}[f(x_1+0,x_2+0)+f(x_1+0,x_2-0)+f(x_1-0,x_2+0)+f(x_1-0,x_2-0)],$$

and the theorem is proved for interior points. The reader will have no difficulty in proving the result stated for boundary points or for the uniform convergence.

7. **Summability $(\Psi_1,\Psi_2,\cdots,\Psi_p)$ of the $p$-fold Fourier series.** In a manner similar to the procedure in § 6, using § 3, we can set up the definition of summability $(\Psi_1,\Psi_2,\cdots,\Psi_p)$ for a $p$-fold series and prove the following theorem. As there is nothing essentially new in the proof, we shall omit it for the sake of brevity.

THEOREM IV. *Let $f(x_1,\cdots,x_p)$ be a function of $p$ real variables, defined in the domain $R$, $-\pi\leqq x_j\leqq\pi$, and such that the $p$-fold integral $\int_R|f(\xi_1,\cdots,\xi_p)|\,d\xi_1\cdots d\xi_p$ exists. Then the Fourier series associated with $f(x_1,\cdots,x_p)$ is summable $(\Psi_1,\cdots,\Psi_p)$ to the sum $(\frac{1}{2})^p\sum f(x_1\pm0,\cdots,x_p\pm0)$ at every point where the limits*

$$\lim_{\sigma=\pm0} f(x_1\pm0,\cdots,x_{j-1}\pm0,\ x_j\pm\sigma,\ \xi_{j+1},\cdots,\xi_p),\qquad j=1,2,\cdots,p,$$

*exist. The summation means the sum of the $2^p$ terms obtained by taking all possible combinations of the positive and negative signs. For $x_j=\pm\pi$, instead of $x_j-0$ and $x_j+0$ read respectively $\pi-0$ and $-\pi+0$. If $R'$, $\alpha_j\leqq x_j\leqq\beta_j$, is any subdomain of $R$ in which $f(x_1,\cdots,x_p)$ is a continuous function with respect to each variable, then in the domain*

*$\alpha_j+\eta \leqq x_j \leqq \beta_j-\eta$, $\eta>0$, the Fourier series associated with $f(x_1, \cdots, x_p)$ is uniformly summable $(\Psi_1, \cdots, \Psi_p)$.*

## III. Special cases and applications, single series.

8. **Summability** $(C1)$. Taking $\varphi_n(\varrho)=1-\varrho/(n-\frac{1}{2})$, $\bar{\varphi}_n(\varrho)=1+\varrho/(n-\frac{1}{2})$, and $\mu=\frac{1}{2}$, it is easy to show that the several conditions of (3) are satisfied, so that Theorem I gives the summability of the Fourier series for the factor $1-k/(n-\frac{1}{2})$, that is for a definition

$$t = \underset{n=\infty}{L} \sum_{k=0}^{n-1} \left(1-\frac{k}{n-\frac{1}{2}}\right)(a_k \cos kx + b_k \sin kx);$$

we shall call this definition $(\bar{C}1)$. We need the following lemma:

LEMMA I. *A definition of summability*

$$t = \underset{n=\infty}{L} \sum_{k=0}^{n-1} \left(1-\frac{k}{n+a}\right) u_k,$$

*where $a$ is not a negative integer, is equivalent* to summability $(C1)$, that is,*

$$t = \underset{n=\infty}{L} \sum_{k=0}^{n-1} \left(1-\frac{k}{n}\right) u_k.$$

The proof of this lemma by the method used by Plancherel† in establishing the equivalence of his definition of summability and summability $(VP)$ is not difficult.

Hence from Theorem I and Lemma I we have proved the summability $(C1)$ of every bounded, absolutely integrable function expanded into a Fourier series. But it is also true that $\psi_n(\varrho)=1-\varrho/(n-\frac{1}{2})$, $\bar{\psi}_n(\varrho)=1+\varrho/(n-\frac{1}{2})$, and $\mu=\frac{1}{2}$ satisfy the conditions of Theorem II. Using Lemma I, we then have a proof of the well known Fejér‡ theorem.

THEOREM V. *Let $f(x)$ be a function of a real variable, defined in the interval $-\pi \leqq x \leqq \pi$, and such that $\int_{-\pi}^{\pi} |f(\xi)|\,d\xi$ exists. Then the Fourier series associated with $f(x)$ is summable $(C1)$ to the sum $\frac{1}{2}[f(x+0)+f(x-0)]$ at all interior points at which the limits $f(x\pm 0)$ exist. At $x=\pm\pi$, the series is summable $(C1)$ to the sum $\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$, providing these limits exist. If $f(x)$ is continuous in a subinterval defined by the inequalities*

* That is, every series summable by either definition is also summable by the other and the sums are the same. See Carmichael, loc. cit., p. 113.

† Plancherel, Rendiconti Palermo, 33 (1912), 41–66. See pp. 43–45.

‡ Fejér, Math. Ann., 58 (1904), 51–69.

*$\alpha \leqq x \leqq \beta$, then in the interval $\alpha+\eta \leqq x \leqq \beta-\eta$, $\eta>0$, the Fourier series associated with $f(x)$ is uniformly summable* $(C1)$.

9. **Summability** $(Cr)$, $r$ **integral,** $r>1$. By the well known theorem, if a series is summable $(Cr)$ it is summable $(Cr')$, $r'>r$, to the same sum, since the Fourier series is summable $(C1)$ it is also summable $(Cr)$ where $r$ is an integer greater than unity. However we shall arrive at this result directly from Theorems I and II and subsidiary lemmas. We shall also get two forms equivalent to $(Cr)$.

Let $\varphi_n(\varrho) = \psi_n(\varrho) = \prod_{l=1}^{r}\left(1-\frac{\varrho-l+1}{n-\frac{1}{2}}\right)$, $\bar{\varphi}_n(\varrho) = \bar{\psi}_n(\varrho) = \prod_{l=1}^{r}\left(1+\frac{\varrho+l-1}{n-\frac{1}{2}}\right)$, and $\mu=\frac{1}{2}$. Then the conditions, both of Theorem I and of Theorem II, are satisfied and the Fourier series is summable by a definition

$$t = \underset{n\doteq\infty}{L}\sum_{k=0}^{n-1}\prod_{l=1}^{r}\left(1-\frac{k-l+1}{n-\frac{1}{2}}\right)(a_k\cos kx+b_k\sin kx);$$

we shall call this summability $(\bar{C}r)$.

LEMMA II. *A definition of summability*

$$t = \underset{n\doteq\infty}{L}\sum_{k=0}^{n-1}\prod_{l=1}^{r}\left(1-\frac{k-l+1}{n+a}\right)u_k,$$

*where $a$ is not a negative integer, is equivalent to summability $(Cr)$, that is,*

$$t = \underset{n\doteq\infty}{L}\sum_{k=0}^{n-1}\prod_{l=1}^{r}\left(1-\frac{k-l+1}{n}\right)u_k.$$

This lemma may be proved like Lemma I. Hence, taking $a=-\frac{1}{2}$, a special case of summability $(\Phi)$ and summability $(\Psi)$ is equivalent to summability $(Cr)$.

LEMMA III. *Summability $(Cr)$ is equivalent to the definition*

$$t = \underset{n\doteq\infty}{L}\sum_{k=0}^{n-1}\left[1-\left(\frac{k}{n}\right)^q\right]^r u_k,$$

*where $q$ is an integer not less than unity.*

This lemma was proved by W. H. Young.* This form of definition can also be obtained from our theory by the aid of Lemma IV which can be

* W. H. Young, Leipziger Berichte, 63 (1911), 369-387.

proved like Lemma I. Let $\varphi_n(\varrho) = \psi_n(\varrho) = \left[1-\left(\frac{\varrho}{n-\frac{1}{2}}\right)^q\right]^r$, $\bar{\varphi}_n(\varrho) = \psi_n(\varrho) = \left[1+\left(\frac{\varrho}{n-\frac{1}{2}}\right)^q\right]^r$, and $\mu = \frac{1}{2}$. The conditions of Theorems I and II are satisfied and the Fourier series is summable by a definition

$$t = \underset{n=\infty}{L} \sum_{k=0}^{n-1}\left[1-\left(\frac{k}{n-\frac{1}{2}}\right)^q\right]^r (a_k \cos kx + b_k \sin kx).$$

LEMMA IV. *Summability by a definition*

$$t = \underset{n=\infty}{L} \sum_{k=0}^{n-1}\left[1-\left(\frac{k}{n}\right)^q\right]^r u_k$$

*is equivalent to summability by a definition*

$$t = \underset{n=\infty}{L} \sum_{k=0}^{n-1}\left[1-\left(\frac{k}{n+a}\right)^q\right]^r u_k,$$

*where $a$ is not a negative integer.*

It will be noticed that this definition for $q = 1$ yields a particular simple form of definition of summability $(Cr)$; it can of course be realized from our theory. As a matter of fact the author was led to the equivalence of this definition (that is with $q = 1$) and $(Cr)$ before he knew of the lemma due to Young. It should also be noticed that Lemma III gives equivalent forms of definition for $(C1)$ by taking $r = 1$. These also can be realized from our theory with the aid of Lemma III.

10. **Summability** $(VP)$. Since the Fourier series is summable $(C1)$ it is also known to be summable $(VP)$.* It can be shown that summability $(VP)$ is also equivalent to a special case of summability $(\Phi)$ and of summability $(\Psi)$. The definition of summability $(VP)$ may be thrown into the form

$$t = \underset{n=\infty}{L}\left[u_0 + \sum_{k=1}^{n-1}\prod_{l=1}^{k}\left(1-\frac{k}{n+l-1}\right)u_k\right].$$

If we take $\varphi_n(\varrho) = \psi_n(\varrho) = \prod_{l=1}^{k}\left(1-\frac{\varrho}{n+l-\frac{3}{2}}\right)$, $\bar{\varphi}_n(\varrho) = \bar{\psi}_n(\varrho) = \prod_{l=1}^{k}\left(1+\frac{\varrho}{n+l-\frac{3}{2}}\right)$, and $\mu = \frac{1}{2}$, it is not difficult to show that the

* Proved independently by Gronwall and C. N. Moore. Gronwall, Comptes Rendus, 158 (1914), 1664–1665. C. N. Moore, ibid., 1774–1775.

conditions of Theorems I and II are satisfied, so that the Fourier series is summable by a definition which we shall call $(\overline{VP})$

$$t = \underset{n=\infty}{L} \left[\frac{1}{2}a_0 + \sum_{k=1}^{n-1}\prod_{l=1}^{k}\left(1-\frac{k}{n+l-\frac{3}{2}}\right)\right](a_k \cos kx + b_k \sin kx).$$

The following easily proved lemma then completes the proof of the summability $(VP)$ of the Fourier series:

LEMMA V. *Summability* $(VP)$ *is equivalent to a definition*

$$t = \underset{n=\infty}{L} \left[u_0 + \sum_{k=1}^{n-1}\prod_{l=1}^{k}\left(1-\frac{k}{n+l+a}\right)\right]u_k,$$

*where* $a$ *is not a negative integer.*

11. **Limitations of the method.** It will be seen that all the definitions of summability considered as special cases of the $(\Phi)$ and $(\Psi)$ definitions of summability are of the type Carmichael* has called "a method of mean values with finite reference". Definitions such as those of Borel and Le Roy, which are methods of "mean values with infinite reference", do not come as special cases under the theory developed here, since the contours $\Gamma_n$ and $\overline{\Gamma}_n$ can enclose only a finite number of integral values of the functions $\varphi_n(\varrho)$, $\overline{\varphi}_n(\varrho)$, $\psi_n(\varrho)$ and $\overline{\psi}_n(\varrho)$, in terms of which the summability factors are defined.

The limit is taken with respect to a variable $n$ which moves over a discrete set of points, namely the positive integers. The Green's function has poles of the first order at the integers while the $\varphi_n(\varrho)$, $\overline{\varphi}_n(\varrho)$, $\psi_n(\varrho)$, and $\overline{\psi}_n(\varrho)$ are analytic in the proper segments of the circle of radius $n-\frac{1}{2}+\gamma$, $\gamma>0$. The limit is taken for a sequence of contours of radius $n-\frac{1}{2}$ as $n$ becomes infinite; if the limit be taken with respect to a continuous variable $\omega$, the method fails. Hence the usual definition of Riesz is not included as a special case of the theory developed.

The restriction is also made in discussing summability $(Cr)$ that $r$ is an integer. If $r$ be taken to be fractional, the condition of analyticity on $\overline{\varphi}_n(\varrho)$ and $\overline{\psi}_n(\varrho)$ does not hold. Hence summability $(Cr)$, $r$ fractional, is not a special case of the $(\Phi)$ and $(\Psi)$ summability.

## IV. Special cases, multiple series.

12. **Monomorphic and polymorphic types of summability.** The summability $(\Psi_1, \cdots, \Psi_p)$ of the $p$-fold series (24) has been defined implicitly as

* Carmichael, loc. cit., p. 101, p. 104.

$$T = \underset{n_1, n_2, \cdots, n_p = \infty}{L} \sum_{k_1, \cdots, k_p = 0}^{n_1 - 1, \cdots, n_p - 1} \alpha^{(1)}_{n_1 k_1} \alpha^{(2)}_{n_2 k_2} \cdots \alpha^{(p)}_{n_p k_p} u_{k_1 k_2 \cdots k_p},$$

where the factor $\alpha^{(j)}_{n_j k_j}$ corresponds to the factor $\alpha_{nk}$ in the definition of summability $(\Psi)$ of the single series. The summability $(\Phi_1, \cdots, \Phi_p)$ of the $p$-fold series is formally the same as the above with the $\alpha^{(j)}_{n_j k_j}$ replaced by $a^{(j)}_{n_j k_j}$.

Therefore, by Theorems III and IV and the special cases of $(\Phi)$ and $(\Psi)$ summability discussed in section III, we can assert immediately the summability of the $p$-fold Fourier series by a large number of types of definition. Thus for a triple Fourier series we have proved the summability $(\overline{C}1, \overline{C}1, \overline{C}1)$, $(\overline{C}1, \overline{C}r, \overline{VP})$, $(\overline{C}1, \overline{C}r, \overline{C}1)$, etc.

It will be seen at once that all these types of definition may be classed under two general headings, namely, those in which the type of the separate factors is uniform, and those in which various types of factors are used. We shall call definitions of summability of the first type "monomorphic", and those of the second type "polymorphic".

Since the summability $(C1, C1)$ of double Fourier series and $(C1, C1, C1)$ of triple Fourier series have been considered by C. N. Moore* and Miss Eversull respectively, we shall state here the theorem for the $p$-fold Fourier series which our theory yields as a special case of summability $(\Psi_1, \cdots, \Psi_p)$.

Explicitly, summability $(\overline{C}1, \overline{C}1, \cdots, \overline{C}1)$ of the $p$-fold series (24) is defined by

$$T = \underset{n_1, n_2, \cdots, n_p = \infty}{L} \sum_{k_1, k_2, \cdots, k_p = 0}^{n_1 - 1, n_2 - 1, \cdots, n_p - 1} \prod_{j=1}^{p} \left(1 - \frac{k_j}{n_j - \frac{1}{2}}\right) u_{k_1 k_2 \cdots k_p}.$$

Then we have

THEOREM VI. *If in Theorem IV $(\Psi_1, \cdots, \Psi_p)$ be replaced by $(\overline{C}1, \cdots, \overline{C}1)$, the resulting theorem is valid.*

* C. N. Moore, Trans. Amer. Math. Soc., 14 (1913), 73–104; Math. Ann., 74 (1913), 555–572. Eversull, Annals of Math., (2), 24 (1922–23), 140–166; Trans. Amer. Math. Soc., 26 (1924), 313–334.

# THE ARITHMETIC OF A GENERAL ALGEBRA.†

By Olive C. Hazlett.

1. **Abstract.** This paper concerns itself with a general associative algebra‡ over any algebraic field $F$ and in particular with its arithmetic. In many ways it is merely an enlargement and extension of a paper read by the writer before the Toronto Congress.§ It uses the definition of an arithmetic of such an algebra which was given by the writer in the paper just mentioned and which replaces the property $R$ of Dickson's definition‖ by $R^*$. Now, if $A$ is an algebra over an algebraic field $F$, it can be regarded as an algebra $A^*$ over the field $\mathfrak{R}$ of rational numbers. In fact, a set of numbers of $A$ over $F$ which has properties $R^*$, $C$, $U$, $M$ coincides with a set of numbers of $A^*$ over $\mathfrak{R}$, which has properties $R$, $C$, $U$, $M$.

It is well-known¶ that an algebra $A$ over $F$ is expressible as the sum of two algebras, $S$ and $N$, where $S$ is a semi-simple subalgebra and $N$ is the maximal invariant nilpotent subalgebra of $A$, and it is here proved that the arithmetic of $A$ is associated with the arithmetic of $S$. This theorem was first proved by Dickson** in his beautiful book on *Algebras and their arithmetics*, but it was proved only for algebras over the field $\mathfrak{R}$ of rational numbers and, moreover, used the normalized units of Cartan and the notion of character, whereas the present proof does not use these units or the notion of character and applies to an algebra over any non-modular field.

2. **Summary of the literature.** For the sake of clearness, we shall

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† Presented to the American Mathematical Society (Kansas City), December 29, 1925.

‡ One of the most fundamental and beautiful memoirs on the subject is: J. H. M. Wedderburn, *On hypercomplex numbers*, Proc. Lond. Math. Soc., ser. 2, vol. 6 (1907), pp. 77–118.

For general treatises on this fascinating subject, see: L. E. Dickson, *Linear algebras*, Cambridge Tract No. 16 (1914), *Algebras and their arithmetics*, University of Chicago Press (1923), (Referred to, in this paper, as *A and A*); Gaetano Scorza, *Corpi numerici e algebre*, Messina (1921).

Important names associated with this line of work are Hamilton, Cayley, Frobenius, B. Peirce, Study, Molien, Scheffers, Wedderburn, Dickson, and many others.

§ Proceedings of the International Mathematical Congress at Toronto (1924).

‖ *A and A*, p. 141; ibid., *A new simple theory of hypercomplex integers*, Journ. de Math., ser. 9, vol. 2 (1923), pp. 281–326 (Referred to as *J*).

¶ Wedderburn, loc. cit., p. 109; *A and A*, p. 125.

** *A and A*, pp. 176–187.

remind the reader of the central definitions and theorems of the theory of algebraic numbers† that he may the more readily compare them with the corresponding definitions and theorems of the arithmetic of an algebra derived in this paper.

If $i$ is a root of an algebraic equation of degree $n$ with rational coefficients which is irreducible in the field $\mathfrak{R}$, then the totality $F$ of numbers of the form $a_0+a_1i+a_2i^2+\cdots+a_{n-1}i^{n-1}$ where the $a$'s are allowed to range freely over the field $\mathfrak{R}$ is closed under the four rational operations (excluding division by zero) and forms a special kind of field called an algebraic field. If $j$ is any number of the field $F$ which satisfies an irreducible equation which is such that the coefficients are rational integers and the coefficient of the highest power of the variable is unity, then $j$ is called an algebraic integer. The totality of algebraic integers of $F$ is closed under addition, subtraction and multiplication. Moreover, every rational number which is an algebraic integer is a rational integer.

If the field $\mathfrak{R}$ of rational numbers in the above definition is replaced by any algebraic field $K$, then the totality of numbers so obtained is called a relative field. Note that a relative field of order $n$ over an algebraic field $F$ of order $m$ can be regarded as an algebraic field of order $mn$. Similarly, we have relative integers.

Now it is well-known that the integers of an algebraic field $F$ (over $\mathfrak{R}$) have a (finite) basis—that is, there exists a finite set of integers of $F$—$j_1, j_2, \cdots, j_n$—such that every algebraic integer in $F$ can be expressed in one and only one way as a number of the form $\sum b_s j_s$, where the $b$'s are rational integers; and conversely, any such number is an algebraic integer of $F$.

Closely related to the theory of algebraic integers and ideals is a well-known theorem‡ due to Gauss: If a polynomial in one variable, $f(x)$, is such that its coefficients are rational integers and the coefficient of the highest power of the variable is unity, and if $f(x)$ is expressible as the product of two polynomials in $x$ where the coefficients of the highest power of the variable is unity in each case, then all coefficients of these factors are rational integers. This, which is usually called Gauss' Lemma, is

† Hilbert, Jahresbericht Deut. Math. Ver., vol. 4 (1894–5), pp. 175–546; French translation (with notes and errata by Humbert and Got), in Annales Fac. des Sci. de Toulouse, ser. 3, vol. 1 (1909), pp. 257–328; 2 (1910), pp. 226–456; 3 (1911), pp. 1–62; translation published in book form, Hermann et Fils (1913). Bachmann, *Zahlentheorie*, vol. 5; Sommer, *Vorlesungen über Zahlentheorie* (1907); Dirichlet, *Vorlesungen über Zahlentheorie* (1894), pp. 434–657; König, *Algebraische Größen* (1903), pp. 1–198, 459–552; Bulletin of National Research Council, Report of Committee on Algebraic Numbers (1923); also *A and A*, pp. 128–140, 200–220.

‡ *Disquisitiones Arithmeticae*, art. 42; König, l. c., pp. 91–93.

equally true if the words "rational integers" are replaced by the words "algebraic integers".

From the above definition, it is evident that the numbers of an algebraic field (or relative field) obey the commutative and associative laws of multiplication. On the other hand, a linear algebra† is any set of numbers of the form $\sum x_s e_s$ where the $x_s$ range independently over any field $F$ and where the $e_s$ are $n$ numbers which are linearly independent with respect to $F$. Multiplication is not assumed to obey either the commutative or associative laws of multiplication and is defined by the equations $e_r e_s = \sum \gamma_{rst} e_t$, where the $\gamma$'s are the numbers of the field $F$. Special cases of linear algebras are fields over $\mathfrak{R}$ and fields over $F$. Another important example is a simple matric algebra of $n^2$ units $e_{i\delta}$, where $n$ is any integer, and where multiplication is defined by the equations $e_{rs} e_{st} = e_{rt}$, $e_{rs} e_{tu} = 0 \, (s \neq t)$.

A modulus of an algebra $A$ is a number which behaves under multiplication with the numbers of $A$ like ordinary unity. An algebra may not have a modulus, but, if it do, then it has only one. Since a modulus, if it exist, will necessarily coincide with the "unity" of the field, $F$, of definition, it will be denoted by 1 in this paper. When $A$ has a modulus, the general number $x = \sum x_s e_s$ of $A$ is a root of the first characteristic equation, $|\sum_i \gamma_{ijk} x_i - \delta_{jk} \omega| = 0$ and also of the second characteristic equation, $|\sum_i \gamma_{jik} x_i - \delta_{jk} \omega| = 0$ where $\delta_{jk}$ is the Kronecker symbol. The equation of lowest degree satisfied by the general number is called the rank equation, and the equation of lowest degree satisfied in $F$ by a particular number, $x$, of the algebra is called the reduced equation of $x$. The irreducible factors of the rank function coincide with the irreducible factors of the characteristic function (both first and second).

Wedderburn‡ proved that an algebra $A$ is the sum of two subalgebras, $S$ and $N$, where $S$ is a semi-simple algebra and where $N$ is a maximal, invariant nilpotent subalgebra. He also proved that a semi-simple subalgebra is the direct sum of simple algebras and that a simple algebra is the direct product of a division algebra and a simple matric algebra.

Lipschitz,§ Hurwitz‖ and du Pasquier¶ gave definitions of a set of integers or arithmetic of an algebra which were more or less satisfactory

† See second footnote of section 1.

‡ L. c., p. 109; *A and A*, p. 125.

§ *Untersuchungen über die Summen von Quadraten* (Bonn, 1886), French translation in Journ. de math., ser. 4, vol. 2 (1886), pp. 393–439.

‖ Gött. Nachr. (1896), pp. 311–346; *Vorlesungen über die Zahlentheorie der Quaternionen*, Berlin (1919).

¶ *Vierteljahrsschrift Naturf. Gesell. Zürich*, vol. 54 (1909), pp. 116–148; *L'enseignement math.*, vol. 17 (1915), pp. 340–343; vol. 18 (1916), pp. 201–260.

for quaternions but which, as Dickson† pointed out, are wholely unsatisfactory for the general algebra. According to Dickson, an arithmetic of an algebra $A$ over $\mathfrak{R}$ is any set of numbers, $x$, of $A$ which possess the following properties:—

*R.* The rank equation of $x$ has all its coefficients rational integers and the coefficients of the highest power of the variable is unity.

*C.* The set is closed under addition, subtraction and multiplication.

*U.* The set contains the modulus.

*M.* The set is a maximal set of all those sets possessing properties *R*, *C* and *U*—that is, it is not contained in any other set possessing properties *R*, *C*, *U*.

As in the theory of algebraic numbers, a unit is an integer $u$ such that there is an integer $v$ satisfying the equation $uv = 1$.

If $Z$ is an integral element and if $U$ is a unit, then $ZU$ and $UZ$ are called right and left associates of $Z$, respectively. If in an algebra $A$ the integral elements $Z$ whose determinant is not zero may be associated with the various integral elements of a subalgebra $S$, we shall say that the latter elements form an arithmetic associated with the arithmetic of $A$. As will be seen from the proof of section 4, since the determination of a unit $U$ such that $ZU = x$ is equivalent to the determination of a unit $V$ such that $Z = xV$ and hence $\Delta(Z) \neq 0$ is equivalent $\Delta(x) \neq 0$. But, if the modulus of $S$ is the modulus of $A$, then a number of $S$ is a nilfactor of $S$ if and only if it is a nilfactor of $A$ and hence $\Delta_s(x) = 0$ is equivalent to $\Delta(x) = 0$. Therefore, by the argument used in section 4, the irreducible factors of $\Delta(Z)$ coincide with the irreducible factors of $\Delta_s(x)$ and thus a number of $A$ has property $R^*$ if and only if the corresponding $x$ has property $R^*$.

Using this definition of an arithmetic Dickson‡ has proved a number of important general theorems about the arithmetic of a general algebra, of which the most interesting from the point of view of this paper are the following:

THEOREM A.§ *The arithmetic of $A = S + N$ is associated with the arithmetic of its semi-simple subalgebra $S$. In other words, we may suppress the properly nilpotent elements of an algebra when studying its arithmetic.*

THEOREM B‖. *The first components of the elements of any maximal set of integral elements (with properties R, C, U) of a direct sum $B \oplus C$ constitute*

† *J*, pp. 289–292; *A and A*, pp. 145–147.

‡ *J*, pp. 316–319; *A and A*, pp. 156–187; Proc. Internat. Math. Cong., Toronto (1924).

§ This is what he calls the fundamental theorem of arithmetics and is proved in *A and A*, pp. 175–187.

‖ This is the first theorem of his section 93, pp. 157–160.

*a set of integral elements of the first component algebra B, and similarly for the second components. Conversely, given the arithmetics of B and of C, we readily determine an arithmetic of $B \oplus C$.*

THEOREM C.† *If a set S of elements of a rational algebra A of order n having a modulus has properties C and U and is itself of order n, we can choose basal units $u_1, \cdots, u_n$ of A belonging to S such that the constants of multiplication are all integers and $u_1$ is the modulus.*

THEOREM D.‡ *Let A be any rational semi-simple algebra of order n having a modulus. Let S be any set of elements of A having properties R, C, U and of order n. Then S has a basis $\omega_1, \cdots, \omega_n$, where $\omega_1$ is the modulus.*

THEOREM E.§ *Let S be a rational simple algebra and let D and M be the rational division sub-algebra and rational simple matric algebra, respectively, of which S is the direct product. Then each element of S may be expressed in the form $\sum d_{ij} e_{ij}$ where the $d_{ij}$ are elements of D and the $e_{ij}$ are the $n^2$ basal units of M. If $\Pi$ is a maximal set of elements of S having properties R and C and containing all the matric units $e_{ij}$ and having the same order $\delta n^2$ as S, then the $d_{ij}$ range independently over a maximal set of elements of D having properties R, C, U and having the same order $\delta$ as D; and conversely.*

COROLLARY.‖ *We know the integral elements of any simple algebra $D \times M$ if we know those of the division algebra D.*

Dickson has also developed the arithmetic of some special cases, applying them to the solution of diophantine equations.¶

But, as the writer remarked in her Toronto paper, Dickson's definition of an arithmetic of an algebra $A$ does not seem to be the one that is most natural or most artistic. To be sure, he actually formulates a definition only for the case where the algebra is over the field $\mathfrak{R}$ of rational numbers, yet he appears to use the same definition for the algebras over the field of complex numbers, as evidenced by his remarks near the beginning of the discussion in each case.**

Accordingly, for reasons given in detail in my Toronto paper, the writer proposed the following

DEFINITION. *Let A be an algebra over any field F, then each element of a set of elements of A shall be called an integer of A over F if the set has the properties $R^*$, C, U, M where $R^*$ is defined thus:*

† This is the theorem of his section 94, pp. 160–161.

‡ This is the theorem of his section 95, pp. 161–163.

§ *A and A*, pp. 165–168.

‖ *Ibid*, p. 168.

¶ *Comptes Rendus, Cong. Internat. Math.* (Strasbourg, 1920), pp. 46–52; *Bull. Amer. Math. Soc.*, vol. 27 (1921), pp. 353–65; *A and A*, pp. 194–199.

** *A* and *A*, p. 184, lines 8–11 from the bottom; *J*, p. 316, lines 19–21.

*$R^*$. For every element of the set, the coefficients of the rank equation are algebraic integers of the field $F$.*

**3. Relation of property $R^*$ to property $R$.** If we consider an algebra $A$ over an algebraic field $F$, the totality of all numbers of $A$ over $F$ form an algebra $A^*$ over the field $\mathfrak{R}$ of rational numbers. First, consider a set of numbers of $A$ over $F$ possessing property $R^*$. Now the general integer $x$ of such an algebra is a root of the rank equation, $\varrho(\omega) = 0$, where the coefficients are algebraic integers of $F$ and the coefficient of the highest power of the variable is unity. Let the distinct conjugates of $\varrho(\omega)$—obtained by replacing the set of coefficients of $\varrho(\omega)$ by the sets of their conjugates, respectively, be denoted by $\varrho^{(0)}, \varrho^{(1)}, \varrho^{(2)}, \cdots$. Then $\bar{\varrho} = \Pi\varrho^{(i)}$ is a polynomial in $\omega$ which is unaltered if the roots of the defining equation for the field $F$ are subjected to any substitution among themselves and thus the coefficients of the various powers of $\omega$ in $\bar{\varrho}$ are rational numbers. Since, moreover, the coefficients of the powers of $\omega$ in $\varrho$ and its conjugates are algebraic integers, then so also are the coefficients in $\bar{\varrho}$. Hence $x$ satisfies an equation in which the coefficients are rational integers and the coefficient of the highest power of $\omega$ is unity. But the rank function for $x$ must be a factor of $\bar{\varrho}$ and hence (by Gauss' lemma) the rank function must have integral coefficients. Thus the general number of $A$ regarded as an algebra $A^*$ over $\mathfrak{R}$ has property $R$.

Conversely, let $x$ be the general number of $A^*$ and let the rank equation be $\bar{\varrho}(\omega) = 0$. In the field $F$, $x$ will satisfy an algebraic equation and we shall understand that the one of lowest degree is $\varrho(\omega) = 0$. By definition, this will be the rank equation for $x$ as a number in $A$ over $F$. Then $\varrho(\omega)$ must be a factor of $\bar{\varrho}(\omega)$; for, if not, then by dividing $\bar{\varrho}$ by $\varrho$ we should obtain an equation of degree lower than the degree of $\varrho$ which is satisfied by $x$. Hence, the rank function for the general number $x$ of $A$ over $F$ is a divisor of the rank function for this same number regarded as a number of $A^*$ over $\mathfrak{R}$. Since this is true for the general number $x$, it is true for any particular $x$. But, if $x$ is an integer of $A^*$ over $\mathfrak{R}$, then its rank function $\bar{\varrho}$ is such that the coefficients of the powers of $\omega$ are rational integers and the coefficient of the highest power of $\omega$ is unity, and thus (by Gauss' lemma) the coefficients of the powers of $\omega$ in $\varrho$ are algebraic integers of $F$ and the coefficient of the highest power of $\omega$ is unity. Hence if $x$ possesses property $R$, it possesses property $R^*$. Thus we have

THEOREM 1. *Let $A$ be an algebra over any algebraic field $F$ which can be regarded as the algebra $A^*$ over the field $\mathfrak{R}$ of rational numbers. Then a set of numbers of $A$ over $F$ which possesses property $R^*$ possesses property $R$ and conversely. Hence if $S$ is a maximal set of numbers of $A$ over $F$*

*possessing properties $R^*$, $C$ and $U$, then $S$ is a maximal set of numbers of $A^*$ over possessing properties $R$, $C$ and $U$; and conversely.*

4. **Fundamental theorem.** In this section we shall prove Theorem $A$ for the general algebra over any field $F$. For this, we need Dickson's†

LEMMA 1. *If $n$ is any nilpotent number of an algebra $A$ possessing a modulus (denoted by 1), then there exists an element $n_1$ such that; $(1+n)(1+n_1) = 1$.*

For $(1+n)(1-n) = 1-n^2$, $(1+n)(1-n)(1+n^2) = 1-n^4$, and, in general, $(1+n)(1-n)(1+n^2)(1+n^4)\cdots(1+n^{2^k}) = 1-n^{2^{k+1}}$. Hence, as soon as $2^{k+1} \geqq \nu$, where $\nu$ is the index of $n$, then $n^{2^{k+1}} = 0$ and the lemma is proved.

Now, if $A$ is any algebra over $F$, then $A = S+N$ where $S$ is a semi-simple subalgebra and $N$ is a maximal invariant nilpotent subalgebra of $A$. Write every number of $A$ in the form $Z = x+X$ where $x$ is in $S$ and $X$ is in $N$. Then we shall prove that, as $Z$ ranges over all the numbers of an arithmetic of $A$ over $F$, then $x$ ranges over all the numbers of an arithmetic of S; and conversely, when $x$ ranges over all the numbers of an arithmetic of $S$ and the $X$ range over all the numbers of $N$ (without restriction), then $Z$ ranges over all the numbers of an arithmetic of $A$.

For convenience, we shall assume that the elements of $A$ are chosen as $e_i\,(i = 1, 2, \cdots, s)$, $E_j\,(j = 1+s, \cdots, \cdots, n+s)$ where the $e_i$ form a basis for $S$ and the $E_j$ form a basis for $N$. We shall use a similar notation for the constants of multiplication. For, since $S$ is a subalgebra, every product $e_i e_j$ contains only the $e_k$ and similarly every product $E_i E_j$ contains only the $E_k$. Also, since $N$ is an invariant subalgebra, every product $e_i E_j$ and every product $E_j e_i$ contains only the $E_k$ and hence the multiplication table is of the form

$$e_i e_j = \sum_k \gamma_{ijk} e_k, \qquad e_i E_j = \sum_k \Gamma_{ijk} E_k,$$
$$E_i e_j = \sum_k \Gamma_{ijk} E_k, \qquad E_i E_j = \sum_k \Gamma_{ijk} E_k.$$

Similarly, we shall write the general number of $A$ in the form $Z = x+X = \sum x_i e_i + \sum X_j E_j$.

Since $S$ is a subalgebra of $A$ and $N$ is an invariant subalgebra, it follows that when a set of numbers $Z$ of $A$ possesses property $C$, so also does the set of their $S$-components. Conversely, if it is assumed that the components of the $Z$ range over $N$ without restriction. Moreover, since the modulus of $A$ is the modulus of $S$, it follows that the set of numbers $Z$ have the property $U$ if and only if the set of their $S$-components have

† *J*, p. 316; *A and A*, p. 179.

that property. Accordingly, to prove that any arithmetic of $A$ is associated with an arithmetic of $S$, we have yet to prove that *i*) when the numbers $Z$ have property $R^*$, then so do their $S$-components, and conversely; *ii*) when the numbers $Z$ have property $M$, then so do their $S$-components, and conversely.

To prove the first of these, we note that the first determinant of the general number $Z$ of $A$ is $\delta(Z) = \delta_s(Z)\delta_n(Z)$, where $\delta_s(Z)$ is the first characteristic determinant of the general number, $x$, of $S$ and where $\delta_n(Z)$ is the determinant

$$\left| \sum_i \Gamma_{ijk} x_i + \sum_i \Gamma_{ijk} X_i - \delta_{jk}\, \omega \right| \quad (j, k = s+1, \cdots, n).$$

A necessary and sufficient condition that a set of numbers $Z$ of $A$ possesses property $R^*$ is that the coefficients of the rank equation $\varrho(Z) = 0$ be algebraic integers in $F$ and hence (by Gauss' lemma) that the coefficients of each irreducible factor of $\varrho(Z)$ in the field $F$ be algebraic integers in $F$. Since each of the irreducible factors of $\varrho(Z)$ is also a factor of $\delta(Z)$ and conversely, property $R^*$ is possessed by a number $Z$ if and only if the coefficents of the powers of $\omega$ in $\delta_s$ and in $\delta_n$ are integers in $F$. But $\delta_s(Z)$ is none other than the first characteristic determinant of the number $x$ of $S$. Hence a number $Z$ of $A$ has property $R^*$ only if its $S$-component has property $R^*$.

To prove the converse, we first show that $\Delta(Z)$ is independent of $X$. For any number $Z = x+X$ of $A$ we shall determine a unit $U = 1+\nu$ (where $\nu$ is a number of $N$) such that $ZU = x$. For $ZU = x+X'$ where

$$X' = \sum_t \left( \sum_{ik} x_i\, U_k\, \Gamma_{ikt} + \sum_{jk} X_j\, U_k\, \Gamma_{jkt} + X_t \right) E_t.$$

Here $i$ ranges over the subscripts pertaining to $S$ and $j, k, t$ range over those pertaining to $N$. Hence $ZU = x$ is equivalent to

$$\sum_{ijk} (x_i\, U_k\, \Gamma_{ikt} + X_j\, U_k\, \Gamma_{jkt}) = -X_t.$$

This system of linear equations has a solution which is unique if and only if $\Delta_n(Z) \neq 0$. Since $U$ is a unit (by Lemma 1), the equation $ZU = x$ is equivalent to $Z = xV$; and if one of these equations has a unique solution, so also has the other. But it can be proved in the same way that $Z = xV$ has a solution which is unique if and only if $\Delta_n(x) \neq 0$. Hence $\Delta_n(Z) \neq 0$ is equivalent to $\Delta_n(x) \neq 0$. Since, moreover, this result is true in $F$ or in any overfield of $F$—inasmuch as we still have, in the overfield, $A = S+N$ where $S$ and $N$ are precisely the same algebras as before and are, respect-

ively, semi-simple and maximal invariant nilpotent subalgebras of $A$—then this result is true in the field $C$ of ordinary complex numbers. Hence the irreducible factors of $\Delta_n(Z)$ coincide with the irreducible factors of $\Delta_n(x)$ and thus $\Delta(Z)$ is independent of $X$.

This means that $Z$ is a nilfactor if and only if its $S$-component, $x$, is a nilfactor of the algebra $A$. But if $x$ is not a nilfactor of the algebra $S$, then it can not be a nilfactor of the algebra $A$. For, since $x$ is not a nilfactor of $S$ and $S$ has a modulus (= the modulus of $A$), $x$ has an inverse, $x'$, Hence, if $xy = 0$ where $y$ is a number of $A$, we should have $y = 0$.

Since, from the foregoing, $Z$ is a nilfactor of $A$ if and only if $x$ is a nilfactor of $S$, it follows that $\Delta(Z) = 0$ is equivalent to $\Delta_s(x) = 0$. By the same argument as was used a few lines above, the irreducible factors of $\Delta(Z)$ coincide with the irreducible factors of $\Delta_s(x)$. But $\Delta(Z-\omega) = \delta(Z)$ and $\Delta_s(x-\omega) = \Delta_s(Z-\omega) = \delta_s(Z)$ and hence the irreducible factors of $\delta(Z)$ coincide with the irreducible factors of $\delta_s(Z)$. Hence we conclude that a number of $A$ has the property $R^*$ if and only if its $S$-component has property $R^*$.

Moreover, since $\delta(Z)$ is independent of $Z$, property $R^*$ puts no restriction whatsoever on the $N$-component of $Z$. Hence, if $\sum(A)$ is a maximal set of numbers of $A$ possessing properties $R^*$, $C$ and $U$, the coördinates of $X$ will range over $F$ without restriction.

Corresponding to any set $\sum(A)$ of numbers of $A$ there will be a definite set $\sum(S)$ of numbers of $S$ consisting of the $S$ components of the numbers of $\sum(A)$. If $\sum(A)$ has property $R^*$, then so also does $\sum(S)$ and conversely. Similarly for property $U$. If $\sum(A)$ has property $C$, then so also has $\sum(S)$; and the convergence is true if the $N$-components of the numbers of $\sum(A)$ range freely over $F$. Finally, if $\sum(A)$ is a maximal set of numbers of $A$ possessing properties $R^*$, $C$, and $U$, then so also is $\sum(S)$. Conversely, if $\sum(S)$ is a maximal set of numbers of $S$ possessing properties $R^*$, $C$ and $U$, then so also is the corresponding $\sum(A)$, provided that the $N$-components of the numbers of $\sum(A)$ range freely over $F$. Hence an arithmetic of $A$ determines a unique arithmetic of $S$, and conversely.

In connection with the foregoing argument, it was proved that, for every number $Z$ of the algebra $A$ of non-zero norm, there could be determined uniquely a unit, $U$, such that $ZU = x$. Hence there is not only a definite correspondence between the numbers of an arithmetic of $A$ and those of an arithmetic of $S$, but this correspondence is given by the process of multiplication by a unit for all numbers $Z$ of non-zero norm, and thus the corresponding numbers of $A$ and $S$ are associated integers.† Hence we have

† *A and A*, pp. 161–163.

THEOREM 2. *Let $A = S + N$ be any algebra over a field $F$, where $S$ is a semi-simple subalgebra and $N$ is a maximal invariant nilpotent subalgebra of $a$. Then an arithmetic of $A$ is associated with a definite arithmetic of $S$.*

5. **Extension of Dickson's other theorems.** Since the proofs in section 93 of Dickson's *Algebras and their arithmetics* hold for any field, the theorems of that section hold here.

Although the proof in his section 94 in one place appears to use a property peculiar to the field $R$ of rational numbers, yet it is at once evident that this is not actually so. For since the $\gamma$'s are now numbers of the field $F$, they are all of the form $\gamma'/d$, where each $\gamma'$ is an integer of $F$ and each $d$ is a rational integer. The rest of the proof is precisely as in section 94 and hence that theorem holds for an algebra over any algebraic field $F$.

The details of section 95 all apply to an algebra over any field $F$ with the exception of the last paragraph, which says that section 86 shows the existence of a basis of order $n$. To prove the theorem of this section, we note that any arithmetic of $A^*$ over $\mathfrak{R}$ has a basis by Dickson's work† and hence there is a finite number of integers of $A$ which are such that all integers of the arithmetic of $A$ are linear functions of these, with coefficients which are rational integers. By the closure property, we can allow these rational integers to range over the algebraic integers of $F$ and the resulting numbers will be integers of $A$ over $F$. Moreover, they will give all the integers of $A$. Hence there is a finite number of integers of $A$ over $F$ which satisfy one of the two requirements for a basis.‡ The second requirement is that the numbers of the basis shall be linearly independent with respect to the field of definition, $F$. But this is not so important as the first one, as it is not used in any essential way, as can be readily verified by referring to Dickson's work. Note, however, that it has not been proved that an arithmetic of $A$ has a basis of the same order as the algebra $A$. Whether or not this is true does not concern us here.

Turning now to section 96, we note that Dickson's proof fails here, but we do not need this theorem in the sequel.

On the other hand, the proof in section 97 (= Theorem E) holds without change for an algebra over any field $F$ except in one step of part (*ii*), where it is proved that the set $\pi$ has property $R^*$. Here we are concerned with an algebra $S$ (= $P$ in Dickson's notation) over any algebraic field $F$, where $S$ is the direct product of a division subalgebra $D$ and a simple matric subalgebra $M$ with $n^2$ basal units $e_{ij}$, where $D$ and $M$ are

† *A and A*, pp. 161–163.

‡ *A and A*, pp. 138, 145.

both over the field $F$. Then each number of $M$ is commutative with each number of $D$ and their moduli coincide with the modulus of $S$. Now $S$ can be regarded as an algebra $S^*$ over the field $\mathfrak{R}$ of rational numbers and similarly with $M$ and $D$. Then Theorem E applies to $S^*$ over $\mathfrak{R}$. If we denote the set $\pi$ of part (*ii*) formed for algebra $S^*$ over $\mathfrak{R}$ by $\pi^*$, then $\pi^*$ has property $R$. Hence, by Theorem 1, the set $\pi$ has property $R^*$. Thus Theorem E is true for an algebra $S$ over any algebraic field $F$, if we make the above reservation as to basis. The Corollary is true without exception. Hence we have proved the following

THEOREM 3. *If $S$ is a semi-simple algebra over any algebraic field $F$ and is the direct sum of simple algebras $S_i$, then the arithmetic of $S$ is known when we know the arithmetic of each $S_i$. Moreover, if $S_i$ is the direct product of a simple matric algebra, $M_i$, and a division algebra, $D_i$, then an arithmetic of $S_i$ is known when an arithmetic of $D_i$ is known and conversely.*

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# SOME RELATIONS BETWEEN A CONTINUOUS CURVE AND ITS SUBSETS.*

By Harry Merrill Gehman.†

In a recent paper,‡ the author has discussed some properties which are necessary and sufficient in order that every subcontinuum of a continuous curve be a continuous curve. S. Mazurkiewicz§ has proved that a continuous curve which contains no simple closed curve is of this type, and R. L. Wilder‖ has proved the more general result that the boundary of a complementary domain of a continuous curve is itself a continuous curve, every subcontinuum of which is a continuous curve. Professor Wilder has proved that this type of continuous curve has certain properties, which we have been able to show in Paper 1, are possessed by all continuous curves which are such that every subcontinuum is a continuous curve, regardless of whether or not they form the boundary of a domain.

In the present paper, we give in Theorem 1, a new necessary and sufficient condition that every subcontinuum of a continuous curve be a continuous curve. Then, by means of Theorem 2, we give in Theorems 3 and 4 certain necessary and sufficient conditions that every subcontinuum of a *continuum* be a continuous curve. Some other related questions are considered in the remaining theorems.

Theorem 1. *If $M$ is a continuous curve, a necessary and sufficient condition that every subcontinuum of $M$ be a continuous curve, is that every subset of $M$ which is irreducibly connected¶ between two points of $M$ is an arc.*

*Proof.* In order to show that the condition is sufficient, we shall show that, if some subcontinuum $K$ of $M$ is not a continuous curve, then $M$ contains an irreducibly connected set which is not an arc.

* Presented to the American Mathematical Society, February 27, and June 12, 1926; received June 19, 1926.

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‡ H. M. Gehman, *Concerning the subsets of a plane continuous curve,* Annals of Math., vol. 27 (1925), pp. 29–46. We shall refer to this hereafter as Paper 1.

§ S. Mazurkiewicz, *Un théorème sur les lignes de Jordan,* Fundamenta Math., vol. 2 (1921), pp. 119–130. See Lemma, p. 123.

‖ R. L. Wilder, *Concerning continuous curves,* Fundamenta Math., vol. 7 (1925), pp. 340–377. See Theorem 11, p. 361.

¶ A set $I$ is said to be *irreducibly connected between $A$ and $B$,* if $I$ is connected and contains both $A$ and $B$, but no proper connected subset of $I$ contains both $A$ and $B$. For a discussion of such sets, see: B. Knaster and C. Kuratowski, *Sur les ensembles connexes* Fundamenta Math., vol. 2 (1921), pp. 206–255.

By means of R. L. Moore's characterization* of a continuum which is not a continuous curve, we can show that $K$ satisfies the following conditions: there exists a simple closed curve $ABCD$ in the plane, and a countable infinity of continua, $W$, $K_1$, $K_2$, $K_3, \cdots$, such that (1) each of these sets is a subset of $K$, has at least one point on $AD$ and at least one point on $BC$, has no points on $AB$ or $CD$, and is a subset of the set $H$ consisting of $ABCD$ and its interior, (2) no two of these sets have a point in common, and no one of them (save possibly $W$) is a proper subset of any connected point set which is common to $K$ and $H$, and (3) the set $W$ is the sequential limiting set of the sequence of sets, $K_1$, $K_2$ $K_3, \cdots$.

From (1) and (2) it follows that if $K'$ is any one of the the sets, $W$, $K_1$, $K_2$, $K_3, \cdots$, then $H-K'$ is disconnected, and consists of at least two maximal connected subsets, one of which contains $AB$, and a second contains $CD$, and no other maximal connected subset contains any points of $W+K_1+K_2+K_3+\cdots$. We can therefore suppose that (4) in the case of $H-W$, only the maximal connected subset which contains $AB$, contains any points of $K_1+K_2+K_3+\cdots$, and in the case of $H-K_i$ ($i=1, 2, 3, \cdots$) the maximal connected subset which contains $AB$, contains only $K_1+K_2+\cdots+K_{i-2}+K_{i-1}$, and the one which contains $CD$, contains $W+K_{i+1}+K_{i+2}+\cdots$.

By R. L. Moore's extension† of a theorem due to Zoretti, we can enclose each of the sets $K_i$ by a simple closed curve $J_i$, which encloses or contains no other points of $W+K_1+K_2+K_3+\cdots$, nor of the arcs $AB$ or $CD$ of $ABCD$, nor of the set of simple closed curves, $J_1+J_2+J_3+\cdots$. Any maximal connected subset of $M$ contained in $J_i$ plus its interior is a simple closed curve‡, and therefore the maximal connected subset that contains $K_i$ is a continuous curve. Since $K_i$ has points on both $AD$ and $BC$, this continuous curve contains an arc joining a point $E_i$ of $AD$ to a point $F_i$ of $BC$, and except for its end points, lying entirely interior to $ABCD$. If in properties (1)–(4), we replace $K$ by $W+E_1F_1+E_2F_2+E_3F_3+\cdots$, and let $K_i$ be the arc $E_iF_i$, then the set $W$, $E_1F_1$, $E_2F_2$, $E_3F_3$, $\cdots$ satisfies properties (1)–(4), with the exception of the latter part of (2) and satisfies this additional property: (5) the continuum $K_i$ is an arc.

* R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bull. Amer. Math. Soc., vol. 29 (1923), pp. 289–302. See especially pp. 296–297. R. L. Wilder, loc. cit., p. 371. Also compare Paper 1, p. 37–38.

† R. L. Moore, *Concerning the separation of point sets by curves*, Proc. Nat. Acad. Sci., vol. 11 (1925), pp. 469–476. See especially Theorem 1. In order to apply the theorem in this case we must add to each of the sets $K_i$, all points of the plane lying in a domain whose boundary is a subset of $K_i$.

‡ Paper 1, Lemma B, p. 34. See also Theorem 6 of this paper.

Let $P$ and $Q$ be two points of $W$ interior to $ABCD$. If we construct about $P$ and $Q$ as centers two non-intersecting circles, both interior to $ABCD$, then, since $M$ is a continuous curve, all points of $M$ sufficiently close to $P$ (or $Q$) can be joined to $P$ (or $Q$) by an arc in $M$ lying within the corresponding circle. By (3) a positive number $n$ can be found such that points $P'$ and $Q'$ exist on the arc $E_n F_n$ which can be joined to $P$ and $Q$ respectively by arcs in $M$ lying within the corresponding circles, and therefore having nothing in common, and lying within the simple closed curve $ABCD$. Each of the arcs $P'P$, $Q'Q$ has points in common with each of the arcs $E_{n+1}F_{n+1}$, $E_{n+2}F_{n+2}, \cdots$. Let $P''$ and $Q''$ be the first points which these arcs have in common with $W$.

Select on $P'P''$ a set of arcs $P_1P_2$, $P_3P_4, \cdots$, such that $P_i$ is a point of $E_{n+i}F_{n+i}$, and such that the arc $P_iP_{i+1}$ has only its end points in common with $E_{n+1}F_{n+1} + E_{n+2}F_{n+2} + \cdots$. Select a similar set of arcs $Q_2Q_3$, $Q_4Q_5, \cdots$ on $Q'Q''$. Consider the set consisting of the arcs $P_1P_2 + P_2Q_2$ (of $E_{n+2}F_{n+2}$) $+ Q_2Q_3 + Q_3P_3$ (of $E_{n+3}F_{n+3}$) $+ P_3P_4 + \cdots$. If we add to this set the point $P''$, which is one of its limit points in $W$, we obtain a set which is irreducibly connected between $P_1$ and $P''$, but which is not an arc because it is not closed, as $Q''$ is a limit point of this set but is not contained in the set.

Therefore the condition of Theorem 1 is sufficient.

By virtue of a theorem of Knaster and Kuratowski,* the points of any irreducibly connected set $I$ can be put into (1—1) correspondence with an interval $(a, b)$ of the $x$-axis in such a way that order is preserved. In order to show that the condition of Theorem 1 is necessary, we shall show that under the given hypotheses, such a correspondence is also continuous, and that therefore $I$ is an arc.

If the correspondence were not continuous, there would exist two sets of corresponding points: $x_1, x_2, x_3, \cdots$, of $(a, b)$, and $X_1, X_2, X_3, \cdots$, of $I$, approaching points $x$ and $Y$, respectively, as sequential limit points, while $x$ and $Y$ are not corresponding points under the given correspondence. Since $(a, b)$ is closed, $x$ is a point of $(a, b)$, but $Y$ is not necessarily a point of $I$. We can assume also that the set $x_1, x_2, x_3, \cdots$, has been selected in such a way that either $x_i$ precedes $x_{i+1}$ on $(a, b)$, for all values of $i$, or else that $x_i$ follows $x_{i+1}$ for all values of $i$, and we shall suppose that the first of these is true.

Since $x$ is a point of $(a, b)$, it has a corresponding point $X$ in $I$. If $Y$ were a point of $I$, it would have a corresponding point $y$ in $(a, b)$, and since $y \neq x$, we can select an integer $n$ such that $y$ does not lie on the

* Loc. cit., Theorem 23, p. 221.

interval $(x_n, x)$ of $(a, b)$. The points $X_n$ and $X$ determine a decomposition* of $I$ into three sets: $I_1, I_2, I_3$, the first of which is irreducibly connected between $A$ and $X_n$, the second between $X_n$ and $X$, and the third between $X$ and $B$. Moreover, these sets have no points or limit points in common, except for $X_n$ in the case of $I_1$ and $I_2$, and except for $X$ in the case of $I_2$ and $I_3$. Since the correspondence between $(a, b)$ and $I$ preserves order, $Y$ is not a point of $I_2$, and yet $I_2$ is the only one of the three sets that contains an infinite number of points of $X_1, X_2, X_3, \cdots$. Therefore $Y$ is not a limit point of $X_1, X_2, X_3, \cdots$, which is contrary to the definition of the point $Y$. Therefore $Y$ cannot be a point of $I$.

The point $X$ is a limit point of points of $I_1 + I_2$, and therefore we can select a sequence of points $P_1, P_2, P_3, \cdots$ of $I_1 + I_2$, such that $P_i$ precedes $P_{i+1}$ for all values of $i$, and such that the sequential limit of the sequence is $X$. Evidently not more than a finite number of points of this sequence can lie between two successive points of the sequence $X_1, X_2, X_3, \cdots$, nor can more than a finite number of points of $X_1, X_2, X_3, \cdots$, lie between two successive points of the sequence $P_1, P_2, P_3, \cdots$. We can select then a sequence of points $X_{m_1}, X_{m_2}, X_{m_3}, \cdots$, from $X_1, X_2, X_3, \cdots$, and a sequence $P_{n_1}, P_{n_2}, P_{n_3}, \cdots$, from $P_1, P_2, P_3, \cdots$, occuring in $I$ in the order $X_{m_1}, P_{n_1}, X_{m_2}, P_{n_2}, \cdots$. For each value of $i$ let $K_i$ be the subset of $I$ irreducibly connected between $X_{m_i}$ and $P_{n_i}$. No one of the sets $K_i$ contains a limit point of any one of the others. The set $K_1'$ is a continuous curve by hypothesis, and therefore contains an arc $X_{m_1} P_{n_1}$. If we remove from the continuous curve $K_2'$ the points that it has in common with this arc, the remainder contains a connected set containing $K_2$ which is arcwise connected.† We can therefore construct an arc $X_{m_2} P_{n_2}$ in $M$ having nothing in common with the arc already constructed. In the same way, for each value of $i$, we can construct an arc $X_{m_i} P_{n_i}$ in $M$ having nothing in common with the arcs previously constructed. The set $M$ therefore contains an infinite number of mutually exclusive arcs $X_{m_i} P_{n_i}$, and since the points $X_{m_i}$ approach $Y$ as a sequential limit and the points $P_{n_i}$ approach $X$ as a sequential limit, there must exist some constant $\epsilon$ such that all the arcs of this set are of diameter greater than $\epsilon$. But this is impossible, by Theorem V of Paper 1, under the given hypotheses. Therefore the condition of Theorem 1 is necessary.

THEOREM 2. *If every proper subcontinuum of a decomposable continuum $M$ is a continuous curve, then $M$ is a continuous curve.*

*Proof.* Since $M$ is decomposable, it can be expressed as the sum of

* Knaster and Kuratowski, loc. cit., Theorems 19 and 22, pp. 219–220.

† R. L. Moore, *Concerning continuous curves in the plane*, Math. Zeitschr., vol. 15 (1922), pp. 254–260.

two of its proper subcontinua, each of which is a continuous curve by hypothesis. It follows that $M$ itself is a continuous curve.

However, an indecomposable continuum may be such that every proper subcontinuum is a continuous curve. An example of such a set has been given by Kuratowski.* In this example, every proper subcontinuum is an arc, and the set therefore shows that pseudo-Jordan curves, as defined by Yoneyama,† may exist,—contrary to a statement by the author in the first abstract of this paper.

In Paper 1, we have considered some necessary and sufficient conditions that every subcontinuum of a continuous curve $M$ be a continuous curve. It is evident that, as a result of Theorem 2 of this paper, it is enough to assume that $M$ is a decomposable continuum, so far as the necessity parts of these theorems are concerned. But this is not possible in the sufficiency parts in all cases, as we shall see below. The following two theorems are consequences of Theorems 1 and 2 of this paper, and Theorems IV, V, and VI of Paper 1.

THEOREM 3. *If every proper subcontinuum of a decomposable continuum $M$ is a continuous curve, then* (1) *$M$ is a continuous curve,* (2) *$M$ contains no $R$-sets,‡* (3) *given any positive number $\varepsilon$, $M$ contains at most a finite number of mutually exclusive continua of diameter greater than $\varepsilon$, and* (4) *every subset of $M$ which is irreducibly connected between two points of $M$, is an arc.*

THEOREM 4. *Either of the following conditions is sufficient in order that every subcontinuum of a continuum $M$ be a continuous curve*: (1) *given any positive number $\varepsilon$, $M$ contains at most a finite number of mutually exclusive continua of diameter greater than $\varepsilon$;* (2) *given any positive number $\varepsilon$, $M$ can be expressed as the sum of a finite number of continua each of diameter less than $\varepsilon$, and each pair having at most a finite number of points in common.*

*Proof.* It was shown in the proof of Theorem VI of Paper 1 that, if condition (2) is true, then condition (1) is also true, but not conversely. Both conditions are stated here, however, since one may be easier to apply to a given continuum than the other. The conclusion is true whenever condition (1) is true, as was shown in the proof of Theorem V of Paper 1, no use being made there of the assumption that $M$ itself was a continuous curve. The truth of Theorem 4 is therefore established.

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* C. Kuratowski, *Théorie des continus irreductibles,* Fundamenta Math., vol. 3 (1922), pp. 209–210, Figure 1.

† K. Yoneyama, *Theory of continuous set of points,* Tôhoku Math. Journal, vol. 12 (1917), p. 157.

‡ An $R$-set is a continuum $N$ containing a continuum of condensation $W$, such that $N - W$ is uniformly connected *im kleinen* relative to $W$. See Paper 1, pp. 32–34.

The sufficient condition given in Theorem IV of Paper 1 is not enough, if $M$ is assumed merely to be a decomposable continuum. The following example shows that $M$ may be a continuum containing no $R$-sets, and yet containing subcontinua which are not continuous curves. The example is a continuum $M$, which is the sum of two continua, $M_1$ and $M_2$, having only one point in common, each of the sets $M_i$ being a continuum every subcontinuum of which is indecomposable.* If $N$ is any subcontinuum of $M$, and $W$ is any continuum of condensation of $N$ then since $N$ is indecomposable, $N-W$ is not connected *im kleinen* at any point, and therefore is not an $R$-set, as was shown in Paper 1. In our example, then, $M$ contains no $R$-sets, but no subcontinuum is a continuous curve.

Nor does it follow from the condition given in Theorem 1 of this paper that every subcontinuum of $M$ is a continuous curve if we assume merely that $M$ is a decomposable continuum. The following example shows that every irreducibly connected subset may be an arc and yet $M$ may contain subcontinua which are not continuous curves. The example is the set of straight line intervals joining (0,0) to (1,0), and joining (0,0) to $(1,1/n)$, for $n = 1, 2, 3, \cdots$.

THEOREM 5. *If $M$ is a continuous curve, and $D$ is a domain whose boundary $B$ consists of a finite number of continuous curves, and $K$ is the continuum obtained by adding limit points to a maximal connected subset of $M$ in $D$, then* (1) *$K$ is a continuous curve, and* (2) *given any positive number $\varepsilon$, there are at most a finite number of continuous curves of this type which are of diameter greater than $\varepsilon$.*

*Proof.* Since $K$ is a subset of $M$, and $B$ is closed, it follows that $K$ is connected *im kleinen* at all points of $D$. If then $K$ is not a continuous curve, any point at which $K$ fails to be connected *im kleinen* is a point of $B$.

By R. L. Moore's theorem, if $K$ is not a continuous curve, there is a simple closed curve $ABCD$ in the plane, and a countable infinity of subcontinua of $K$: $W, K_1, K_2, \cdots$, satisfying conditions (1)–(4) given in the proof of Theorem 1.

Let $P$ and $Q$ be two points of $W$ interior to $ABCD$. We have just shown that $W$ is a subset of $B$, and therefore $P$ and $Q$ are points of the same maximal connected subset $B_1$ of $B$. We can therefore put about each of these points a circle interior to $ABCD$, such that each circle contains or encloses no points of the other, and no points of $B-B_1$. Since $M$ is a continuous curve, there exist in $M$ two arcs $P'P$ and $Q'Q$ within the circles about $P$ and $Q$ respectively, where $P'$ and $Q'$ are points of

* B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Math., vol. 3 (1922), pp. 247–286.

some continuum $K_n$ of the sequence $K_1, K_2, K_3, \cdots$. Let $P''$ and $Q''$ be the first points which these arcs have in common with $W$. We can select on the arc $P'P''$, a set of points $P_1, P_2, P_3, \cdots$ such that $P_i$ is the last point which $P'P''$ has in common with $K_{n+i}$, and we can select a similar set $Q_1, Q_2, Q_3, \cdots$ on $Q'Q''$. These points must all be points of $B$. The points $P''$ and $Q''$ are limit points of the sets $P_1, P_2, P_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ respectively, and since $B_1$ is a continuous curve, for some fixed value of $i$ there must exist arcs $P_iP''$ and $Q_iQ''$ in $B_1$ and within the circles about $P$ and $Q$, respectively.

These arcs have points in common with all but a finite number of the continua $K_1, K_2, K_3, \cdots$, and on account of condition (2), it is evident that, if we remove from $K$ the points which it has in common with these two arcs, the remainder is not connected. This is impossible, as by the definition of $K$ this remainder is a connected subset of $M$ plus certain of its limit points, and such a set is always connected.

Therefore $K$ is a continuous curve, and the truth of (1) of Theorem 5 is established.

The maximal connected subset of $M+B$ that contains $M$ is a continuous curve and, if we remove from it the points that it has in common with $B$, the number of maximal connected subsets of the remainder (in the totality of domains complementary to $B$) that are of diameter greater than any given positive number, is finite by a generalization of a theorem due to W. L. Ayres.* Since the diameter of a set is not increased by adding its limit points, it is evident that (2) is also true.

THEOREM 6. *If $M$ is a continuous curve and $B$ is the boundary of a domain $D$, whose exterior consists of a finite number of domains, the boundary of each being a continuous curve, then* (1) *every maximal connected subset of $M$ in $D+B$ is a continuous curve, and* (2) *given any positive number $\varepsilon$, there are at most a finite number of continuous curves of this type which are of diameter greater than $\varepsilon$.*

*Proof.* The set of points $B_1$ which forms the common boundary of $D$ and a domain $E$ of the exterior of $D$, is a subset of $B$, and a continuous curve, by hypothesis. Every point of $D$ is therefore separated from every point of $E$ by a simple closed curve lying in $B_1$,† and since $D$ and $E$ are

* W. L. Ayres, *Concerning the arcs and domains of a continuous curve,* (abstract), Bull. Amer. Math. Soc., vol. 32 (1926), p. 37. Although Mr. Ayres' hypothesis is that $N$ is a continuous curve, the conclusion of the theorem is also true if $N$ consists of a finite number of continuous curves. While ignorant of Mr. Ayres' proof, we had proved the same theorem and had presented it to the Society in the first abstract of this paper.

† R. L. Moore, *Concerning continuous curves in the plane,* loc. cit., Theorem 5, p. 260. See also, R. L. Moore, *Concerning the separation of point sets by curves,* loc. cit., footnote 5, p. 475.

connected, it follows that $B_1$ is identical with this simple closed curve. Therefore the boundary of the exterior of $D$ consists of a finite number of simple closed curves. Let us denote this boundary by $B'$.

In Lemma B of Paper 1, we have proved part (1) of this theorem for the special case where $B$ is a simple closed curve. As was shown there, a continuum $K$ which is a maximal connected subset of $M$ in $D + B$ is connected *im kleinen* at all points which are not points of $B'$. If then $K$ is not a continuous curve, the continuum of condensation $W$ at points of which $K$ fails to be connected *im kleinen*, must be a subset of $B'$. Let us now select on $W$ a point $P$, such that if $XY$ is any arc in $B'$ containing $P$ as an interior point, then $P$ is not a limit point of $B'-XY$. Since $K$ is a subset of $M$, and $M$ is connected *im kleinen* at $P$, we obtain the same contradiction as that obtained in the proof of Lemma B of Paper 1. Therefore $K$ is a continuous curve, and part (1) of Theorem 6 is true.

Suppose part (2) were false, and there were an infinite number of maximal connected subsets of $M$ in $D + B$ which are of diameter greater than some fixed positive number $\varepsilon$. Then, since we have just proved that each of these is a continuous curve, we can construct in each an arc of diameter greater than $\varepsilon$, thus obtaining an infinite collection of arcs such that points of no two of them can be joined by an arc in $M$ and in $D + B$. Then, as in the proof of Theorem V of Paper 1, we finally arrive at the sequence of arcs: $G_n H_n$, $G_{n+1} H_{n+1}$, $\cdots$ having the properties described there,* whose sequential limiting set $W$ is a continuum which is a subset of $M$.

Let $P$ be a point which is not a limit point of the end points of the arcs $G_n H_n$, $G_{n+1} H_{n+1}$, $\cdots$, and which is either not a point of $B'$ or, in case $W$ is a subset of $B'$, is a point of $B'$ such that, if $XY$ is any arc in $B'$ containing $P$ as an interior point, then $P$ is not a limit point of $B'-XY$. In this latter case, we can select an arc $QPR$ of $B'$ such that no point of the arc, except possibly $Q$ or $R$, is a limit point of $B'-QPR$ or of $W-QPR$. Having selected $P$ as described above, let $C$ be a circle about $P$, excluding all end points of the arcs $G_n H_n$, $G_{n+1} H_{n+1}$, $\cdots$, and excluding all points of $B'$ except possibly points of the arc $QPR$. Then, since $M$ is connected *im kleinen* at $P$, there exists an arc $Q'P'$ in $M$ within $C$, where $Q'$ is a point of one of the arcs $G_k H_k$, and $P'$ is a point on $W$, and is the only point which the arc $Q'P'$ has in common with $W$. This arc will have points in common with infinitely many arcs of the set $G_n H_n$, $G_{n+1} H_{n+1}$, $\cdots$, while the only point it can possibly have in common with $B'$ is $P'$. The entire arc $Q'P'$ therefore lies in $D + B$, and there-

* Paper 1, pp. 40–41.

fore infinitely many of the arcs of the set $G_n H_n$, $G_{n+1} H_{n+1}, \cdots$, lie in the same maximal connected subset of $M$ in $D + B$, which is contrary to the method of their selection. Therefore part (2) of Theorem 6 is true.

THEOREM 7. *If $M$ and $N$ are continuous curves and $N$ is a subset of $M$, and if $X$ is any maximal connected subset of $M—N$, then $X'$ is a continuous curve.*

*Proof.* Any maximal connected subset $X$ of $M—N$ lies in a domain $D$ complementary to $N$, and the boundary of $D$ is a continuous curve $B$ which is a subset of $N$. Then by Theorem 5 the continuum $X'$ is a continuous curve.

THEOREM 8. *If $M$ and $N$ are continuous curves and $N$ is a subset of $M$, and if $K$ is a set consisting of $N$ and any collection of maximal connected subsets of $M—N$, then $K$ is a continuous curve.*

*Proof.* R. L. Wilder* has shown that the number of maximal connected subsets of $M—N$ is countable and that no point of any one of these maximal connected subsets $X$ is a limit point of any subset of $M—X$. Since each of the sets $X$ is closed, save for limit points in $N$, it follows that any set $K$ consisting of $N$ and a collection of maximal connected subsets of $M—N$, is a continuum. By Theorem 7 and the theorem of Ayres mentioned in the proof of Theorem 5, it follows that continuum $K$ can be expressed as the sum of a collection of continuous curves which satisfy certain conditions† under which the continuum is a continuous curve. Therefore $K$ is a continuous curve.

* Loc. cit., Theorem 9, p. 360.

† H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Math., vol. 27 (1926), pp. 381–384.

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# TRANSFORMATIONS OF ONE PRINCIPAL EQUATION INTO ANOTHER.

By R. Garver.*

By a principal equation of degree $n$ we mean one of the form

$$x^n + a_3 x^{n-3} + a_4 x^{n-4} + \cdots + a_{n-2} x^2 + a_{n-1} x + a_n = 0. \tag{1}$$

Any equation of degree $n$ can be reduced to this form by a Tschirnhaus transformation (linear or quadratic) involving at most one square root.† We shall consider Tschirnhaus transformations on the roots of (1) of type

$$y = k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_{n-2} x^2 + k_{n-1} x + k_n. \tag{2}$$

If (2) is to transform (1) into another principal equation, the two conditions $\sum y = \sum y^2 = 0$ must be satisfied, where by $\sum y$ we mean $\sum_{i=1}^{n} y(x_i)$, the $x_i$ being the $n$ roots of (1). If these two conditions are satisfied, the coefficients of $y^{n-1}$ and $y^{n-2}$ in the transformed equation will be zero. $\sum y$ is secured directly by summing (2) for the roots of (1), and $\sum y = 0$ may be expressed as a condition on $k_n$:

$$k_n = -\frac{1}{n} \sum_{i=1}^{n-3} k_i s_{n-i}, \tag{3}$$

where $s_j$ represents $\sum x_i^j$.

We obtain $\sum y^2$ by squaring (2) and summing; it is linear in $k_{n-1}$ since $s_2$ is zero. We may then express $\sum y^2 = 0$ as a condition on $k_{n-1}$:

$$k_{n-1} = \frac{-\sum_{\lambda=2}^{2n-3} \sum_{i}{}' k_i k_{\lambda-i} s_{2n-\lambda} - n k_n^2}{2 \sum_{j=1}^{n-2} k_j s_{n+1-j}}, \tag{4}$$

where by $\sum'$ we mean that $i$ goes from 1 up to the smaller of $n$ and $\lambda-1$, except that neither $i$ nor $\lambda-i$ takes on the value $n-1$, nor does $i$ take on values which would make $\lambda-i$ greater than $n$. The other $k_i$ in (2) may be assigned arbitrary values, provided that $k_n$ and $k_{n-1}$ are then computed by means of (3) and (4).

* Received August 30, 1926.

† See Dickson, *Modern Algebraic Theories*, 211-212.

For actual work, for instance we might be attempting the transformation of an equation (which could be taken in the form (1)) to a normal form which was also principal, transformations of certain special types are more convenient than those of form (2). One such type is

$$y = u\varphi + v\psi, \tag{5}$$

where $u$ and $v$ are constants, and $\varphi$ and $\psi$ are polynomials of degree not greater than $n-1$ which possess the five properties*

$$\sum \varphi = \sum \varphi^2 = \sum \psi = \sum \psi^2 = \sum \varphi\psi = 0, \tag{6}$$

where, as above, $\sum \varphi$ means $\sum_{i=1}^{n} \varphi(x_i)$. If we have two polynomials possessing these five properties, any linear combinations of them will likewise; we shall take $\varphi$ and $\psi$ in the convenient reduced forms

$$\varphi = b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_{n-2} x^2 + b_{n-1} x + b_n \quad (b_1 = 0,\ b_2 = 1), \tag{7}$$

$$\psi = c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_{n-2} x^2 + c_{n-1} x + c_n \quad (c_1 = 1,\ c_2 = 0), \tag{8}$$

where we write in the coefficients $b_1$, $b_2$, $c_1$, $c_2$ in order to simplify the form of later formulas.

The five properties (6) impose five conditions on the coefficients of $\varphi$ and $\psi$. The first, $\sum \varphi = 0$, can be solved for $b_n$ in terms of the other $b_i$, and $\sum \psi = 0$ can be solved in like manner for $c_n$. The results are similar to (3):

$$b_n = -\frac{1}{n} \sum_{i=1}^{n-3} b_i s_{n-i}, \tag{9}$$

$$c_n = -\frac{1}{n} \sum_{i=1}^{n-3} c_i s_{n-i}. \tag{10}$$

The equation $\sum \psi^2 = 0$ can be considered as determining $c_{n-1}$ in terms of the preceding $c_i$ (and $c_n$, which has already been fixed by (10)). This gives

$$c_{n-1} = \frac{-\sum_{\lambda=2}^{2n-3} \sum_{i}{}' c_i c_{\lambda-i} s_{2n-\lambda} - n c_n^2}{2 \sum_{j=1}^{n-2} c_j s_{n+1-j}}, \tag{11}$$

where $\sum'$ is defined as in (4).

* For examples of the use of functions possessing such properties see Dickson, *Modern Algebraic Theories*, 213, 215, and Brioschi, *Opere Matematiche*, 293-299.

The other two equations, $\sum \varphi^2 = 0$ and $\sum \varphi\,\psi = 0$, can both be solved linearly for $b_{n-1}$:

$$b_{n-1} = \frac{-\sum_{\lambda=2}^{2n-3} \sum_{i}{}' b_i\, b_{\lambda-i}\, s_{2n-\lambda} - n\, b_n^2}{2\sum_{j=1}^{n-2} b_j\, s_{n+1-j}}, \tag{12}$$

$$b_{n-1} = \frac{-\sum_{\lambda=2}^{2n-3} \sum_{i}{}'' c_i\, b_{\lambda-i}\, s_{2n-\lambda} - n\, b_n\, c_n}{\sum_{j=1}^{n-2} c_j\, s_{n+1-j}}, \tag{13}$$

where by $\sum'''$ we mean that $i$ goes from 1 up to the smaller of $n$ and $\lambda-1$, except that $\lambda-i$ does not take on the value $n-1$, nor does $i$ take on values which would make $\lambda-i$ greater than $n$. If we equate the right members of (12) and (13) we obtain a quadratic in $b_{n-2}$, with coefficients involving the $b_i$ (except $b_{n-1}$ and $b_{n-2}$) and the $c_i$. Using (9), (10), (11) and the assigned values of $b_1, b_2, c_1, c_2$ we have $b_{n-1}$ expressed in terms of $b_3, \cdots, b_{n-3}, c_3, \cdots, c_{n-2}$. Then using (12) or (13) to determine $b_{n-1}$, we have five coefficients, $c_{n-1}, c_n, b_{n-2}, b_{n-1}, b_n$ determined* in terms of $b_3, \cdots, b_{n-3}, c_3, \cdots, c_{n-2}$, which may be assigned arbitrary values (except that denominators in (11), (12), (13) must not be made zero).

We have at once the

THEOREM. *Any transformation $y = u\varphi + v\psi$, where $\varphi$ and $\psi$ are polynomials of form* (7) *and* (8) *which possess the five properties $\sum \varphi = \sum \varphi^2 = \sum \psi = \sum \psi^2 = \sum \varphi\,\psi = 0$ transforms principal equations into principal equations.*

For we have

$$\sum y = u\sum \varphi + v\sum \psi = 0,$$
$$\sum y^2 = u^2 \sum \varphi^2 + 2uv \sum \varphi\,\psi + v^2 \sum \psi^2 = 0.$$

As an illustration of the convenience of transformations of this type let us consider the reduction of the quintic to the Bring-Jerrard normal form† $y^5 + dy + e = 0$, which is itself a principal equation. We may take the quintic in the form

$$x^5 + a_3 x^2 + a_4 x + a_5 = 0, \tag{14}$$

* Hence for $n < 5$, polynomials $\varphi$ and $\psi$ of type (7) and (8) possessing the five properties (6) do not in general exist, since there are five conditions on fewer than five coefficients. For $n = 4$, for example, the five conditions are independent, and there are only four coefficients. In fact the only polynomials, both of degree less than 4, which satisfy the five conditions without restrictions on the coefficients of (1) are the pair $\varphi = 0$, $\psi = x$, or vice versa.

† See Dickson, 212.

and $\varphi$ and $\psi$ in the form

$$\varphi = x^3 + b_3 x^2 + b_4 x + b_5, \tag{15}$$

$$\psi = x^4 + c_3 x^2 + c_4 x + c_5, \tag{16}$$

where $c_4$, $c_5$, $b_3$, $b_4$, $b_5$ are determined in terms of $c_3$ and known quantities by means of (9)–(13). We can take $c_3 = 0$ unless $a_5$ (and hence $s_5$) is zero, for in that case the denominator $2s_5 + 2c_3 s_3$ of (11) would be zero, as would that of (13).

Then as we have seen above

$$y = u\varphi + v\psi$$

transforms (14) into another principal equation. Further

$$y^3 = u^3\varphi^3 + 3u^2 v\varphi^2\psi + 3uv^2\varphi\psi^2 + v^3\psi^3, \tag{17}$$

and the condition $\sum y^3 = 0$ is a cubic in $u/v$ with coefficients which can be computed. Hence for suitable choice of $u/v$, $\sum y^3$ can be made zero, and the coefficient of $y^2$ in the transformed equation will then also be zero.

It would also be an advantage if we could say that we suffer no loss of generality in limiting an investigation to transformations of the particular type $y = u\varphi + v\psi$. Hence the

THEOREM. *Every Tschirnhaus transformation of degree $n-1$ which transforms a sufficiently general* principal equation of degree $n$ $(n>4)$ into another principal equation can be expressed in the form $y = u\varphi + v\psi$, where $\varphi$ and $\psi$ are polynomials of form* (7) *and* (8) *which possess the five properties $\sum\varphi = \sum\varphi^2 = \sum\psi = \sum\psi^2 = \sum\varphi\psi = 0$.*

For, taking $\varphi$ of form (7), we can write

$$y = k_2\varphi + k_1\left(\frac{y}{k_1} - \frac{k_2}{k_1}\varphi\right), \tag{18}$$

where $b_{n-2}$, $b_{n-1}$, $b_n$ can be determined so that $\sum\varphi = \sum\varphi^2 = \sum\varphi y = 0$. For this purpose we use (9), (12), and an equation similar to (13), obtained from it by replacing the $c_i$ by the $k_i$. Now $\left(\frac{y}{k_1} - \frac{k_2}{k_1}\varphi\right)$ has its second coefficient zero, and we have

* Denominators in the various formulas must be different from zero.

$$\sum\left(\frac{y}{k_1}-\frac{k_2}{k_1}\varphi\right) = \frac{\sum y}{k_1}-\frac{k_2}{k_1}\sum\varphi \qquad = 0,$$

$$\sum\left(\frac{y}{k_1}-\frac{k_2}{k_1}\varphi\right)^2 = \frac{\sum y^2}{k_1^2}-\frac{2k_2}{k_1^2}\sum\varphi y+\frac{k_2^2}{k_1^2}\sum\varphi^2 = 0.$$

Hence $\left(\frac{y}{k_1}-\frac{k_2}{k_1}\varphi\right)$ is a $\psi$ function of type (8) with $\sum\psi=\sum\psi^2=0$. We then have

$$y = k_2\varphi+k_1\psi \qquad \left(\psi \equiv \frac{y}{k_1}-\frac{k_2}{k_1}\varphi\right), \tag{19}$$

with $\sum\varphi=\sum\varphi^2=\sum\psi=\sum\psi^2=0$. Further

$$\sum\varphi\psi = \frac{\sum\varphi y}{k_1}-\frac{k_2}{k_1}\sum\varphi^2 = 0, \tag{20}$$

which proves the theorem.

In a similar manner we may prove the

COROLLARY. *Every Tschirnhaus transformation of degree* $m$ $(4\leqq m\leqq n-1)$ *which transforms a sufficiently general principal equation into another principal equation can be expressed in the form* $y=u\varphi+v\psi$, *where* $\varphi$ *and* $\psi$ *are polynomials of form* (7) *and* (8) *with* $n-1$ *replaced by* $m$, $n-2$ *by* $m-1$, *etc., which possess the five properties* $\sum\varphi=\sum\varphi^2=\sum\psi=\sum\psi^2=\sum\varphi\psi=0$.

# ON RELATED MAXIMA AND MINIMA.*

By Norman Miller.

1. **Introduction.** The problem of finding the relative maxima and minima of a function of several variables when the variables are subject to one or more auxiliary conditions is familiar to students of the differential calculus. We have in the general case $n$ functions of which one is to be minimized subject to the condition that the others remain constant. By choosing in succession each of the $n$ functions as the one to be minimized we have $n$ separate problems and the consideration of the relative character of the extremes obtained at a minimizing point is a matter of some interest. Attention has been called to this problem by J. K. Whittemore† who has discussed in detail the relation between the extremes of two functions of two variables, each being minimized under the condition that the other is constant. The present paper deals less exhaustively with the case of $n$ functions of $n$ variables. In the second section certain sufficient conditions are obtained for determining the character of the extremes of the several functions at a critical point. The third section is devoted to the discussion of a problem which possesses considerable interest in itself and which illustrates the application of the general criterion. This problem may be regarded as a generalization of the familiar rectangle problem‡ viz., to determine the shape of the rectangle of given perimeter and maximum area *or* to determine the shape of the rectangle of given area and minimum perimeter. The fact that these two problems have the same solution may be stated analytically in the form—If $xy = a$ is a maximum subject to the condition $x + y = b$ ($x$ and $y$ being assumed positive) then $x + y = b$ is a minimum subject to the condition $xy = a$—and in this form it serves best as an introduction to the general problem.

2. **General criterion.** Consider a set of $n$ functions of $n$ real variables,

$$u_i(x_1, x_2, \cdots, x_n), \qquad i = 1, 2, \cdots, n,$$

which at a certain point $P$, $(x_1', x_2', \cdots, x_n')$ satisfy the following hypotheses:

(1) Each function possesses partial derivatives of the first and second orders with regard to all its variables.

* Received June 10, 1926; presented to the American Mathematical Society September 9, 1926.

† "Reciprocity in a Problem of Relative Maxima and Minima", American Journal of Mathematics XLIII, 1921, pp. 271–290.

‡ Also referred to by Whittemore, l. c.

(2) The Jacobian of the $n$ functions vanishes

$$J \equiv \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0.$$

(3) The Jacobian has at least one column the cofactors of whose elements are all different from zero. There is no loss in generality in our supposing, as we shall, that the first column possesses this property.

Suppose now that the values of $n-1$ of the functions are computed at the point $P$,

$$u_i(x_1', x_2', \cdots, x_n') \equiv c_i, \qquad i = 2, 3, \cdots, n,$$

and consider the problem of finding the relative extremes of the function $u_1$ when the variables are subject to the conditions

$$u_i(x_1, x_2, \cdots, x_n) = c_i, \qquad i = 2, 3, \cdots, n. \tag{1}$$

Choose $x_1$ as the single independent variable. At a critical point the derivatives of all the functions with regard to $x_1$ vanish.

$$\sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} \cdot \frac{dx_j}{dx_1} = 0, \qquad i = 1, 2, \cdots, n.$$

For consistency of this set of equations in the derivatives $dx_j/dx_1$ the eliminant, which is the Jacobian of the $n$ functions, must be zero. It follows that the critical points are determined by the equation $J = 0$ together with the set of equations (1). Hence the point $P \equiv (x_1', x_2', \cdots, x_n')$ is one critical point. The character of this critical point (maximum or minimum) will be determined by the sign of the second derivative $d^2u_1/dx_1^2$. The calculation gives

$$\frac{d^2u_1}{dx_1^2} = \sum_{j,k=1}^{n} \frac{\partial^2 u_1}{\partial x_j \partial x_k} \cdot \frac{dx_j}{dx_1} \cdot \frac{dx_k}{dx_1} + \sum_{j=2}^{n} \frac{\partial u_1}{\partial x_j} \cdot \frac{d^2x_j}{dx_1^2}.$$

We shall use the abbreviation

$$U_i = \sum_{j,k=1}^{n} \frac{\partial^2 u_i}{\partial x_j \partial x_k} \cdot \frac{dx_j}{dx_1} \cdot \frac{dx_k}{dx_1}.$$

Then

$$\frac{d^2u_1}{dx_1^2} = U_1 + \sum_{j=2}^{n} \frac{\partial u_1}{\partial x_j} \cdot \frac{d^2x_j}{dx_1^2}.$$

The second derivatives of $u_2, u_3, \cdots, u_n$ have similar expressions and these vanish. This gives a set of $n$ equations,

$$
\begin{aligned}
U_1 - \frac{d^2 u_1}{dx_1^2} + \sum_{j=2}^{n} \frac{\partial u_1}{\partial x_j} \cdot \frac{d^2 x_j}{dx_1^2} &= 0, \\
U_2 \quad + \sum_{j=2}^{n} \frac{\partial u_2}{\partial x_j} \cdot \frac{d^2 x_j}{dx_1^2} &= 0, \\
U_3 \quad + \sum_{j=2}^{n} \frac{\partial u_3}{\partial x_j} \cdot \frac{d^2 x_j}{dx_1^2} &= 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
U_n \quad + \sum_{j=2}^{n} \frac{\partial u_n}{\partial x_j} \cdot \frac{d^2 x_j}{dx_1^2} &= 0
\end{aligned} \tag{2}
$$

of which the eliminant is

$$
\begin{vmatrix}
U_1 - \dfrac{d^2 u_1}{dx_1^2} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\
U_2 & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
U_n & \dfrac{\partial u_n}{\partial x_2} & \dfrac{\partial u_n}{\partial x_3} & \cdots & \dfrac{\partial u_n}{\partial x_n}
\end{vmatrix} = 0.
$$

This equation determines $\dfrac{d^2 u_1}{dx_1^2}$ viz.,

$$
\frac{d^2 u_1}{dx_1^2} = \frac{\begin{vmatrix}
U_1 & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\
U_2 & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\
\cdots & \cdots & \cdots & \cdots \\
U_n & \dfrac{\partial u_n}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_n}
\end{vmatrix}}{\begin{vmatrix}
\dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\dfrac{\partial u_n}{\partial x_2} & \dfrac{\partial u_n}{\partial x_3} & \cdots & \dfrac{\partial u_n}{\partial x_n}
\end{vmatrix}}
$$

which may be abbreviated,

$$
\frac{d^2 u_1}{dx_1^2} = \frac{D}{\left[\dfrac{\partial u_1}{\partial x_1}\right]},
$$

where $\left[\frac{\partial u_i}{\partial x_1}\right]$ denotes the cofactor of $\frac{\partial u_i}{\partial x_1}$ in $J$. The scope of the investigation will now be restricted by imposing a fourth hypothesis viz.,

(4) At the point $P$, the determinant $D$ is different from zero. It follows that $u_1$ has at the point $P$ an extreme whose value may be denoted by $c_1$,

$$u_1(x_1', x_2', \cdots, x_n') = c_1.$$

If now we propose the problem of minimizing the function $u_2$, the variables being subject to the $n-1$ conditions

$$u_i(x_1, x_2, \cdots, x_n) = c_i, \qquad i = 1, 3, 4, \cdots, n,$$

we get as one critical point, the point $P$. Moreover the derivatives $dx_i/dx_1$ $(i = 2, 3, \cdots, n)$ and therefore the functions $U_i$ have, at this point, the same values as in the previous problem. Also by analogy with the result already obtained (interchanging $u_1$ and $u_2$)

$$\frac{d^2 u_2}{dx_1^2} = \frac{\begin{vmatrix} U_2 & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ U_1 & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ U_n & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}}{\begin{vmatrix} \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \cdots & \frac{\partial u_3}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}} = \frac{D}{\left[\frac{\partial u_2}{\partial x_1}\right]}.$$

According to the hypotheses this derivative exists and is different from zero; consequently the value $u_2 = c_2$ is a maximum or a minimum.

The same method applied to the function $u_k$ shows that the value $c_k$ is an extreme of this function when the variables are subject to the conditions imposed by the $n-1$ equations

$$u_i = c_i, \qquad i = 1, 2, \cdots, k-1, k+1, \cdots, n.$$

Moreover the character of the extreme is determined by the sign of the quantity

$$\frac{D}{\left[\frac{\partial u_k}{\partial x_1}\right]}.$$

The expressions for the derivatives $d^2u_i/dx_1^2$ which test the character of the extremes at $(x_1', x_2', \cdots, x_n')$ in the $n$ separate problems have the same numerators. Their relative signs therefore depend on the denominators which are the cofactors of the elements in the first column of the Jacobian $J$. The character of the extremes will be the same for those functions for which the corresponding cofactors have the same sign.

3. **Example.** The rectangle problem mentioned in the introduction is an illustration of the foregoing theory when applied to the two functions of two variables representing the perimeter and area of a rectangle. A generalization in three dimensions would be concerned with three functions representing the sum of the edges, the surface and the volume of a rectangular parallelepiped. The generalization may evidently be carried to $n$ dimensions. As, however, the extension is direct we shall, in order to avoid the cumbersome notation of $n$ variables, give the analysis for the case $n = 4$. Consider then the four symmetric sums,

$$\begin{aligned} u_1 &= x+y+z+t, \\ u_2 &= xy+xz+xt+yz+yt+zt, \\ u_3 &= xyz+xyt+xzt+yzt, \\ u_4 &= xyzt. \end{aligned}$$

Equate to zero the Jacobian of these functions

$$J \equiv \begin{vmatrix} 1 & 1 & 1 & 1 \\ y+z+t & x+z+t & x+y+t & x+y+z \\ yz+yt+zt & xz+xt+zt & xy+xt+yt & xy+xz+yz \\ yzt & xzt & xyt & xyz \end{vmatrix} = 0.$$

As an equation in $x$ this is a cubic with the three roots $x = y$, $x = z$, $x = t$. If therefore we select arbitrarily different values for $y$, $z$ and $t$ and then for $x$ choose any one of these values, say $x = y$, the Jacobian will vanish. For this set of values of the variables let the values $c_1$, $c_2$, $c_3$, $c_4$ of the successive functions be calculated. Provided that the determinant $D$ of section 2 does not vanish at the point chosen, each of these constants is a relative extreme for its function subject to the conditions

imposed by the other three equations $u_i = c_i$. For a general point $(x, y, z, t)$ the determinant $D$ has the following value, where primes denote differentiation with respect to $x$,

$$2\begin{vmatrix} 0 & 1 & 1 & 1 \\ y'+z'+t'+y'z'+y't'+z't' & x+z+t & x+y+t & x+y+z \\ (z+t)y'+(y+t)z'+(y+z)t' \\ +(x+t)y'z'+(x+z)y't' \\ +(x+y)z't' & xz+xt+zt & xy+xt+yt & xy+xz+yz \\ zty'+ytz'+yzt'+xzy't' \\ +xyz't'+xty'z' & xzt & xyt & xyz \end{vmatrix}.$$

At the critical point the derivatives of $u_1$, $u_2$, $u_3$, $u_4$ vanish; moreover $x = y$. This leads to the following simplifications respecting the elements of the first column

$$\begin{aligned} y'+z'+t' &= -1, \\ (z+t)y'+(y+t)z'+(y+z)t' &= -z-t, \\ zty'+ytz'+yzt' &= -zt. \end{aligned}$$

Hence at the critical point

$$D = 2\begin{vmatrix} 0 & 1 & 1 & 1 \\ -1+y'z'+y't'+z't' & y+z+t & 2y+t & 2y+z \\ -z-t+(y+t)y'z' \\ +(y+z)y't'+2yz't' & yz+yt+zt & y^2+2yt & y^2+2yz \\ -zt+yzy't'+y^2z't' \\ +yty'z' & yzt & y^2t & y^2z \end{vmatrix}$$

which, by simplification and expansion reduces to the form

$$D = 2(y-z)^2(y-t)^2(z-t).$$

Since $y$, $z$ and $t$ are three different numbers it follows that $D \neq 0$ at the point in question. Hence, subject to the auxiliary conditions already explained, each function $u_i$ has a relative extreme $c_i$ at this point.

The investigation of the character of these extremes requires a knowledge of the cofactors of the elements of the first column of $J$. These are the 3-rowed determinants with proper signs, of the matrix,

$$\begin{Vmatrix} 1 & 1 & 1 \\ y+z+t & 2y+t & 2y+z \\ yz+yt+zt & y^2+2yt & y^2+2yz \\ yzt & y^2t & y^2z \end{Vmatrix}.$$

Denote by $D_1$, $D_2$, $D_3$, $D_4$ the determinants obtained by omitting in order the 1st, 2nd, 3rd and 4th rows and affixing the signs $+$, $-$, $+$, $-$.

$$D_4 = -\begin{vmatrix} 1 & 1 & 1 \\ y+z+t & 2y+t & 2y+z \\ yz+yt+zt & y^2+2yt & y^2+2yz \end{vmatrix}$$

which by an easy reduction becomes

$$-\begin{vmatrix} 1 & 1 & 1 \\ z+t & y+t & y+z \\ zt & yt & yz \end{vmatrix} = -\Phi,$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ y+z+t & 2y+t & 2y+z \\ yzt & y^2t & y^2z \end{vmatrix} = y\Phi,$$

$$D_2 = -\begin{vmatrix} 1 & 1 & 1 \\ yz+yt+zt & y^2+2yt & y^2+2yz \\ yzt & y^2t & y^2z \end{vmatrix} = -y^2\Phi,$$

$$D_1 = \begin{vmatrix} y+z+t & 2y+t & 2y+z \\ yz+yt+zt & y^2+2yt & y^2+2yz \\ yzt & y^2t & y^2z \end{vmatrix} = y^3\Phi.$$

$\Phi \neq 0$ since $y$, $z$ and $t$ are three different numbers. Hence the four cofactors are proportional to $y^3$, $-y^2$, $y$, $-1$. It follows that, if $y$ is positive, the signs of these cofactors alternate while, if $y$ is negative, the signs are all the same. In the former case maxima and minima alternate for the series of functions at the point in question while in the latter case the extremes are all of one kind.

It is clear that the method applies to the case of $n$ variables $x_1, x_2, \cdots, x_n$ and the result will be merely stated. Form the $n$ symmetric sums

$$
\begin{aligned}
u_1 &= \sum x_i, \\
u_2 &= \sum x_i\, x_j && (i \neq j), \\
u_3 &= \sum x_i\, x_j\, x_k, && (i, j, k \text{ all different}), \\
&\cdots\cdots \\
u_n &= x_1\, x_2 \cdots x_n.
\end{aligned}
$$

Let $n-1$ *different* constant values be assigned to the $n$ variables, two of the variables assuming the same value, and let the values of the functions be calculated for this point

$$u_i = c_i, \qquad i = 1, 2, \cdots, n.$$

Then the constant $c_i$ is a relative extreme for the function $u_i$ subject to to the conditions imposed by the other $n-1$ equations. Moreover, if the one value which is assigned to two variables is positive, the extremes of the $u$'s will be alternately maxima and minima while, if this value is negative, the extremes will be all maxima or all minima.

It follows that if the character of any one of the extremes mentioned in the theorem just stated is known the character of all the extremes is known. There remains to be solved the problem of determining the character of a particular extreme. We shall therefore direct attention to the last function $u_n$ and examine its variation by a method which throws additional light on the problem.

If the numbers $x_1, x_2, \cdots, x_n$ are regarded as the roots of an algebraic equation of the $n$th degree, then the functions $u_i$ are the elementary symmetric functions of these roots. Hence if

$$u_i = c_i, \qquad i = 1, 2, \cdots, n-1, \tag{3}$$

then the $x$'s must satisfy the equation

$$x^n - c_1 x^{n-1} + c_2 x^{n-2} - \cdots + (-1)^{n-1} c_{n-1} x + (-1)^n u_n = 0. \tag{4}$$

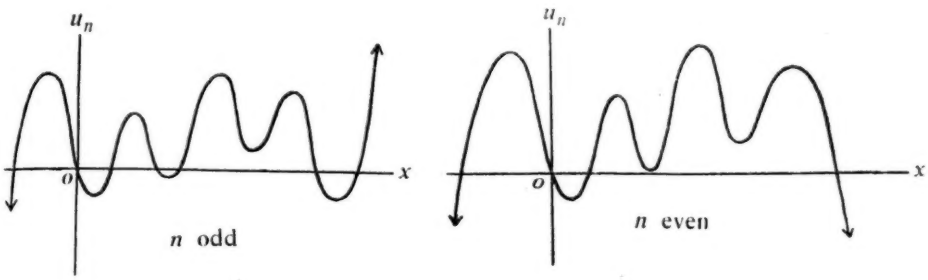
Hence under the conditions (3) $u_n$ is a function of the single variable $x$, viz.,

$$u_n = (-1)^{n-1} \left\{ x^n - c_1 x^{n-1} + c_2 x^{n-2} - \cdots + (-1)^{n-1} c_{n-1} x \right\}.$$

The extremes of $u_n$ require that $du_n/dx = 0$, in other words that equation (4) and its derived equation should have a common root, which leads again to the condition that at least two of the $n$ values of $x$ must be equal.* If

* This method of deriving the condition was pointed out to me by Prof. C. F. Gummer.

the graph of $u_n$ be plotted against $x$ then according as $n$ is odd or even the curve may be represented schematically as follows.




In case $n$ is odd $u_n$ has at most $\frac{n-1}{2}$ relative maxima and $\frac{n-1}{2}$ relative minima, in case $n$ is even at most $\frac{n}{2}$ relative maxima and $\frac{n}{2}-1$ relative minima. The condition that to a given value of $u_n$, say $u_n = a$, should correspond $n$ real values of $x$ is the condition that the line $u_n = a$ should cut the graph in $n$ real points. This will be true only when there are $n-1$ distinct extremes and when the smallest relative maximum is greater than the greatest relative minimum. If this be the case, then the smallest relative maximum of $u_n$ is the absolute maximum of $u_n$, if the condition be imposed that all the variables $x_1, x_2, \cdots, x_n$ are real. Again, if this condition holds, let $x = x_1'$ be the value of $x$ which gives the smallest relative maximum and let a horizontal tangent be drawn to the curve at this point. This line will cut the curve in $n-2$ other distinct points, whose abscissas may be denoted by $x_3', x_4', \cdots, x_n'$. The set of values $x_1', x_2' = x_1', x_3', x_4', \cdots, x_n'$ for the $n$ original variables furnish the maximum value of $u_n$ subject to equations (3). A similar method gives the values of the $n$ variables which make $u_n$ a minimum.

If, then, a set of real values of the variables $x_i$ (two and only two variables being equal) is known to furnish an extreme of $u_n$, the character of the extreme can be determined by observing the position in which the two equal values occur when the $n$ values are arranged in order of magnitude. For the first relative extreme (counting from the left) of $u_n$ is in every case a maximum, hence the abscissa corresponding to any relative maximum (minimum) will occur in an odd (even) position in the series of abscissas for which $u_n$ has this value. It follows that the statement of the foregoing theorem regarding the $n$ symmetric sums may be completed as follows: If the value which is assigned to two variables occupies an odd (even) position, counting from the smallest, in the order of magnitude of the $n-1$ constants then the extreme of $u_n$ is a maximum (minimum).

This completed theorem, in case the variables are all positive, may be phrased in geometrical language, the function $u_i$ representing essentially the $i$-dimensional hypersurface content of a rectangular parallelepiped in $n$ dimensions. In this sense it is a generalization of the rectangle problem of two dimensions.

It may not be amiss to add a numerical illustration of the 3-dimensional case. If $x, y, z$ denote the edges of a rectangular box then the functions

$$\begin{aligned} u &= 4(x+y+z), \\ v &= 2(xy+yz+zx), \\ w &= xyz \end{aligned}$$

denote the sum of the edges, the surface and the volume. If $w$ be extremized under the conditions $u = 96$, $v = 360$ then, regarded as a function of one edge $x$,

$$w = x^3 - 24x^2 + 180x,$$

and the extremizing values are $x = 6$ and $x = 10$. If $x = 6$ the 3 edges are 6, 6, 12 for which $w = 432$. If $x = 10$ the 3 edges are 10, 10, 4 for which $w = 400$. The general theorem then leads to the following conclusion in which the term "box" is understood to mean rectangular parallelepiped. A box $6\times6\times12$ has a greater volume than any other box with the same surface and sum of edges, a smaller surface than any other box with the same volume and sum of edges and a greater sum of edges than any other box with the same surface and volume. A corresponding statement in which "greater" and "smaller" are interchanged applies to the box whose dimensions are $10\times10\times4$.

QUEENS UNIVERSITY.
January, 1926.

# ON FREDHOLM'S INTEGRAL EQUATIONS, WHOSE KERNELS ARE ANALYTIC IN A PARAMETER.*

By J. D. Tamarkin.

**1.** The reciprocal of the kernel $K(x, \xi)$ of a Fredholm's integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u(\xi)\, d\xi$$

is the function $\mathfrak{R}(x, \xi, \lambda)$ which is determined (uniquely) by the equations

$$\mathfrak{R}(x, \xi, \lambda) + K(x, \xi) = \lambda \int_a^b \mathfrak{R}(x, s, \lambda)\, K(s, \xi)\, ds$$
$$= \lambda \int_a^b K(x, s)\, \mathfrak{R}(s, \xi, \lambda)\, ds.$$

After Fredholm, it is well known that, under certain restrictions upon $K(x, \xi)$, the reciprocal $\mathfrak{R}(x, \xi, \lambda)$ is meromorphic in $\lambda$.

This fact, in the symmetric case, has been proved by Hilbert†, the only assumption being, that the integral

$$\int_a^b \int_a^b |K(x, \xi)|^2\, dx\, d\xi$$

exists. The asymmetric case required some further restrictions as to the kernel $K(x, \xi)$.

As far as we are aware, it was Hobson,‡ who carried through Fredholm's method under the most general suppositions concerning the kernel $K(x, \xi)$, namely either

(a) The successive iterated kernels

$$K_1(x, \xi) \equiv K(x, \xi),$$

$$K_2(x, \xi) \equiv \int_a^b K(x, s)\, K(s, \xi)\, ds, \quad \cdots, \quad K_{n-1}(x, \xi) \equiv \int_a^b K(x, s)\, K_{n-2}(s, \xi)\, ds$$

are unlimited, but the kernel $K_n(x, \xi)$ is limited on the square

$$a \leqq x \leqq b; \qquad a \leqq \xi \leqq b$$

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* Received August 22, 1926.

† *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Gött. Nachr. 1906, pp. 157–227 (vierte Mitteilung) = monograph under the same title, Leipzig und Berlin, B. G. Teubner, 1912, pp. 109–156.

‡ *On the linear integral equation*, Proc. Lond. Math. Soc., (2) 13 (1914), pp. 307–340.

and the integrals

$$\int_a^b |K(x, \xi)|\, d\xi, \qquad \int_a^b K|(x, \xi)|\, dx,$$
$$\int_a^b |K_2(x, \xi)|\, dx, \quad \cdots, \quad \int_a^b |K_{n-1}(x, \xi)|\, dx$$

exist and are bounded on $(a, b)$; or else

(b) The kernel $K(x, \xi)$ is of the form:

$$K(x, \xi) = \mu(x)\, \nu(\xi) \frac{Q(x, \xi)}{|x - \xi|^{\alpha}},$$

where $\alpha < \frac{1}{2}$, $Q(x, \xi)$ is bounded, and $\mu(x)$, $\nu(x)$, if not bounded, are such that the product $\mu(x)\, \nu(x)$ is absolutely integrable on $(a, b)$.

Either of Hobson's suppositions (a), (b) in some respects is more general, but in other respects less general than the Hilbert's condition above. The situation may change essentially if, instead of the classical Fredholm's equation, we have a more general one:

$$u(x) = f(x) + \int_a^b K(x, \xi, \lambda)\, u(\xi)\, d\xi, \tag{1}$$

where the kernel $K(x, \xi, \lambda)$ is analytic in $\lambda$. The reciprocal of the kernel $K(x, \xi, \lambda)$, which now is determined by the equations:

$$\begin{aligned} \Re(x, \xi, \lambda) + K(x, \xi, \lambda) &= \int_a^b \Re(x, s, \lambda)\, K(s, \xi, \lambda)\, ds \\ &= \int_a^b K(x, s, \lambda)\, \Re(s, \xi, \lambda)\, ds \end{aligned} \tag{2}$$

may not exist at all, for instance, in the case of the equation

$$u(x) = f(x) + [\varphi(x) + \lambda \psi(x)] \int_a^b \varphi(\xi)\, u(\xi)\, d\xi; \qquad \int_a^b [\varphi(x)]^2\, dx = 1;$$
$$\int_a^b \varphi(x)\, \psi(x)\, dx = 0.$$

The reciprocal cannot exist here and the homogeneous equation

$$u(x) = \int_a^b K(x, \xi, \lambda)\, u(\xi)\, d\xi,$$

for all values of $\lambda$, possesses the evident solution

$$u(x) = \varphi(x) + \lambda \psi(x).$$

A slight modification in Hobson's conditions and arguments is sufficient, however, to prove that the reciprocal of the kernel $K(x, \xi, \lambda)$ is meromorphic in $\lambda$, *if only it exists for at least one value of* $\lambda$.

The purpose of this note is to establish the same result under a condition which reduces to that of Hilbert in the case of the kernel $\lambda K(x, \xi)$, and also to discuss a more general case when the kernel $K(x, \xi, \lambda)$ is itself meromorphic in $\lambda$. The line of attack is quite different from that of Fredholm and Hobson; our method is based upon the ideas of Hilbert* and H. von Koch,† who applied the method of infinite determinants to Hilbert's equations.

**2.** First, let us indicate the formal part of the method. We denote by

$$\omega_1(x),\ \omega_2(x),\ \cdots,\ \omega_n(x),\ \cdots \tag{3}$$

any closed set of functions, orthogonal and normalized on $(a, b)$. If $f(x)$ and $\varphi(x)$ are any two functions which are integrable together with their squares on $(a, b)$ and if

$$f_n = \int_a^b f(x)\,\omega_n(x)\,dx, \qquad \varphi_n = \int_a^b \varphi(x)\,\omega_n(x)\,dx$$

denote the Fourier's coefficients of $f(x)$ and $\varphi(x)$ with respect to the set (3), then we have the so called Parseval's formula:

$$\int_a^b f(x)\,\varphi(x)\,dx = \sum_{n=1}^{\infty} f_n\,\varphi_n.$$

The inequality

$$\sum_{n=1}^{\infty} |f_n| \cdot |\varphi_n| \leqq \sqrt{\int_a^b |f(x)|^2\,dx \cdot \int_a^b |\varphi(x)|^2\,dx}$$

shows that the series of the right-hand member is absolutely convergent. If the function $\varphi(x)$ (but not $f(x)$) depends on any number of parameters the integral

$$\int_a^b |\varphi(x)|^2\,dx$$

* Grundzüge etc., Fünfte Mitteilung, Gött. Nachr. 1906, pp. 439–480 = monograph, pp. 174–185.

† *Sur la convergence des déterminants infinis,* Rend. del Circolo Mat. di Palermo, 28, 1909, pp. 255–266 (263–264). The method of H. von Koch has been used by J. Mollerup, in order to express the Fredholm's determinant of the iterated kernel $K_2(x, \xi)$ in form of an infinite determinant: *Sur l'identité du déterminant de Fredholm et d'un déterminant infini de v. Koch,* Bull. des Sci. Math. (2) 36, 1912, pp. 130–136. Under more special suppositions the same has been done by J. Marty: *Transformation d'un déterminant infini en un déterminant de Fredholm,* Bull. des Sci. Math. (2) 33, 1909, pp. 296–300.

being uniformly bounded, then the series $\sum_{n=1}^{\infty} f_n \varphi_n$ converges uniformly. Now we set

$$(4)\quad \begin{cases} k_i(x,\lambda) = \int_a^b K(x,\xi,\lambda)\,\omega_i(\xi)\,d\xi, \qquad k_i'(\xi,\lambda) = \int_a^b K(x,\xi,\lambda)\,\omega_i(x)\,dx, \\ k_{ij}(\lambda) = \int_a^b \int_a^b K(x,\xi,\lambda)\,\omega_i(\xi)\,\omega_j(x)\,d\xi\,dx \\ \qquad = \int_a^b k_j'(\xi,\lambda)\,\omega_i(\xi)\,d\xi = \int_a^b k_i(x,\lambda)\,\omega_j(x)\,dx, \\ \mathfrak{k}_i(x,\lambda) = \int_a^b \mathfrak{K}(x,\xi,\lambda)\,\omega_i(\xi)\,d\xi. \end{cases}$$

Then by virtue of Parseval's formula

$$\int_a^b \mathfrak{K}(x,s,\lambda)\,K(s,\xi,\lambda)\,ds = \sum_{j=1}^{\infty} k_j'(\xi,\lambda)\,\mathfrak{k}_j(x,\lambda).$$

Multiplying (2) by $\omega_i(\xi)$ and integrating over $(a, b)$, we get a system of equations for the unknown coefficients $\mathfrak{k}_i(x,\lambda)$:

$$(5)\qquad \mathfrak{k}_i(x,\lambda) + k_i(x,\lambda) = \sum_{j=1}^{\infty} k_{ij}(\lambda)\,\mathfrak{k}_j(x,\lambda).$$

The problem now is to solve this system and to prove that the coefficients $\mathfrak{k}_i(x,\lambda)$ determine a function, which satisfies the equations (2).

**3.** The system (5) will be solved by absolutely convergent infinite determinants.* We shall use the following sufficient condition for the simultaneous absolute convergence of the infinite determinant

$$(6)\quad \boldsymbol{A} = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots \\ A_{21} & A_{22} & \cdots & A_{2j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{i1} & A_{i2} & \cdots & A_{ij} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \qquad A_{ij} = \delta_{ij} + a_{ij}, \qquad \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j, \end{cases}$$

together with its minors of all orders:

(•) *The series*

$$(7)\qquad \sum_{i,j=1}^{\infty} |a_{ij}|^2, \qquad \sum_{i=1}^{\infty} |a_{ii}|$$

*are convergent.*

All the "usual" rules of the theory of finite determinants can be applied to infinite determinants.

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* See footnote † p. 129, and also: *Sur un théorème de M. Hilbert*, Math. Ann. 69 (1910), pp. 266–283. F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, Paris, Gauthier-Villars, 1913. We give below some necessary details and complements.

Let $\boldsymbol{A}_{ij}$ denote the cofactor of the element $A_{ij}$ in the determinant $\boldsymbol{A}$. If we set

$$\boldsymbol{A}^{(n)} = \begin{vmatrix} A_{11} & \cdots & A_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ A_{n1} & \cdots & A_{nn} \end{vmatrix}$$

and denote by $\boldsymbol{A}_{ij}^{(n)}$ the cofactor of the element $A_{ij}$ in $\boldsymbol{A}^{(n)}$, then

$$\boldsymbol{A}^{(n)} \to \boldsymbol{A}, \qquad \boldsymbol{A}_{ij}^{(n)} \to \boldsymbol{A}_{ij} \quad \text{as} \quad n \to \infty.$$

It has been proved by Hadamard* that if

$$\Delta = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is any determinant of the $n$th order and if

$$s_i^2 = \sum_{r=1}^{n} |a_{ir}|^2, \qquad \sigma_j^2 = \sum_{r=1}^{n} |a_{rj}|^2 \qquad (i, j = 1, 2, \cdots, n),$$

then

$$|\Delta|^2 \leqq s_1^2 s_2^2 \cdots s_n^2 \quad \text{and} \quad |\Delta|^2 \leqq \sigma_1^2 \sigma_2^2 \cdots \sigma_n^2.$$

In our case this gives:

$$(8) \qquad |\boldsymbol{A}^{(n)}|^2 \leqq \left\{ \begin{array}{l} \displaystyle\prod_{j=1}^{n} \sum_{i=1}^{n} |A_{ij}|^2 \leqq \prod_{j=1}^{\infty} \left\{ 1 + 2|a_{jj}| + \sum_{i=1}^{\infty} |a_{ij}|^2 \right\} \\ \displaystyle\prod_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}|^2 \leqq \prod_{i=1}^{\infty} \left\{ 1 + 2|a_{ii}| + \sum_{j=1}^{\infty} |a_{ij}|^2 \right\} \end{array} \right\} \leqq \boldsymbol{\Pi}^2,$$

where $\boldsymbol{\Pi}$ is a positive constant which depends only on the values of the series (7). In the same way:

$$(9) \quad |\boldsymbol{A}_{ij}^{(n)}|^2 \leqq \boldsymbol{\Pi}^2 \sum_{k \neq j}^{\infty} |A_{jk}|^2 = \boldsymbol{\Pi}_{ij}'^2; \quad |\boldsymbol{A}_{ij}^{(n)}|^2 \leqq \boldsymbol{\Pi}^2 \sum_{k \neq i}^{\infty} |A_{ki}|^2 = \boldsymbol{\Pi}_{ij}''^2 \quad (i \neq j).$$

Hence

$$(10) \qquad |\boldsymbol{A}| \leqq \boldsymbol{\Pi}, \quad |\boldsymbol{A}_{ii}| \leqq \boldsymbol{\Pi}, \quad |\boldsymbol{A}_{ij}| \leqq \boldsymbol{\Pi}_{ij}', \quad |\boldsymbol{A}_{ij}| \leqq \boldsymbol{\Pi}_{ij}'' \quad (i \neq j).$$

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* *Résolution d'une question relative aux déterminants*, Bull. de Sci. Math. (2) 17, 1893, pp. 240–246. A very simple proof of Hadamard's theorem, in the case of complex $a_{ij}$, is given in the book by G. Vivanti, *Elementi della teoria delle equazioni integrali lineari*, U. Hoepli, Milano, 1916, pp. 41–44.

The inequalities (10) assure the absolute convergence of any determinant which is obtained from $\boldsymbol{A}$ by replacing any row or any column by a sequence

$$a_1, a_2, \cdots, a_n, \cdots$$

for which the sum $\sum_{n=1}^{\infty} |a_n|^2$ is convergent. The same argument proves further that the bordered determinant

$$(11) \qquad \boldsymbol{A}_{x,y} \equiv \begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_j & \cdots \\ y_1 & A_{11} & A_{12} & \cdots & A_{1j} & \cdots \\ y_2 & A_{21} & A_{22} & \cdots & A_{2j} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_i & A_{i1} & A_{i2} & \cdots & A_{ij} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = -\sum_{i,j=1}^{\infty} \boldsymbol{A}_{ij}\, y_i\, x_j$$

is also absolutely convergent, if the series

$$\sum_{j=1}^{\infty} |x_j|^2, \qquad \sum_{i=1}^{\infty} |y_i|^2$$

are convergent. Hence the double series of the right-hand member is absolutely convergent. The inequalities (8), being applied to the determinants $\boldsymbol{A}_{x,y}$ and

$$\boldsymbol{A}^{(n)}_{x,y} \equiv \begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_n \\ y_1 & A_{11} & A_{12} & \cdots & A_{1n} \\ y_2 & A_{21} & A_{22} & \cdots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y_n & A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}$$

show:

$$(12) \qquad \left\{ \begin{matrix} |\boldsymbol{A}^{(n)}_{x,y}|^2 \\ |\boldsymbol{A}_{x,y}|^2 \end{matrix} \right\} \leqq \begin{cases} \sum_{j=1}^{\infty} |x_j|^2 \prod_{i=1}^{\infty} \left\{ |y_i|^2 + \sum_{j=1}^{\infty} |A_{ij}|^2 \right\} \\ \sum_{i=1}^{\infty} |y_i|^2 \prod_{j=1}^{\infty} \left\{ |x_j|^2 + \sum_{i=1}^{\infty} |A_{ij}|^2 \right\} \end{cases}$$

Hence the determinant $\boldsymbol{A}_{x,y}$, as a linear form in either of two sets of quantities

$$(13) \qquad x_1, x_2, \cdots, x_j, \cdots; \qquad y_1, y_2, \cdots, y_i, \cdots$$

satisfies the inequalities:

$$(14) \qquad |\boldsymbol{A}_{x,y}|^2 \leq \sum_{j=1}^{\infty} |x_j|^2 \cdot M_1(y); \qquad |\boldsymbol{A}_{x,y}|^2 \leq \sum_{i=1}^{\infty} |y_i|^2 \cdot M_2(x),$$

where $M_1(y)$ does not depend on $x_1, x_2, \cdots$ and $M_2(x)$ does not depend on $y_1, y_2, \cdots$.

Any linear form

$$l(z) \equiv \sum_{\nu=1}^{\infty} l_\nu z_\nu \tag{15}$$

in infinitely many variables $z_1, z_2, \cdots, z_\nu, \cdots$ is called bounded (German: beschränkt), if, the sequence $z_1, z_2, \cdots$ being arbitrary, we have for all $n$:

$$\left| \sum_{\nu=1}^{n} l_\nu z_\nu \right| \leqq M \sqrt{\sum_{\nu=1}^{n} |z_\nu|^2},$$

where $M$ is a positive constant which does not depend on $n$. It is well known* that a necessary and sufficient condition that the linear form (15) be bounded is that the series $\sum_{\nu=1}^{\infty} |l_\nu|^2$ be convergent; in this case

$$\sum_{\nu=1}^{\infty} |l_\nu|^2 \leqq M^2.$$

Consequently $A_{x,y}$ is a bounded linear form in either of two sets (13); this yields the convergence of either of the series

$$\sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} A_{ij} x_j \right|^2 \leqq \prod_{j=1}^{\infty} \left[ |x_j|^2 + \sum_{i=1}^{\infty} |A_{ij}|^2 \right];$$

$$\sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} A_{ij} y_i \right|^2 \leqq \prod_{i=1}^{\infty} \left[ |y_i|^2 + \sum_{j=1}^{\infty} |A_{ij}|^2 \right],$$

for any two sequences (13) with convergent sums $\sum_{j=1}^{\infty} |x_j|^2$, $\sum_{i=1}^{\infty} |y_i|^2$.

**4.** We are now going to consider a system of infinitely many equations in infinitely many unknowns:

$$\sum_{j=1}^{\infty} A_{ij} \zeta_j = c_i, \qquad (i = 1, 2, \cdots), \tag{16}$$

where the matrix

$$\begin{pmatrix} A_{11} \cdots A_{1n} \cdots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{pmatrix}$$

satisfies the condition (*).

* Cfr. E. Hellinger and O. Toeplitz, *Grundlagen für eine Theorie der unendlichen Matrizen*, Math. Ann. 69, 1910, pp. 289–330 (294–295).

By a "solution" of this system is meant any sequence

$$\zeta_1, \zeta_2, \cdots, \zeta_\nu, \cdots$$

for which the sum

$$\sum_{i=1}^{\infty} |\zeta_i|^2 \tag{17}$$

converges and which satisfies the equations (16).

It is readily seen that the convergence of the series

$$\sum_{i=1}^{\infty} |c_i|^2 \tag{18}$$

is *necessary* for the existence of a solution of (16); this is shown by the inequalities, easy to prove:*

$$|c_i| \leqq \sum_{j=1}^{\infty} |A_{ij}\,\zeta_j| \leqq |\zeta_i| + \sum_{j=1}^{\infty} |a_{ij}\,\zeta_j|,$$

$$|c_i|^2 \leqq |\zeta_i|^2 + \sum_{j=1}^{\infty} |\zeta_j|^2 \sum_{j=1}^{\infty} |a_{ij}|^2 + 2|\zeta_i| \sum_{j=1}^{\infty} |a_{ij}\,\zeta_i|,$$

$$\sum_{i=1}^{\infty} |c_i|^2 \leqq \sum_{i=1}^{\infty} |\zeta_i|^2 \left[1 + \sum_{i,j=1}^{\infty} |a_{ij}|^2\right] + 2\sqrt{\sum_{i=1}^{\infty} |\zeta_i|^2 \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}\,\zeta_j|\right)^2}$$

$$\leqq \sum_{i=1}^{\infty} |\zeta_i|^2 \left[1 + \sqrt{\sum_{i,j=1}^{\infty} |a_{ij}|^2}\right]^2.$$

We have to consider two cases:

1°. Suppose that the determinant $\boldsymbol{A}$ of the system (16) is $\neq 0$. In this case the convergence of the series (18) is also *sufficient* for the existence of a solution of (16) which is uniquely determined by the formulas:

$$\zeta_i = \frac{\boldsymbol{A}_i}{\boldsymbol{A}} \qquad (i = 1, 2, \cdots) \tag{19}$$

where

$$\boldsymbol{A}_i = \sum_{j=1}^{\infty} \boldsymbol{A}_{ji}\, c_j \tag{20}$$

denotes the determinant which is obtained from $\boldsymbol{A}$ by replacing the elements of the $i$th column by the sequence $c_1, c_2, \cdots$. It is obvious from what has been said before, that the series (20) is absolutely convergent, as well as the series

$$\sum_{i=1}^{\infty} |\zeta_i|^2 = \frac{1}{|\boldsymbol{A}|^2} \sum_{i=1}^{\infty} \left|\sum_{j=1}^{\infty} \boldsymbol{A}_{ji}\, c_j\right|^2.$$

* This may be done by a repeated application of Cauchy's inequality:

$$\left|\sum \alpha_\nu \beta_\nu\right|^2 \leqq \sum |\alpha_\nu|^2 \sum |\beta_\nu|^2.$$

This shows that the substitution of the expressions (19) in (16) yields a determinant of the type $\boldsymbol{\Delta}_{x,y}$, where $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ are replaced by $A_{i1}, A_{i2}, \cdots$ and $c_1, c_2, \cdots$ respectively. Changing the order of summation, which is permissible because of the absolute convergence, we see at once:

$$\sum_{j=1}^{\infty} A_{ij}\,\zeta_j = \frac{1}{\boldsymbol{\Delta}} \sum_{j=1}^{\infty} A_{ij} \sum_{k=1}^{\infty} A_{kj}\,c_k = \sum_{k=1}^{\infty} c_k \cdot \frac{1}{\boldsymbol{\Delta}} \sum_{j=1}^{\infty} \boldsymbol{\Delta}_{kj}\,A_{ij} = c_i.$$

At the same time we obtain a limitation for the sum (17):

$$(21)\qquad \sum_{i=1}^{\infty} |\zeta_i|^2 = \frac{1}{|\boldsymbol{\Delta}|^2} \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} \boldsymbol{\Delta}_{ji}\,c_j \right|^2 \leqq \prod_{i=1}^{\infty} \left\{ |c_i|^2 + \sum_{j=1}^{\infty} |A_{ij}|^2 \right\} \leqq \frac{e^{a+c}}{|\boldsymbol{\Delta}|},$$

where

$$(22)\qquad a = \sum_{i,j=1}^{\infty} |a_{ij}|^2 + 2 \sum_{i=1}^{\infty} |a_{ii}|; \qquad c = \sum_{i=1}^{\infty} |c_i|^2$$

depend only on the values of the series (22). The inequality (21) seems to be new; it may present some importance in the applications of infinite determinants.

2°. Suppose now that $\boldsymbol{\Delta} = 0$. It has been proved by H. von Koch that the condition $\boldsymbol{\Delta} = 0$ is necessary and sufficient for the existence of a solution of the homogeneous system

$$(23)\qquad \sum_{j=1}^{\infty} A_{ij}\,\zeta_j = 0,$$

for which $\sum_{j=1}^{\infty} |\zeta_j|^2 \neq 0$. It has been proved also that, if $\boldsymbol{\Delta} = 0$, there exists such an integer $r$ that $\boldsymbol{\Delta}$ and all its minors of the orders $1, 2, \cdots (r-1)$ are zero, but at least one of the minors of the $r$th order is different from zero. In this case the homogeneous system (23) admits of exactly $r$ linearly independent solutions and the same is true of the transposed system

$$\sum_{j=1}^{\infty} A_{ji}\,\zeta_j = 0.$$

H. von Koch has found also the necessary and sufficient conditions for the compatibility of the nonhomogeneous system (16) (when $\boldsymbol{\Delta} = 0$) and has given the most general expression for the solution of the system (16).

**5.** A very important observation must now be made, concerning the case when the coefficients $A_{ij}$, $c_i$ of the system (16) are functions of a certain parameter $z$. Suppose that $A_{ij}(z)$, $c_i(z)$ are analytic in $z$ on an

open domain $(\mathfrak{D})$ of the $z$-plane. Suppose also that the condition $(*)$ is satisfied, that the series

$$\sum_{i=1}^{\infty} |c_i(z)|^2$$

is convergent on $(\mathfrak{D})$ and, which is essential, that the sums

$$\sum_{i,j=1}^{\infty} |a_{ij}(z)|^2, \qquad \sum_{i=1}^{\infty} |a_{ii}(z)|, \qquad \sum_{i=1}^{\infty} |c_i(z)|^2 \tag{24}$$

are bounded on any fixed closed subregion $(\mathfrak{D}_0)$ of $(\mathfrak{D})$. By virtue of (8), (9) the determinants $\boldsymbol{A}^{(n)}(z)$ and $\boldsymbol{A}_{ij}^{(n)}(z)$ are analytic on $(\mathfrak{D})$ and uniformly bounded on $(\mathfrak{D}_0)$, which shows, by virtue of an important theorem of Montel*, that the limiting functions $\boldsymbol{A}(z)$ and $\boldsymbol{A}_{ij}(z)$ are also analytic on $(\mathfrak{D})$ and that the convergence of $\boldsymbol{A}^{(n)}(z)$ to $\boldsymbol{A}(z)$ and of $\boldsymbol{A}_{ij}^{(n)}(z)$ to $\boldsymbol{A}_{ij}(z)$ is uniform on $(\mathfrak{D}_0)$. Consequently the solution $\zeta_i(z)$ of the system

$$\sum_{j=1}^{\infty} A_{ij}(z)\zeta_j(z) = c_i(z) \qquad (i = 1, 2, \cdots) \tag{25}$$

is meromorphic on $(\mathfrak{D})$, under the condition that $\boldsymbol{A}(z)$ is not identically zero. The poles of $\zeta_i(z)$, if there be any, must coincide with the roots of the denominator $\boldsymbol{A}(z)$. Suppose that $\boldsymbol{A}(z)$ is not identically zero; every fixed region $(\mathfrak{D}_0)$ contains but a finite number of the roots of $\boldsymbol{A}(z)$, and the functions $\zeta_i(z)$ are analytic on every closed subregion $(\mathfrak{D}_1)$ of $(\mathfrak{D}_0)$ which contains no roots of $\boldsymbol{A}(z)$. Finally, it is readily seen that the determinant $\boldsymbol{A}_{x,y}(z)$ is also analytic on $(\mathfrak{D}_1)$ and meromorphic on $(\mathfrak{D})$, under the condition that

$$x_1(z),\ x_2(z),\ \cdots; \qquad y_1(z),\ y_2(z),\ \cdots$$

are any two sequences, which are analytic on $(\mathfrak{D})$ and for which the sums

$$\sum_{i=1}^{\infty} |x_i(z)|^2, \qquad \sum_{i=1}^{\infty} |y_i(z)|^2 \tag{26}$$

are bounded on $(\mathfrak{D}_0)$.

**6.** Let us return to our problem. The following theorem will be established:

THEOREM 1. *Suppose the kernel $K(x, \xi, \lambda)$ of the integral equation*

$$u(x) = f(x) + \int_a^b K(x, \xi, \lambda)\, u(\xi)\, d\xi \tag{27}$$

*is analytic in $\lambda$ on an open domain $(\Lambda)$ of the $\lambda$-plane and for almost all points $(x, \xi)$ of the square:*

$$a \leqq x \leqq b; \qquad a \leqq \xi \leqq b.$$

* *Leçons sur les séries de polynomes a une variable complexe*, Paris, Gauthier-Villars, 1910, pp. 18–19.

*Suppose that the integrals*

$$\int_a^b |K(x, \xi, \lambda)|^2 d\xi, \qquad \int_a^b |K(x, \xi, \lambda)|^2 dx \tag{28}$$

*exist for almost all values of $x$ and $\xi$ respectively, and that, on every closed subregion $(\Lambda_0)$ of $(\Lambda)$:*

$$\int_a^b |K(x, \xi, \lambda)|^2 d\xi \leqq F_0(x), \qquad \int_a^b |K(x, \xi, \lambda)|^2 dx \leqq F_0(\xi), \tag{29}$$

*$F_0(x)$ being a positive function which depends only on $(\Lambda_0)$ and which is integrable on $(a, b)$.* Then the reciprocal $\mathfrak{K}(x, \xi, \lambda)$ of the kernel $K(x, \xi, \lambda)$ either is meromorphic on $(\Lambda)$ for almost all values of $x, \xi$, or does not exist at all.*

In the special case when $K(x, \xi, \lambda)$ is an integral function in $\lambda$, the reciprocal $\mathfrak{K}(x, \xi, \lambda)$, if it exists for at least one value of $\lambda$, must be equal to a quotient of two integral functions in $\lambda$. In the case of the classical Fredholm's equation the restrictions of Theorem 1 reduce to those of Hilbert (1).

**7.** The conditions of Theorem 1 are not sufficient to ensure the system (5) satisfying the condition (•), but they are sufficient if, in the system (5), the kernel $K(x, \xi, \lambda)$ is replaced by any kernel of the form:

$$K_0(x, \xi, \lambda) \equiv \int_a^b K^{(1)}(x, s, \lambda) K^{(2)}(s, \xi, \lambda) ds \tag{30}$$

where $K^{(1)}(x, \xi, \lambda)$, $K^{(2)}(x, \xi, \lambda)$ are any two kernels which satisfy the same conditions of integrability as the kernel $K(x, \xi, \lambda)$. This is the reason why we proceed to establish Theorem 1 for the kernel $K_0(x, \xi, \lambda)$ instead of for $K(x, \xi, \lambda)$. Under the conditions of Theorem 1 the double integrals $\int_a^b \int_a^b |K^{(i)}(x, \xi, \lambda)|^2 dx\, d\xi$ $(i = 1, 2)$ exist for all values of $\lambda$ in $(\Lambda)$ and, on $(\Lambda_0)$, we have:

$$\int_a^b \int_a^b |K^{(i)}(x, \xi, \lambda)|^2 dx\, d\xi \leqq N_0^* \qquad (i = 1, 2). \tag{31}$$

The values of $x$ and of $\xi$ for which the integrals

$$\int_a^b |K^{(i)}(x, \xi, \lambda)|^2 d\xi, \qquad \int_a^b |K^{(i)}(x, \xi, \lambda)|^2 dx \qquad (i = 1, 2) \tag{32}$$

* In what follows we designate by $F_0(x)$ any function of this type so that, for instance, instead of $cF_0(x)$, where $c$ is a constant, we shall write simply $F_0(x)$. The notation $N_0$ will be used in order to designate any constant, which is determined by locating the subregion $(\Lambda_0)$, but which does not depend on $\lambda$ in $(\Lambda_0)$.

do not exist, shall be called singular values, all other values of $x, \xi$ nonsingular. The singular values constitute a set of measure zero, and for all nonsingular values of $x, \xi$ the kernel $K_0(x, \xi, \lambda)$ certainly exists and is analytic in $\lambda$ on $(A)$. Only nonsingular values of $x, \xi$ shall be considered in what follows.

Using Schwarz's inequality it is readily seen:

$$(33)\quad \begin{cases} |K_0(x,\xi,\lambda)|^2 \leqq F_0(x)\,F_0(\xi), \\ \int_a^b |K_0(x,\xi,\lambda)|^2 d\xi \leqq F_0(x), \qquad \int_a^b |K_0(x,\xi,\lambda)|^2 dx \leqq F_0(\xi),\dagger \\ \int_a^b\int_a^b |K_0(x,\xi,\lambda)|^2 dx\,d\xi \leqq N_0 \text{ on } (A_0). \end{cases}$$

Let $\mathfrak{K}^*(x, \xi, \lambda)$ be the reciprocal of the kernel $K_0(x, \xi, \lambda)$. We set, as in **2**:

$$(34)\quad \begin{cases} K_i(x,\lambda) = \int_a^b K_0(x,\xi,\lambda)\,\omega_i(\xi)\,d\xi, \quad K_i'(\xi,\lambda) = \int_a^b K_0(x,\xi,\lambda)\,\omega_i(x)\,dx, \\ K_{ij}(\lambda) = \int_a^b\int_a^b K_0(x,\xi,\lambda)\,\omega_i(\xi)\,\omega_j(x)\,d\xi\,dx = \int_a^b K_j'(\xi,\lambda)\,\omega_i(\xi)\,d\xi \\ \qquad\qquad = \int_a^b K_i(x,\lambda)\,\omega_j(x)\,dx, \\ \mathfrak{k}_i^*(x,\lambda) = \int_a^b \mathfrak{K}^*(x,\xi,\lambda)\,\omega_i(\xi)\,d\xi, \quad \mathfrak{k}_i'^*(\xi,\lambda) = \int_a^b \mathfrak{K}^*(x,\xi,\lambda)\,\omega_i(x)\,dx. \end{cases}$$

The system of **2** is replaced by the system

$$(35)\qquad \mathfrak{k}_i^*(x,\lambda) + K_i(x,\lambda) = \sum_{j=1}^{\infty} K_{ij}(\lambda)\,\mathfrak{k}_j^*(x,\lambda), \quad (i = 1, 2, \cdots).$$

We denote

$$-K_{ij}(\lambda) \equiv a_{ij}(\lambda); \qquad K_i(x,\lambda) \equiv c_i(\lambda); \qquad -\mathfrak{k}_i^*(x,\lambda) \equiv \zeta_i(\lambda),$$

and proceed to test the conditions $(*)$ of **3**. By virtue of Parseval's formula

$$\sum_{i,j=1}^{\infty} |a_{ij}(\lambda)|^2 = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |K_{ij}(\lambda)|^2 = \sum_{j=1}^{\infty}\int_a^b |K_j'(\xi,\lambda)|^2 d\xi \ddagger$$
$$= \int_a^b d\xi \sum_{j=1}^{\infty} |K_j'(\xi,\lambda)|^2 = \int_a^b d\xi \int_a^b |K_0(x,\xi,\lambda)|^2\,dx \leqq N_0.$$

† See footnote * p. 137.

‡ We can invert the order of summation and integration by virtue of a well known property of Lebesgue's integrals: If a sequence of integrable functions $\varphi_1(x), \cdots, \varphi_n(x), \cdots$ converges almost everywhere on $(\alpha, \beta)$, and if $|\varphi_n(x)| \leqq \varphi(x)$, where $\varphi(x)$ is a positive integrable function which does not depend on $n$, then: $\int_\alpha^\beta \varphi_n(x)\,dx \to \int_\alpha^\beta \varphi(x)\,dx$, as $n \to \infty$. See, for instance, De la Vallée-Poussin, *Cours d'Analyse infinitésimale*, t. I (3rd ed.), 1914, p. 264.

The same Parseval's formula shows also:

$$\sum_{i=1}^{\infty} |\, c_i(\lambda)\,|^2 = \sum_{i=1}^{\infty} |\, K_i(x,\lambda)\,|^2 = \int_a^b |\, K_0(x,\xi,\lambda)\,|^2 d\xi \leqq F_0(x),$$

for all nonsingular values of $x$; finally, using Schwarz's inequality:

$$\begin{aligned}\sum_{i=1}^{n} |\, a_{ii}(\lambda)\,| &= \sum_{i=1}^{n} |\, K_{ii}(\lambda)\,| \\ &= \sum_{i=1}^{n} \left| \int_a^b \int_a^b \int_a^b K^{(1)}(x,s,\lambda)\, K^{(2)}(s,\xi,\lambda)\, \omega_i(x)\, \omega_i(\xi)\, dx\, d\xi\, ds \right| \\ \leqq \int_a^b ds \sqrt{\int_a^b |\, K^{(1)}(x,s,\lambda)\,|^2 dx \int_a^b |\, K^{(2)}(s,\xi,\lambda)\,|^2 d\xi} &\leqq \int_a^b F_0(s)\, ds = N_0.\end{aligned}$$

The conditions of **3–5** thus being satisfied, we see at once that the determinant $D_0(\lambda)$ of the system (35) is absolutely convergent, together with all its minors of all the orders, all these functions being analytic on $(A_0)$.

Suppose that $D_0(\lambda)$ is not identically zero. Let $(A_1)$ be any fixed closed subregion of $(A_0)$ which does not contain any root of $D_0(\lambda)$. The solution

$$\zeta_i(\lambda) = -\mathfrak{k}_i^*(x,\lambda) = \frac{D_i(x,\lambda)}{D_0(\lambda)} \tag{36}$$

of the system (35) is then analytic on $(A_1)$ and the sum

$$\sum_{i=1}^{\infty} |\, \zeta_i(\lambda)\,|^2 \tag{37}$$

is bounded on $(A_1)$, for fixed nonsingular value of $x$.

**8.** Suppose now that $D_0(\lambda) \neq 0$. By virtue of the Riesz-Fisher theorem,† there exists a function $\Phi(x,\xi,\lambda)$, which is uniquely determined, for almost all values of $\xi$, by the conditions:

$$\int_a^b \Phi(x,\xi,\lambda)\, \omega_i(\xi)\, d\xi = \mathfrak{k}_i^*(x,\lambda) \qquad (i = 1, 2, \cdots), \tag{38}$$

and for which

$$\sum_{i=1}^{\infty} |\, \mathfrak{k}_i^*(x,\lambda)\,|^2 = \int_a^b |\, \Phi(x,\xi,\lambda)\,|^2 d\xi.$$

We shall prove that this function $\Phi(x,\xi,\lambda)$ satisfies the equation

$$\Phi(x,\xi,\lambda) + K_0(x,\xi,\lambda) = \int_a^b \Phi(x,s,\lambda)\, K_0(s,\xi,\lambda)\, ds. \tag{39}$$

† Cfr. T. Lalesco, *Introduction à la théorie des équations intégrales*, pp. 95–96.

Applied to the functions $\Phi(x, \xi, \lambda)$, $K_0(x, \xi, \lambda)$ Parseval's formula shows:

$$\int_a^b \Phi(x, s, \lambda) K_0(s, \xi, \lambda)\, ds = \sum_{j=1}^{\infty} K_j'(\xi, \lambda) \mathfrak{k}_j^*(x, \lambda).$$

Multiplication by $\omega_i(\xi)$ and integration yield then:

$$\begin{aligned}\int_a^b \omega_i(\xi)\, d\xi \int_a^b \Phi(x, s, \lambda) K_0(s, \xi, \lambda)\, ds &= \int_a^b \omega_i(\xi)\, d\xi \sum_{j=1}^{\infty} K_j'(\xi, \lambda) \mathfrak{k}_j^*(x, \lambda) \\ &= \sum_{j=1}^{\infty} \mathfrak{k}_j^*(x, \lambda) \int_a^b K_j'(\xi, \lambda) \omega_i(\xi)\, d\xi \dagger \\ &= \sum_{j=1}^{\infty} K_{ij}(\lambda) \mathfrak{k}_j^*(x, \lambda) = \mathfrak{k}_i^*(x, \lambda) + K_i(x, \lambda),\end{aligned}$$

because of (35). Hence, the functions

$$\int_a^b \Phi(x, s, \lambda) K_0(s, \xi, \lambda))\, ds; \qquad \Phi(x, \xi, \lambda) + K_0(x, \xi, \lambda)$$

have the same Fourier's coefficients with respect to the closed set

$$\omega_1(\xi), \quad \omega_2(\xi), \quad \cdots, \quad \omega_i(\xi), \quad \cdots.$$

Consequently the equation (39) is satisfied for almost all values of $\xi$. The exceptional values of $\xi$ (which constitute a pointset of measure zero) shall be considered also as singular. If we set $\mathfrak{K}^*(x, \xi, \lambda) = \Phi(x, \xi, \lambda)$, the equation

$$(40) \qquad \mathfrak{K}^*(x, \xi, \lambda) = -K_0(x, \xi, \lambda) + \int_a^b \mathfrak{K}^*(x, s, \lambda) K_0(s, \xi, \lambda)\, ds$$

is thus established for all nonsingular values of $x$ and $\xi$, that is almost everywhere. The right-hand member of (40) may be written as follows:

$$\begin{aligned}&-K_0(x, \xi, \lambda) + \sum_{j=1}^{\infty} K_j'(\xi, \lambda) \mathfrak{k}_j^*(x, \lambda) \\ (41) \qquad &= \frac{1}{D(\lambda)} \begin{vmatrix} -K_0(x, \xi, \lambda) & K_1'(\xi, \lambda) & K_2'(\xi, \lambda) & \cdots \\ K_1(x, \lambda) & A_{11}(\lambda) & A_{12}(\lambda) & \cdots \\ K_2(x, \lambda) & A_{21}(\lambda) & A_{22}(\lambda) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.\end{aligned}$$

If we remember what has been said in (5) we see at once that this expression is meromorphic on $(A)$.

We may start as well from the system

$$(35') \qquad \mathfrak{k}_i'^*(\xi, \lambda) + K_i'(\xi, \lambda) = \sum_{j=1}^{\infty} K_{ji} \mathfrak{k}_j'^*(\xi, \lambda) \quad (i = 1, 2, \cdots),$$

† See footnote ‡ p. 138.

instead of from system (35). The corresponding function $\mathfrak{R}^{**}(x, \xi, \lambda)$ which is determined by the conditions

$$\int_a^b \mathfrak{R}^{**}(x, \xi, \lambda)\, \omega_i(x)\, dx = \eta_i^*(\xi, \lambda) \qquad (i = 1, 2, \ldots) \tag{38'}$$

will satisfy the relations

$$\mathfrak{R}^{**}(x, \xi, \lambda) = -K_0(x, \xi, \lambda) + \int_a^b K_0(x, s, \lambda)\, \mathfrak{R}^{**}(s, \xi, \lambda)\, ds$$

$$= \frac{1}{D(\lambda)} \begin{vmatrix} -K(x, \xi, \lambda) & K_1(\xi, \lambda) & K_2(x, \lambda) & \ldots \\ K_1'(\xi, \lambda) & A_{11}(\lambda) & A_{21}(\lambda) & \ldots \\ K_2'(\xi, \lambda) & A_{12}(\lambda) & A_{22}(\lambda) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}, \tag{41'}$$

for all nonsingular values of $x, \xi$. The determinants of the right-hand members of (41) and of (41′) being equal, the functions $\mathfrak{R}^*(x, \xi, \lambda)$, $\mathfrak{R}^{**}(x, \xi, \lambda)$ coincide almost everywhere. This proves that the function $\mathfrak{R}^*(x, \xi, \lambda)$ satisfies both equations of the reciprocal kernel

$$\mathfrak{R}^*(x, \xi, \lambda) + K_0(x, \xi, \lambda) = \int_a^b \mathfrak{R}^*(x, s, \lambda)\, K_0(s, \xi, \lambda)\, ds$$

$$= \int_a^b K_0(x, s, \lambda)\, \mathfrak{R}^*(s, \xi, \lambda))\, ds. \tag{42}$$

**9.** Take any integral equation

$$u(x) = f(x) + \int_a^b \bar{K}(x, \xi)\, u(\xi)\, d\xi, \tag{43}$$

whose kernel $\bar{K}(x, \xi)$ satisfies Hilbert's condition:

$$\int_a^b \int_a^b |\bar{K}(x, \xi)|^2\, dx\, d\xi \tag{44}$$

exists. The reciprocal $\bar{\mathfrak{R}}(x, \xi)$ of the kernel $\bar{K}(x, \xi)$ is the function which satisfies the equations

$$\bar{\mathfrak{R}}(x, \xi) + \bar{K}(x, \xi) = \int_a^b \bar{\mathfrak{R}}(x, s)\, \bar{K}(s, \xi)\, ds, \tag{45}$$

$$\bar{\mathfrak{R}}(x, \xi) + \bar{K}(x, \xi) = \int_a^b \bar{K}(x, s)\, \bar{\mathfrak{R}}(s, \xi)\, ds \tag{46}$$

simultaneously. By the "solution" of the equation (43) is meant any function $u(x)$ for which the integral

$$\int_a^b |u(x)|^2\, dx \tag{47}$$

exists and which satisfies the equation (43), $f(x)$ being any function, integrable together with its square.

Accordingly we shall require the existence of the integral

$$\int_a^b \int_a^b |\bar{\mathfrak{R}}(x, \xi)|^2 dx\, d\xi. \tag{48}$$

It is readily seen that (45) is a *sufficient* condition for the uniqueness of the solution of (43). Namely, multiplying (43) by $\bar{\mathfrak{R}}(t, x)$ and integrating we find that, if the solution of (43) exists, it must be equal to

$$u(x) = f(x) - \int_a^b \bar{\mathfrak{R}}(x, \xi) f(\xi)\, d\xi. \tag{49}$$

On the other hand, (46) is a *necessary and sufficient* condition for the possibility of the representation of a solution of (43) in the form (49) for an *arbitrary* $f(x)$. This can be shown by a direct substitution of (49) in (43).

Hence the reciprocal $\bar{\mathfrak{R}}(x, \xi)$ cannot exist if the homogeneous equation

$$u(x) = \int_a^b \bar{K}(x, \xi)\, u(\xi)\, d\xi \tag{50}$$

has a solution which is not zero almost everywhere, or if the nonhomogeneous equation (43) has no solution for an arbitrary $f(x)$. The same remains true if we replace the equation (43) by the associated equation

$$v(x) = f(x) + \int_a^b \bar{K}(\xi, x)\, v(\xi)\, d\xi. \tag{43'}$$

This can be proved simply by permuting the letters $x$ and $\xi$. It is also obvious that, if the reciprocal $\bar{\mathfrak{R}}(x, \xi)$ exists, it is unique. Finally it may be easily proved that, if the homogeneous equation (50) has a solution $\not\equiv 0$, a *necessary* condition for the compatibility of the nonhomogeneous equation (43) is given by

$$\int_a^b f(x)\, v(x)\, dx = 0, \tag{51}$$

where $v(x)$ denotes any solution of the associated homogeneous equation. All the facts mentioned above are well known, but usually they are established under somewhat different suppositions concerning the kernel $\bar{K}(x, \xi)$.*

**10.** Now Theorem I may be easily established for the kernel $K_0(x, \xi, \lambda)$. It follows from **8** that, if $D(\lambda) \neq 0$, the reciprocal of the kernel $K_0(x, \xi, \lambda)$

* Hilbert (footnote * p. 129) extended the classical results of the theory of Fredholm to the kernels of the type $\bar{K}(x, \xi)$, using the theory of bilinear forms in infinitely many variables. The same will be done below by more elementary methods.

is represented by the function $\mathfrak{R}^*(x, \xi, \lambda)$ which is meromorphic on $(\Delta)$. If $D(\lambda) = 0$, the reader may show exactly in the same fashion as above, that the homogeneous integral equations

$$u(x) = \int_a^b K_0(x, \xi, \lambda) u(\xi) d\xi; \qquad v(x) = \int_a^b K_0(\xi, x, \lambda) v(\xi) d\xi$$

possess the same number of linearly independent solutions, which are not zero almost everywhere, so that the reciprocal can not exist for such values of $\lambda$. This implies that each root of the denominator $D(\lambda)$ represents a pole of the reciprocal $\mathfrak{R}^*(x, \xi, \lambda)$ and that this reciprocal does not exist at all, if $D(\lambda)$ is identically zero.

**11.** So much for the kernel $K_0(x, \xi, \lambda)$. We return to the kernel $K(x, \xi, \lambda)$ and to the integral equation

$$u(x) = f(x) + \int_a^b K(x, \xi, \lambda) u(\xi) d\xi \tag{27}$$

of Theorem 1. We shall prove that *if $\lambda$ is any fixed value for which the reciprocal $\mathfrak{R}(x, \xi, \lambda)$ of the kernel $K(x, \xi, \lambda)$ exists, then it is always possible to find such an integer $m$ for which the reciprocal $\mathfrak{R}_m(x, \xi, \lambda)$* of the *$m$th iterated kernel $K_m(x, \xi, \lambda)$ also exists.*

For this purpose we introduce an auxiliary parameter $\varrho$ and we consider the integral equation

$$u(x) = \varphi(x) = \varrho \int_a^b K(x, \xi, \lambda) u(\xi) d\xi. \tag{52}$$

We set

$$k(x, \xi) \equiv K(x, \xi, \lambda)$$

and denote by $\mathfrak{k}(x, \xi)$ the reciprocal of the kernel $k(x, \xi)$. Our supposition is, that this reciprocal exists, so that the homogeneous equation

$$u(x) = \varrho \int_a^b k(x, \xi) u(\xi) d\xi \tag{53}$$

has no solution for $\varrho = 1$. We have to show that the homogeneous equation

$$u(x) = \int_a^b k_m(x, \xi) u(\xi) d\xi; \qquad k_m(x, \xi) \equiv K_m(x, \xi, \lambda), \tag{54}$$

has no solution for at least one value of the integer $m \neq 1$. If the contrary is the case, then for any given $m$ there exists a function $\varphi_m(x)$ which is not zero almost everywhere and which satisfies the equation

$$\varphi_m(x) = \int_a^b k_m(x, \xi) \varphi_m(\xi) d\xi. \qquad (m = 2, 3, \cdots). \tag{55}$$

On setting*

$$\psi_{m,\varepsilon}(x) = \varphi_m(x) + \varepsilon\int_a^b k(x,\xi)\varphi_m(\xi)\,d\xi + \cdots + \varepsilon^{m-1}\int_a^b k_{m-1}(x,\xi)\varphi_m(\xi)\,d\xi, \qquad \varepsilon^m = 1, \tag{56}$$

we see at once that $\psi_{m,\varepsilon}(x)$ is a solution of the integral equation

$$\psi_{m,\varepsilon}(x) = \varepsilon\int_a^b k(x,\xi)\,\psi_{m,\varepsilon}(\xi)\,d\xi.$$

If in (56) we substitute, instead of $\varepsilon$, all the distinct roots $\varepsilon_1, \varepsilon_2 \cdots \varepsilon_m$ of the equation $\varepsilon^m = 1$, we must obtain at least one function $\not\equiv 0$, because

$$m\varphi_m(x) = \sum_{(\varepsilon)}\psi_{m,\varepsilon}(x),$$

and $\varphi_m(x) \not\equiv 0$. The corresponding value of $\varepsilon$ is certainly $\neq 1$. Thus we have established the existence of infinitely many characteristic values of the integral equation (53) which are distinct and which are situated on the circle $|\varrho| = 1$. An immediate substitution shows that, if $\varrho_0$ is a characteristic value of the equation (53), $\varrho_0^2$ is a characteristic value of the equation

$$u(x) = \varrho^2\int_a^b k_2(x,\xi)\,u(\xi)\,d\xi. \tag{56}$$

Hence we obtain infinitely many characteristic values of the equation (56) which are distinct and are situated on the circle $|\varrho| = 1$. This is impossible, however, because, the kernel $\varrho^2 k_2(x,\xi)$ being of the type $K_0(x,\xi,\varrho)$ the reciprocal $\mathfrak{k}_2(x,\xi,\varrho)$ of $\varrho^2 k_2(x,\xi)$ either does not exist at all, or must be meromorphic on the whole $\varrho$-plane. The former case is certainly excluded here, because the integral

$$\int_a^b\int_a^b |\varrho^2 k_2(x,\xi)|^2\,dx\,d\xi \tag{57}$$

may be made arbitrarily small by taking $|\varrho|$ sufficiently small, which implies the convergence of Neumann's series for the reciprocal $\mathfrak{k}_2(x,\xi,\varrho)$.† Hence $\mathfrak{k}_2(x,\xi,\varrho)$ is meromorphic in $\varrho$, and consequently it can not have infinitely many poles on the circle $|\varrho| = 1$.

* E. Schmidt, *Zur Theorie der linearen und nichtlinearen Integralgleichungen*, Math. Ann. 64, 1907, pp. 161–174 (162–164).

† E. Schmidt, *Zur Theorie der linearen und nichtlinearen Integralgleichungen*, Math. Ann. 63 (1907), pp. 433–476 (448).

**12.** We exclude the case when the reciprocal of the kernel $K(x, \xi, \lambda)$ exists for no values of $\lambda$, in which case Theorem 1 is trivial. Thus we may assume that for a certain fixed value of integer $m$ the reciprocal of the kernel $K_m(x, \xi, \lambda)$ exists for some values of $\lambda$. The kernel $K_m(x, \xi, \lambda)$ being of the type $K_0(x, \xi, \lambda)$ the results of **10** may be applied to this kernel, whence the reciprocal $\mathfrak{R}_m(x, \xi, \lambda)$ of $K_m(x, \xi, \lambda)$ is meromorphic on $(\Delta)$. Let $\lambda$ be a fixed value which is different from any pole of $\mathfrak{R}_m(x, \xi, \lambda)$. If $u(x)$ is a solution of the equation (27), the same function must be a solution of the equation

$$u(x) = f_m(x) + \int_a^b K_m(x, \xi, \lambda)\, u(\xi)\, d\xi, \tag{58}$$

where

$$f_m(x) = f(x) + \int_a^b K(x, \xi, \lambda)\, f(\xi)\, d\xi + \cdots + \int_a^b K_{m-1}(x, \xi, \lambda) f(\xi)\, d\xi \tag{59}$$

which is readily proved by successive substitutions. On setting

$$\begin{aligned} \Omega(x, \xi, \lambda) \equiv & -K(x, \xi, \lambda) - \cdots - K_{m-1}(x, \xi, \lambda) + \mathfrak{R}_m(x, \xi, \lambda) \\ & + \int_a^b \mathfrak{R}_m(x, s, \lambda)\, K(s, \xi, \lambda)\, ds + \cdots + \int_a^b \mathfrak{R}_m(x, s, \lambda)\, K_{m-1}(s, \xi, \lambda)\, ds \end{aligned} \tag{60}$$

we bring the (unique) solution of (58) to the form:

$$\begin{aligned} u(x) &= f_m(x) - \int_a^b \mathfrak{R}_m(x, \xi, \lambda) f_m(\xi)\, d\xi \\ &= f(x) - \int_a^b \Omega(x, \xi, \lambda) f(\xi)\, d\xi. \end{aligned} \tag{61}$$

Conversely we can prove that (61) represents a solution of (27). In order to do this we shall proceed to show that $\Omega(x, \xi, \lambda)$ satisfies both equations

$$\Omega(x, \xi, \lambda) + K(x, \xi, \lambda) = \int_a^b \Omega(x, s, \lambda)\, K(s, \xi, \lambda)\, ds, \tag{62}$$

$$\Omega(x, \xi, \lambda) + K(x, \xi, \lambda) = \int_a^b K(x, s, \lambda)\, \Omega(s, \xi, \lambda)\, ds \tag{63}$$

for the reciprocal of the kernel $K(x, \xi, \lambda)$, so that, denoting this reciprocal by $\mathfrak{R}(x, \xi, \lambda)$, we have:

$$\mathfrak{R}(x, \xi, \lambda) = \Omega(x, \xi, \lambda).$$

The equation (62) may be verified by a direct substitution of (60) in (62). The relation (63) in the classical theory usually is deduced from the Neumann's series, simultaneously with the relation (62). In our case this

can not be done immediately and we have to find some other way of establishing (63). For this purpose we again introduce another parameter $\varrho$ and the kernel $k(x, \xi, \varrho) \equiv \varrho K(x, \xi, \lambda)$ and its iterated kernels $k_2(x, \xi, \varrho)$, $k_3(x, \xi, \varrho)$, $\cdots$ which reduce to $K(x, \xi, \lambda)$, $K_2(x, \xi, \lambda)$, $\cdots$ for $\varrho = 1$.* For sufficiently small $|\varrho|$ the reciprocals

$$\mathfrak{k}(x, \xi, \varrho), \qquad \mathfrak{k}_2(x, \xi, \varrho), \cdots$$

of all these kernels exist and we must have

$$\mathfrak{k}(x, \xi, \varrho) = \omega(x, \xi, \varrho) \equiv -k(x, \xi, \varrho) - \cdots - k_{m-1}(x, \xi, \varrho) + \mathfrak{k}_m(x, \xi, \varrho)$$
$$+ \int_a^b \mathfrak{k}_m(x, s, \varrho)\, k(s, \xi, \varrho)\, ds + \cdots + \int_a^b \mathfrak{k}_m(x, s, \varrho)\, k_{m-1}(s, \xi, \varrho)\, ds.$$

Hence for small $|\varrho|$:

$$\omega(x, \xi, \varrho) + k(x, \xi, \varrho) = \int_a^b \omega(x, s, \varrho)\, k(s, \xi, \varrho)\, ds, \tag{64}$$
$$\omega(x, \xi, \varrho) + k(x, \xi, \varrho) = \int_a^b k(x, s, \varrho)\, \omega(s, \xi, \varrho)\, ds.$$

The left- and right-hand members of the equations (64) are meromorphic on the whole $\varrho$-plane, the same being true of $\mathfrak{k}_m(x, \xi, \varrho)$. The equations (64) must hold true for all values of $\varrho$ for which all the functions in question are analytic, in particular for $\varrho = 1$, which is not a pole of $\mathfrak{k}_m(x, \xi, \varrho)$, by hypothesis. Hence (63) is established; at the same time we have established the existence of the reciprocal $\mathfrak{K}(x, \xi, \lambda)$ for all values of $\lambda$ which are different from any of the poles of $\mathfrak{K}_m(x, \xi, \lambda)$.

**13.** The expression $\Omega(x, \xi, \lambda)$ obviously is meromorphic on $(A)$. Let us see now what happens when $\lambda = \lambda_0$ is a pole of $\mathfrak{K}_m(x, \xi, \lambda)$. We have two cases to consider: $\lambda_0$ is or is not a pole of $\Omega(x, \xi, \lambda)$.

1°. Suppose that $\lambda_0$ is not a pole of $\Omega(x, \xi, \lambda)$. The function

$$\Omega_0(x, \xi) \equiv \Omega(x, \xi, \lambda_0)$$

then is perfectly determined; it certainly satisfies all our conditions of integrability as well as the equations for the reciprocal, so that the reciprocal $\mathfrak{K}(x, \xi, \lambda_0)$ exists in this case and it is represented by $\Omega(x, \xi, \lambda_0)$.

2°. Suppose now that $\lambda_0$ is also a pole of the function $\Omega(x, \xi, \lambda)$. We proceed to show that the reciprocal of the kernel $K(x, \xi, \lambda)$ can not exist in this case, that the homogeneous equations

$$u(x) = \int_a^b K(x, \xi, \lambda_0)\, u(\xi)\, d\xi; \qquad v(x) = \int_a^b K(\xi, x, \lambda_0)\, v(\xi)\, d\xi$$

* The notation here differs slightly from that of 11.

possess solutions $u(x)$, $v(x)$ which are not zero almost everywhere, and, finally, that the nonhomogeneous equation (27) has no solution if $f(x)$ is arbitrary.

Let $r$ be the multiplicity of the pole $\lambda_0$ of $\Omega(x, \xi, \lambda)$, so that in the neighbourhood of $\lambda = \lambda_0$ we have an expansion

$$\mathfrak{R}(x, \xi, \lambda) = \Omega(x, \xi, \lambda) = \frac{\chi_r(x, \xi)}{(\lambda - \lambda_0)^r} + \cdots + \frac{\chi_1(x, \xi)}{\lambda - \lambda_0} + \boldsymbol{P}(x, \xi, \lambda - \lambda_0)$$

where $\chi_r(x, \xi) \not\equiv 0$ and $\boldsymbol{P}(x, \xi, \lambda - \lambda_0)$ is analytic. On multiplying the equations (62), (63) by $(\lambda - \lambda_0)$ and making $\lambda \to \lambda_0$ we have:

$$\chi_r(x, \xi) = \int_a^b \chi_r(x, s)\, K(s, \xi, \lambda_0)\, ds = \int_a^b K(x, s, \lambda_0)\, \chi_r(s, \xi)\, ds$$

which makes all assertions above almost evident.

Thus, if the reciprocal $\mathfrak{R}(x, \xi, \lambda)$ of the kernel $K(x, \xi, \lambda)$ exists for at least one value of $\lambda$, it is equal to a function $\Omega(x, \xi, \lambda)$ which is meromorphic on $(\mathcal{A})$. Hence Theorem 1 is proved.

An easy application of the elegant method due to W. A. Hurwitz* will enable the reader to extend other classical results of the theory of Fredholm (so called 1st, 2nd, and 3rd theorems of Fredholm) to the kernel $K(x, \xi, \lambda)$ of our Theorem 1. It ought to be noted also that Theorem 1 remains true, if in the conditions of the theorem the kernel $K(x, \xi, \lambda)$ is replaced by any iterated kernel $K_m(x, \xi, \lambda)$ supposing, however, that the functions $K(x, \xi, \lambda), \cdots, K_{m-1}(x, \xi, \lambda)$ are integrable.

**14.** We go over now to a more general case where the kernel $K(x, \xi, \lambda)$ is meromorphic itself, under the restriction, however, that the poles are independent of $x$ and $\xi$. We shall establish

THEOREM 2. *Suppose that the kernel $K(x, \xi, \lambda)$ of the integral equation*

$$u(x) = f(x) + \int_a^b K(x, \xi, \lambda)\, u(\xi)\, d(\xi) \tag{27}$$

*satisfies the following conditions*:

1°. *$K(x, \xi, \lambda)$ is meromorphic in $\lambda$ on an open domain $(\mathcal{A})$ of the $\lambda$-plane, the poles being independent of $x, \xi$ $(a \leqq x \leqq b,\ a \leqq \xi \leqq b)$.*

2°. *The conditions of Theorem 1 are satisfied on any open subregion of $(\mathcal{A})$, which contains no poles of $K(x, \xi, \lambda)$.*

3°. *The coefficients of the negative powers of $(\lambda - \lambda_0)$ in the expansion of $K(x, \xi, \lambda)$ in neighbourhood of any pole $\lambda_0$:*

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* *On the pseudo-resolvent to the kernel of an integral equation*, Trans. Amer. Math. Soc. 13 (1912), pp. 405–418.

$$K(x,\xi,\lambda)=\frac{g_\mu(x,\xi)}{(\lambda-\lambda_0)^\mu}+\cdots+\frac{g_1(x,\xi)}{\lambda-\lambda_0}+P(x,\xi,\lambda-\lambda_0) \tag{65}$$

*are represented by sums of a finite number of products of a function of $x$ alone by a function of $\xi$ alone:*

$$g_s(x,\xi)=\sum_{\nu=1}^{\sigma}\varphi_\nu^{(s)}(x)\,\psi_\nu^{(s)}(\xi)^* \qquad (s=1,2,\cdots,\mu), \tag{66}$$

*each factor being absolutely integrable together with its square on $(a,b)$. Then the reciprocal of the kernel $K(x,\xi,\lambda)$ either is meromorphic on $(\Delta)$ or does not exist at all.*

The conditions of Theorem 1 being satisfied on every open subregion $(\bar{\Delta})$ of $(\Delta)$, which is free from poles of $K(x,\xi,\lambda)$, we see at once that the reciprocal $\Re(x,\xi,\lambda)$ is meromorphic on $(\Delta)$. Let $\lambda_0$ be a pole of $K(x,\xi,\lambda)$ and $(c)$ a circle around $\lambda_0$, whose radius $\delta$ may be taken arbitrarily small. Theorem 2 will be proved if we can show that $\Re(x,\xi,\lambda)$ is meromorphic on $(c)$.

Again we may confine ourselves to the consideration of the kernel of the type $K_0(x,\xi,\lambda)$ because, if the reciprocal of the kernel $K(x,\xi,\lambda)$ exists, the same is true of the reciprocal of a certain iterated kernel $K_m(x,\xi,\lambda)$, and if we prove that the reciprocal $\Re_m(x,\xi,\lambda)$ is meromorphic, the same will be true of $\Re(x,\xi,\lambda)$ by virtue of (60).

Hence we may assume that $K(x,\xi,\lambda)$ itself is of the type $K_0(x,\xi,\lambda)$. The formulas of **8** applied to the kernel $K(x,\xi,\lambda)$ yield:

$$\Re(x,\xi,\lambda)=\frac{D(x,\xi,\lambda)}{D(\lambda)}, \tag{67}$$

where

$$D(\lambda)=\begin{vmatrix}1-k_{11}(\lambda) & -k_{12}(\lambda) & \cdots\\ -k_{21}(\lambda) & 1-k_{22}(\lambda) & \cdots\\ \cdots & \cdots & \cdots\end{vmatrix},$$

$$D(x,\xi,\lambda)=\begin{vmatrix}-K(x,\xi,\lambda) & k_1'(\xi,\lambda) & k_2'(\xi,\lambda) & \cdots\\ k_1(x,\lambda) & 1-k_{11}(\lambda) & -k_{12}(\lambda) & \cdots\\ k_2(x,\lambda) & -k_{21}(\lambda) & 1-k_{22}(\lambda) & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{vmatrix} \tag{68}$$

and

$$k_i(x,\lambda)=\int_a^b K(x,\xi,\lambda)\,\omega_i(\xi)\,d\xi,\quad k_i'(\xi,\lambda)=\int_a^b K(x,\xi,\lambda)\,\omega_i(x)\,dx,$$
$$k_{ij}(\lambda)=\int_a^b\int_a^b K(x,\xi,\lambda)\,\omega_i(\xi)\,\omega_j(x)\,d\xi\,dx. \tag{69}$$

* Some of the factors $\varphi_\nu^{(s)}(x)$, $\psi_\nu^{(s)}(x)$ may be equal zero.

It seems at first sight that $\lambda_0$ must be an essential singularity of $\mathfrak{R}(x, \xi, \lambda)$ because the numerator and the denominator of (67) seem to contain infinitely many negative powers of $(\lambda - \lambda_0)$. This is not, however, the case, and we are able to prove that there exists a fixed positive integer $\varkappa$ such that either of the products

$$(\lambda - \lambda_0)^{\varkappa} D(\lambda), \qquad (\lambda - \lambda_0)^{\varkappa} D(x, \xi, \lambda)$$

is analytic on $(c)$. If this is true, the reciprocal $\mathfrak{R}(x, \xi, \lambda)$ is equal to a quotient of two functions analytic on $(c)$, and hence it is itself meromorphic on $(c)$.

**15.** The proof is based upon a curious property of certain determinants of a special type:

LEMMA. *Suppose that the elements $\alpha_{ij}(z)$ of a determinant $\Delta(z)$ of the nth order are functions in a parameter $z$, and that*

$$\alpha_{ij}(z) = \sum_{s=1}^{m-1} z^s c_{ij}^{(s)} + z^m \theta_{ij}(z),$$

$$c_{ij}^{(s)} = \sum_{\nu=1}^{\sigma} p_{i\nu}^{(s)} q_{j\nu}^{(s)},$$

*where $m$ and $\sigma$ are given integers, such that*

$$m \geqq 1, \qquad \sigma \geqq 1, \qquad m\sigma < n,$$

*$p_{i\nu}^{(s)}$, $q_{j\nu}^{(s)}$ are given constants and $\theta_{ij}(z)$ are given functions which are analytic in the neighbourhood of the origin. Then $\Delta(z)$ has the root $z = 0$ of multiplicity which is at least*

$$mn - \frac{m(m+1)}{2}\sigma.$$

We shall use the notation $\Delta_{n_0, m_0, \sigma_0}(z)$ to designate any determinant of the type of $\Delta(z)$, in which the integers $n, m, \sigma$ are replaced by $n_0, m_0, \sigma_0$ respectively. Let $\pi_{n_0, m_0, \sigma_0}$ denote the multiplicity of the root $z = 0$ of the determinant $\Delta_{n_0, m_0, \sigma_0}(z)$. We have to prove that

$$\pi_{n, m, \sigma} \geqq mn - \frac{m(m+1)}{2}\sigma.$$

The coefficient $c_{ij}^{(s)}$ can be considered as a linear form in $\sigma$ quantities

$$q_{j\nu}^{(s)}. \qquad (\nu = 1, 2 \cdots \sigma).$$

If we set

$$l_i^{(s)}(x) \equiv \sum_{\nu=1}^{\sigma} p_{i\nu}^{(s)} x_\nu,$$

we obviously have:

$$c_{ij}^{(s)} = l_i^{(s)}(q_j^{(s)}).$$

Now, take the constant term $c_{ij}^{(0)}$ of the element $a_{ij}(z)$. Since the number of linearly independent forms in $\sigma$ quantities is exactly $\sigma$, we may suppose, without loss of generality, that all the linearly independent forms of the set

$$l_i^{(0)}(x) \qquad (i = 1, 2, \cdots, n)$$

are contained in the $\sigma$ first rows of the determinant $\Delta(z)$. The forms

$$l_{\sigma+1}^{(0)}(x), \cdots, l_n^{(0)}(x),$$

then, are linear combinations of the first $\sigma$ forms

$$l_1^{(0)}(x), \cdots, l_\sigma^{(0)}(x).$$

Hence, by taking suitable linear combinations of the first $\sigma$ rows of $\Delta(z)$ and by subtracting them from the remaining $(n-\sigma)$, we can reduce to zero the constant terms of the elements

$$a_{ij}(z) \qquad (i = \sigma+1, \cdots, n;\ j = 1, 2, \cdots, n).$$

The character of the coefficients $c_{ij}^{(s)}$ $(s>0)$ obviously will not be changed by such an operation: the forms

$$l_1^{(s)}(x), \cdots, l_n^{(s)}(x) \qquad (s>0)$$

will merely be replaced by their linear combinations

$$\overline{l}_1^{(s)}(x), \cdots, \overline{l}_n^{(s)}(x)$$

and the coefficients $c_{ij}^{(s)}$ will be replaced respectively by the coefficients

$$\overline{c}_{ij}^{(s)} = l_i^{(\overline{s})}(q_j^{(s)}) \quad (s = 1, 2, \cdots, m-1;\ i, j = 1, 2, \cdots, n).$$

After this transformation the determinants of the $(n-\sigma)$th order, which are contained in the matrix of the last $(n-\sigma)$ rows of $\Delta(z)$, obviously are of

the form $\varrho^{n-\sigma}\Delta_{n-\sigma,m-1,\sigma}(z)$, and the expansion of $\Delta(z)$ in the elements of the last $(n-\sigma)$ rows shows at once:

$$\Delta(z) = \varrho^{n-\sigma} \text{ times a linear combination of } \Delta_{n-\sigma,m-1,\sigma}(z).$$

This proves that

$$\pi_{n,m,\sigma} \geqq n-\sigma+\pi_{n-\sigma,m-1,\sigma}.$$

On substituting $(n-\sigma)$ instead of $n$ and $(m-1)$ instead of $m$, $\cdots$ it is readily seen that

$$\pi_{n,m,\sigma} \geqq (n-\sigma)+(n-2\sigma)+\cdots+[n-(m-1)\sigma]+\pi_{n-(m-1)\sigma,1,\sigma}.$$

Now the same argument as before shows that

$$\pi_{k,1,\sigma} \geqq k-\sigma \qquad (k \geqq \sigma).$$

Hence

$$\pi_{n,m,\sigma} \geqq (n-\sigma)+\cdots+(n-m\sigma) = mn-\frac{m(m+1)}{2}\sigma.$$

As a check on our lemma, note that for the determinant

$$\Delta(z) = \begin{vmatrix} z\theta_{11}+p_1q_1 & z\theta_{12}+p_1q_2 & \cdots & z\theta_{1n}+p_1q_n \\ z\theta_{21}+p_2q_1 & z\theta_{22}+p_2q_2 & \cdots & z\theta_{2n}+p_2q_n \\ \cdots & \cdots & \cdots & \cdots \\ z\theta_{n1}+p_nq_1 & z\theta_{n2}+p_nq_2 & \cdots & z\theta_{nn}+p_nq_n \end{vmatrix}, \qquad \theta_{ij} \text{ const.}$$

it may be proved immediately that

$$\Delta(z) = z^{n-1}\left\{\Theta+\sum_{i,j=1}^{n} p_iq_j\,\Theta_{ij}\right\}, \qquad \Theta = \begin{vmatrix} \theta_{11}\cdots\theta_{1n} \\ \cdots\cdots \\ \theta_{n1}\cdots\theta_{nn} \end{vmatrix},$$

where $\Theta_{ij}$ denotes the cofactor of the element $\theta_{ij}$ in $\Theta$.

**16.** In order to apply the lemma above to the proof of Theorem 2, we may observe that the radius $\delta$ can be taken so small that the circle $(c)$ is in $(\mathcal{A})$ and that expansion (65) can be used. On substituting (65) in (69) we see

$$(\lambda-\lambda_0)^{\mu}k_{ij}(\lambda) = \sum_{s=1}^{\mu}(\lambda-\lambda_0)^{\mu-s}\sum_{\nu=1}^{\sigma}\int_a^b \varphi_\nu^{(s)}(x)\,\omega_j(x)\,dx\int_a^b \psi_\nu^{(s)}(\xi)\,\omega_i(\xi)\,d\xi + P_{ij}(\lambda-\lambda_0).$$

Hence it is obvious that the product

$$(\lambda-\lambda_0)^{\mu n}\,\Delta^{(n)}(\lambda); \qquad \Delta^{(n)}(\lambda)=\begin{vmatrix} 1-k_{11}(\lambda) & -k_{12}(\lambda) & \cdots & -k_{1n}(\lambda) \\ -k_{21}(\lambda) & 1-k_{22}(\lambda) & \cdots & -k_{2n}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is a determinant of the type treated in our lemma, the only difference being that $z$ is replaced by $(\lambda-\lambda_0)$. Setting $\varkappa=\dfrac{\mu(\mu+1)}{2}\sigma$, we see at once that $(\lambda-\lambda_0)^{\varkappa}\,\Delta^{(n)}(\lambda)$ is analytic on $(c)$. If we observe that $\Delta^{(n)}(\lambda)$ and therefore $(\lambda-\lambda_0)^{\varkappa}\,\Delta^{(n)}(\lambda)$ are uniformly bounded on the contour of $(c)$ and that

$$(\lambda-\lambda_0)^{\varkappa}\,\Delta^{(n)}(\lambda)\to(\lambda-\lambda_0)^{\varkappa}\,\Delta(\lambda) \text{ on } (c),$$

except at the point $\lambda=\lambda_0$, an immediate application of the theorem of Montel* will prove that $(\lambda-\lambda_0)^{\varkappa}\,\Delta(\lambda)$ is also analytic on $(c)$. The same argument may be applied to the determinant

$$\Delta^{(n)}(x,\xi,\lambda)=\begin{vmatrix} -K(x,\xi,\lambda) & k_1'(\xi,\lambda) & \cdots & k_n'(\xi,\lambda) \\ k_1(\xi,\lambda) & 1-k_{11}(\lambda) & \cdots & -k_{1n}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

with the result that $(\lambda-\lambda_0)^{\varkappa}\,\Delta(x,\xi,\lambda)$ is analytic on $(c)$. Theorem 2 thus is proved.

The supposition 3° of Theorem 2 is essential for the meromorphic character of the reciprocal of the kernel $K(x,\xi,\lambda)$. If this restriction is removed, the reciprocal $\Re(x,\xi,\lambda)$ may have $\lambda_0$ as an essential singularity; this is illustrated by the example of the integral equation

$$u(x)=f(x)+\frac{1}{\lambda}\int_a^b K(x,\xi)\,u(\xi)\,d\xi,$$

whose kernel $K(x,\xi)$ is definite symmetric: the characteristic values of this equation have zero as limiting point, so that $\lambda=0$ is an essential singularity for the reciprocal of $\dfrac{1}{\lambda}K(x,\xi)$. On the other hand it may happen that the reciprocal of a meromorphic kernel $K(x,\xi,\lambda)$ is analytic on the whole $\lambda$-plane; for instance, if $K(x,\xi,\lambda)$ is a reciprocal of another analytic kernel $K'(x,\xi,\lambda)$: the reciprocal of $K(x,\xi,\lambda)$ coincides with $K'(x,\xi,\lambda)$.

* See footnote * p. 136.

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# ON THE EXISTENCE OF THE ABSOLUTE MINIMUM IN SPACE PROBLEMS OF THE CALCULUS OF VARIATIONS.*

By Lawrence M. Graves.†

One purpose of this paper is to make more accessible to American readers part of the theory developed by Tonelli in his *Fondamenti di Calcolo delle Variazioni.*‡ Tonelli has discussed semi-continuity, sufficient conditions for the existence of the absolute minimum, necessary conditions for a minimum, sufficient conditions for a relative minimum, and the existence and uniqueness of extremals joining two points, obtaining many new theorems and extending the theory to cover a great variety of cases. For many theorems already standard in the Calculus of Variations, Tonelli has greatly weakened the hypotheses, thus making a distinct advance in the theory. Out of this great mass of material the present paper selects a single line of theorems, relating to the existence of the absolute minimum for problems in ordinary or non-parametric form. Other important parts of the theory of Tonelli may possibly be presented in later papers.

Instead of restricting attention to integrals involving a single function $y(x)$ we shall consider integrals depending on $k$ functions $y_1(x) \cdots y_k(x)$, that is, we shall work in space of $k+1$ dimensions. Not all of Tonelli's theorems extend readily to higher spaces, — I have selected some that do. Incidentally I have modified and simplified some of the proofs.

A large part of Tonelli I is taken up with the author's theory of Lebesgue integration.§ We shall assume here the familiar theory of Lebesgue integration, as presented, e. g., by Bliss, de la Vallée Poussin, or Caratheodory.‖ Another point of Tonelli's presentation from which we shall diverge is his use of Cipolla's device¶ of sequences of sets to avoid using the Zermelo

* Received September 13, 1926.

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‡ The two volumes of this work which have appeared will be referred to in the sequel as Tonelli I and Tonelli II.

For an explanation of Tonelli's point of view, see a note by him in Bull. Amer. Math. Soc., 31 (1925), p. 163. Also see the review by Dresden, Bull. Amer. Math. Soc., 32 (1926), p. 381.

§ Tonelli has since presented a still further altered and simplified theory in *„Sulla nozione di integrale"*, Annali di Mat. (4), vol. 1 (1923–24), pp. 105–145.

‖ Bliss, Bull. Amer. Math. Soc., 24 (1917), pp. 1–47; de la Vallée Poussin, *Intégrales de Lebesgue*, etc.; Caratheodory, *Vorlesungen über reelle Funktionen.*

¶ Cipolla, *Sul postulato di Zermelo e la teoria dei limiti delle funzioni*, Atti della Accad. Gioenia in Catania, (5), vol. 6 (1913), Memoir V.

postulate. While I am fully in sympathy with Tonelli's point of view with regard to the Zermelo postulate, it seems desirable here to simplify the notations and proofs as much as possible in order to expose more clearly the main line of reasoning and the principal results. Those readers who wish to avoid all use of infinite arbitrary choice can readily insert Cipolla's device in the proofs, and can also assume Tonelli's theory of Lebesgue integration.

## I. Notations and definitions.

We shall retain a large part of the notation used by Tonelli. Some changes are necessitated by the extension of the theory to higher spaces.

1. Points $(y_1 \cdots y_k)$ in $k$-dimensional $y$-space will be denoted freely by $y$ or by $y_i$. Following the tensor analysis notation we shall omit summation signs, indicating summation by equating indices. E. g.,

$$y_{1i}\, y_{2i} \equiv \sum_{i=1}^{k} y_{1i}\, y_{2i}, \qquad \eta_i f_{y_i' y_j'}\, \eta_j \equiv \sum_{i,j=1}^{k} \eta_i f_{y_i' y_j'}\, \eta_j .$$

The principal $k$-partite symbols used are $y$, $y'$, $f_{y'}$.

By the notation $\|y\|$ we shall mean the greatest of the absolute values $|y_i|$.

2. A *curve*

$$C: \quad y = y(x), \qquad (a, b),$$

consists of $k$ single-real-valued continuous functions $(y_1(x) \cdots y_k(x)) \equiv y(x)$ defined on a finite closed interval $a \leqq x \leqq b$.* As a singular case we may have $a = b$. In all other cases, the left hand end of the interval $(a, b)$ is always written first. The interval $(a, b)$ may be different for different curves.

For the discussion of limits, Tonelli defines various types of neighborhoods $(\varrho)$ of a curve $C$.† It is more symmetrical to define the distance of two curves, following the notions of Fréchet, and to define neighborhoods $(C)_\varrho$ in terms of distance. Uniform convergence, limit in terms of the "proper" neighborhoods $(\varrho)$ of Tonelli, and limit in terms of distance, are equivalent notions.

As a preliminary we define the distance $\|\varphi_1, \varphi_2\|$ of two functions

$$\varphi_1(x), \quad (a_1, b_1), \qquad \varphi_2(x), \quad (a_2, b_2),$$

* Cf. Tonelli I, p. 347, where $y(x)$ is required to be *absolutely* continuous.

† I, pp. 71, 348.

to be the maximum of the quantities

$|a_1 - a_2|$,
$|b_1 - b_2|$,
$|\varphi_1(x) - \varphi_2(x)|$ on the common interval of definition,
$|\varphi_1(a_1) - \varphi_2(x)|$ on $a_2 \leqq x \leqq a_1$, if $a_2 < a_1$,
$|\varphi_1(x) - \varphi_2(a_2)|$ on $a_1 \leqq x \leqq a_2$, if $a_1 < a_2$,
$|\varphi_1(b_1) - \varphi_2(x)|$ on $b_1 \leqq x \leqq b_2$, if $b_1 < b_2$,
$|\varphi_1(x) - \varphi_2(b_2)|$ on $b_2 \leqq x \leqq b_1$, if $b_2 < b_1$.

Then we define the distance $\|C_1, C_2\|$ of two curves

$$C_1: \quad y = y_1(x), \quad (a_1, b_1), \qquad C_2: \quad y = y_2(x), \quad (a_2, b_2),$$

to be the greatest of the $k$ numbers $\|y_{1i}, y_{2i}\|$. The distance $\|C_1, C_2\|$ may as easily be defined directly if desired.

The distance $\|C_1, C_2\|$ has the four properties characteristic of Fréchet's distance function, viz.,

1. $\|C_1, C_2\| \geqq 0$,
2. $\|C_2, C_1\| = \|C_1, C_2\|$,
3. $\|C_1, C_2\| = 0$ if and only if $C_1$ coincides with $C_2$,
4. $\|C_1, C_3\| \leqq \|C_1, C_2\| + \|C_2, C_3\|$.

It should be remembered that $\|y\|$ and $\|C_1, C_2\|$ are distinct symbols with different meanings.

We define the *neighborhood* $(C_0)_\varrho$ of a curve $C_0$ to be the class of all curves $C$ for which $\|C, C_0\| < \varrho$. The neighborhood $(\varphi_0)_\varrho$ of a function $\varphi_0$ is defined similarly.

**3. Curves of accumulation.*** We say that a set $K$ of curves $C$ has a curve of accumulation $C_0$ in case every neighborhood $(C_0)_\varrho$ contains curves of $K$ distinct from $C_0$. A sequence $\{C_n\}$ has a curve of accumulation $C_0$ in case every neighborhood $(C_0)_\varrho$ contains an infinity of curves of the sequence (not necessarily distinct from $C_0$).

Corresponding definitions hold, of course, with curves replaced by functions.

As an indication of how Cipolla's idea of sequences of sets enters, we say that a sequence $\{K_n\}$ of sets of curves has a curve of accumulation $C_0$ in case every neighborhood $(C_0)_\varrho$ contains curves from an infinity of the sets $K_n$.

* Tonelli I, pp. 73–75 and 348.

4. **Equi-continuity and absolute continuity.** Related to the idea of the total variation of a function is that of the variation of the function on sets of intervals of measure $\lambda$, or the *$\lambda$-variation.** For a function $\varphi$ on $(a, b)$, we define

$$V_\varphi(\lambda) \equiv \bar{B} \sum_j |\varphi(b_j) - \varphi(a_j)|, \qquad (\lambda > 0),$$
$$V_\varphi(0) \equiv 0,$$

where the symbol $\bar{B}$ stands for the upper bound of the sum taken for every finite set of non-overlapping sub-intervals $(a_j, b_j)$ of $(a, b)$. the sum of whose lengths is less than or equal to $\lambda$.

The *oscillation* on intervals of length $\lambda$, or *$\lambda$-oscillation*, is defined by

$$O_\varphi(\lambda) \equiv \bar{B} |\varphi(x_2) - \varphi(x_1)|, \qquad (\lambda > 0),$$
$$O_\varphi(0) \equiv 0,$$

where the upper bound $\bar{B}$ is taken for all intervals $(x_1, x_2)$ of length less than or equal to $\lambda$.

The functions $V_C$ and $O_C$ for a curve

$$C: \quad y = y(x), \qquad (a, b),$$

are defined in the same way, the expression $|\varphi(b_j) - \varphi(a_j)|$ being replaced by $\|y(b_j) - y(a_j)\|$, etc.

The continuity of the function $O_\varphi$ is equivalent to the continuity of the function $\varphi$ itself. In case the function $V_\varphi$ is continuous, the function $\varphi$ is *absolutely continuous,*† and has a derivative $\varphi'$ almost everywhere on $(a, b)$. Similarly, in case the function $V_C$ for a curve $C$ is continuous, we say that $C$ is absolutely continuous.

Now consider a set $W$ of functions $\varphi$. We define

$$O_W(\lambda) \equiv \bar{B} O_\varphi(\lambda), \qquad V_W(\lambda) \equiv \bar{B} V_\varphi(\lambda),$$

where the upper bounds are taken with respect to all functions $\varphi$ in $W$. In the same way, for a set $K$ of curves $C$ we define

$$O_K(\lambda) \equiv \bar{B} O_C(\lambda), \qquad V_K(\lambda) = \bar{B} V_C(\lambda).$$

In case the function $O_W$ is continuous, the functions of the set $W$ are said to be *equi-continuous.*‡ In case the function $V_W$ is continuous,

* For this terminology, and for the properties of the $\lambda$-variation, cf. Caratheodory, *Vorlesungen über reelle Funktionen*, pp. 510 ff.

† Cf.. Tonelli I, p. 63; Caratheodory, loc. cit., p. 513.

‡ Cf. Ascoli, *Le curve limiti di una varietà data di curve*, Mem. R. Acc. dei Lincei, 18 (1883), pp. 521-586; Arzelà, *Sulle serie di funzioni*, Mem. R. Acc. Sci. Ist. Bologna, 8 (1899-1900), p. 176; Tonelli I, p. 77.

we shall say that the functions of the set $W$ are *equi-absolutely continuous.*

We say that the functions of a set $W$ are *uniformly bounded* in case there exists a constant $M$ such that

$$|a|<M, \qquad |b|<M, \qquad |\varphi(x)|<M,$$

for every function

$$\varphi(x), \qquad (a, b),$$

in $W$.

Definitions similar to all the above hold for classes $K$ of curves $C$.

5. **Approximate convergence.*** Consider a sequence $\{\varphi_n\}$ and a function $\varphi$,

$$\varphi_n(x), \qquad (a_n, b_n), \qquad \varphi(x), \qquad (a, b),$$

all of which are measurable on their respective intervals of definition. Denote by $E_{n\varrho}$ the set of points in the interval common to the two intervals of definition for which

$$|\varphi_n(x)-\varphi(x)|>\varrho.$$

Then we say that the sequence $\{\varphi_n\}$ *converges approximately* to $\varphi$ in case, for every pair of positive numbers $\varepsilon$ and $\varrho$ there exists an integer $N$ such that, when $n>N$ we have

$$m E_{n\varrho}<\varepsilon, \qquad |a_n-a|<\varepsilon, \qquad |b_n-b|<\varepsilon.$$

Here the notation $mE$ denotes the Lebesgue measure of the set $E$.

6. **The integral $I_C$.** We consider a function $f(x, y, y')$ of $(2k+1)$ arguments, defined in a certain domain $R$ of the $(x, y, y')$ space. The domain $R$ consists of all points $(x, y, y')$ for which $(x, y)$ is in a domain $A$ of the $(k+1)$-dimensional $(x, y)$ space. The domain $A$, which is the fundamental set of points in which admissible curves are to lie,† is an arbitrary set of points of the $(x, y)$ space, restricted only by the condition that the part of $A$ contained in an arbitrary hypersphere is *closed.* The function $f$ and its $k$ partial derivatives $f_{y_i'}$ are supposed to be defined and continuous on the domain $R$.

If the continuous curve

$$C: \quad y = y(x), \qquad (a, b),$$

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* This notion was introduced by F. Riesz under the name of "*convergence en mesure*". See the paper *Sur les suites de fonctions mesurables*, Comptes Rendus, 148 (1909), p. 1303; Tonelli I, p. 92.

† Called "campo $A$" by Tonelli. See I, p. 201.

lies in $A$, and if $y'_{0i}(x)$ are $k$ functions measurable on $(a, b)$, then the functions $f(x, y(x), y'_0(x))$ and $f_{y'_i}(x, y(x), y'_0(x))$ are measurable on $(a, b)$.* To show this we may take the intermediate step of showing that the functions $f(x, y(x), y')$, etc., are continuous in their $(k+1)$ arguments together, and then apply the theorem in Caratheodory.

An *ordinary curve*

$$C: \quad y = y(x), \qquad (a, b),$$

is one such that: 1) the points of $C$ belong to the domain $A$; 2) $C$ is absolutely continuous; 3) the integral

$$I_C = \int_a^b f(x, y(x), y'(x))\, dx$$

exists (in the sense of Lebesgue), the integrand being set equal to zero on the set of points of measure zero where the derivatives $y'_i(x)$ do not exist finitely.

The following two conditions on the integral $I_C$ are used as hypotheses in most of the theorems we shall derive.

(I) *For all points* $(x, y)$ *in* $A$ *and all points* $y'$ *and* $\bar{y}'$,

$$\mathcal{E}(x, y, y', \bar{y}') \geqq 0,$$

*where* $\mathcal{E}$ *is the Weierstrassian function, defined by*

$$\mathcal{E}(x, y, y', \bar{y}') \equiv f(x, y, \bar{y}') - f(x, y, y') - (\bar{y}'_i - y'_i) f_{y'_i}(x, y, y').$$

(II) *For every bounded and closed set* $\bar{A}$ *contained in* $A$ *there exist constants* $M > 1$ *and* $\alpha > 1$ *such that, if* $(x, y)$ *is in* $\bar{A}$ *and* $\| y' \| > M$, *then*

$$f(x, y, y') > \| y' \|^{\alpha}.$$

In the plane case, Tonelli does not require the hypothesis (II) for his theorems on semi-continuity,† but for the existence theorems condition (II) seems to be essential even in the plane.

In case the second partial derivatives $f_{y'_i y'_j}$ exist and are continuous, a condition equivalent to (I) is that we have $\eta_i f_{y'_i y'_j} \eta_j \geqq 0$ for every $(x, y, y')$ in $R$ and every point $\eta$. In this case Tonelli calls the integral $I_C$ *quasi-regular positive*. However, condition (I) is the form actually used in the proof of the theorems we shall consider. The other terms used by

* See Caratheodory, *Vorlesungen über reelle Funktionen*, p. 377; Tonelli I, p. 349.

† I, pp. 383–400.

Tonelli to characterize various types of integrals,—*regular, quasi-regular normal,* etc.,*—will not concern us here.

Hamilton's principle, when applied to certain problems in dynamics, gives rise to an integral $I_C$ satisfying the conditions (I) and (II). The integrand will be of the form

$$f(x, y, y') = y'_i g_{ij}(x, y) y'_j + h(x, y),$$

where the first term is a positive definite form in $y'$ for every point $(x, y)$ in the domain $A$.

7. **Closed classes** $K$. If $K$ is a class of ordinary curves $C$, we say that $K$ is *closed* in case $K$ contains all its curves of accumulation which are ordinary curves.†

As examples of closed classes $K$ we may mention:

1) $K$ consists of all ordinary curves joining two fixed points having different abscissas.

2) $K$ consists of all ordinary curves joining a fixed point to a fixed closed set of points.

3) $K$ consists of all ordinary curves joining two fixed points and having a third point in a fixed closed set of points.

4) $K$ consists of all ordinary curves joining two fixed points and lying on a given closed surface or hypersurface.

The great degree of generality of the classes $K$ engenders a corresponding degree of generality in the existence theorems of § 13. The determination of a minimizing curve for the integral $I_C$ in a class $K$ falling under example 4) is a problem of Lagrange with finite side conditions. Such a problem arises from the application of Hamilton's principle to problems in mechanics involving holonomic constraints.‡

8. **Semi-continuity.**§ We define lower semi-continuity, which is the only kind we shall talk about. For upper semi-continuity and maxima of $I_C$ the signs and inequalities are reversed.

We say that the integral $I_C$ has *lower semi-continuity* at an ordinary curve $C_0$ in case: for every positive $\epsilon$ there exists a positive $\delta$ such that, for every ordinary curve $C$ in $(C_0)_\delta$ we have

$$I_C > I_{C_0} - \epsilon.$$

In case $I_C$ has lower semi-continuity at every ordinary curve $C$, we say simply that $I_C$ has lower semi-continuity.

* I, pp. 360, 361.

† These classes are called "complete" by Tonelli (II, p. 281). The word "closed" is in accord with the established usage of point set theory and the theory of abstract sets.

‡ Cf. Bolza, *Vorlesungen über Variationsrechnung*, p. 554.

§ Tonelli I, p. 361.

## II. Preliminary theorems.

9. ASCOLI'S THEOREM.* *If the functions of the sequence* $\{\varphi_n(x)\}$ *are equi-continuous and uniformly bounded, then there exists at least one function of accumulation* $\varphi_0$ *of the sequence.*†

COROLLARY. *If the curves of the sequence* $\{C_n\}$ *are equi-continuous and uniformly bounded, then there exists at least one curve of accumulation* $C_0$ *of the sequence.*

To prove the corollary we apply the theorem to the sequence of functions $\{y_{n1}\}$. Then there is a subsequence $\{y_{n_m 1}\}$ approaching a function of accumulation $y_{01}$ uniformly. Then we apply the theorem to the sequence $\{y_{n_m 2}\}$, and so on.

LEMMA 1. *Let* $C_0$ *be a curve of accumulation of a set* $K$ *of curves* $C$. *If the curves of* $K$ *are equi-continuous, we have* $O_{C_0}(\lambda) \leqq O_K(\lambda)$; *if equi-absolutely continuous, we have* $V_{C_0}(\lambda) \leqq V_K(\lambda)$.

This lemma follows from a consideration of the inequality

$$\begin{aligned}(1)\qquad \|y_0(x_1)-y_0(x_2)\| &\leqq \|y_0(x_1)-y(x_1)\|+\|y(x_1)-y(x_2)\| \\ &+\|y(x_2)-y_0(x_2)\| \leqq 2\|C_0, C\|+O_K(x_1-x_2),\end{aligned}$$

where $x_2<x_1$, and $y(x_1)$ is to be replaced by $y(b)$ in case $x_1>b$, and similarly at the left hand end. Since there are curves $C$ in every neighborhood of $C_0$, we have

$$\|y_0(x_1)-y_0(x_2)\| \leqq O_K(x_1-x_2).$$

To show $V_{C_0}(\lambda) \leqq V_K(\lambda)$, we have to consider a finite number of inequalities like (1).

10. **Properties of measurable functions and Lebesgue integrals.** SCHWARZ INEQUALITY (generalized). *If* $\alpha>1$, *and* $f$ *and* $g$ *are functions of* $x$ *measurable on the measurable set* $E$, *then*

$$\left|\int_E fg\,dx\right| \leqq \left(\int_E |f|^\alpha dx\right)^{\frac{1}{\alpha}} \left(\int_E |g|^{\frac{\alpha}{\alpha-1}} dx\right)^{\frac{\alpha-1}{\alpha}}$$

*provided the integrals on the right hand side exist.*‡

* See Ascoli, *Le curve limiti di una varietà data di curve*, Mem. R. Acc. dei Lincei, 18 (1883) pp. 521–586.

† The proof will be omitted here. See: Arzelà, *Sulle serie di funzioni*, Mem. R. Acc. Sci. Ist. Bologna, 8 (1899–1900), pp. 176 ff; Fréchet, *Sur quelques points du Calcul Fonctionel*, Palermo Rendiconti, 22 (1906) pp. 11 ff; Roussel, Journ. de Math., 5 (1926), p. 395.

For a proof applicable to the very general types of classes $K$ we are considering, see Tonelli I, pp. 78–85. If one restricts attention to classes $K$ of functions all defined on the same interval, the earlier proofs are adequate. Tonelli's theorems pay attention also to sets and sequences of sets of functions $\varphi$.

‡ Cf. Tonelli I, p. 165; F. Riesz, Math. Annalen, 69 (1910), p. 456.

LEMMA 2. *If the function $\varphi$ is absolutely continuous on $(a, b)$, then for every $\lambda \geqq 0$ we have*

$$V_\varphi(\lambda) \leqq \bar{B} \int_E |\varphi'(x)|\, dx \leqq 2V_\varphi(\lambda),$$

*where the upper bound of the integral is taken over all measurable sets $E$ of $(a, b)$ having measure less than or equal to $\lambda$.**

For the next lemma it will be convenient to define "step functions". We say that a function $\varphi$ on $(a, b)$ is a *step function* in case $(a, b)$ is divisible into a finite number of sub-intervals interior to each of which $\varphi$ is constant.

LEMMA 3. *If the function $\varphi$ is measurable on $(a, b)$, then there exists a sequence $\{\varphi_n\}$ of step functions converging approximately (§ 5) to $\varphi$. If the function $\varphi$ is bounded, the sequence $\{\varphi_n\}$ may be required to have the same bounds.*

As a basis for the proof of this lemma we take the property of measurable functions on which Tonelli founds his theory of Lebesgue integration, viz.

If $\varphi(x)$ is measurable on $(a, b)$ then for every positive $\epsilon$ there exists a closed set $E$ in $(a, b)$, of measure $> (b - a - \epsilon)$ such that the function $\varphi$, regarded as defined on $E$, is continuous.†

Corresponding to $\epsilon = 1/n$, let $E_n$ be one of the closed sets of the theorem just stated. The complement of $E_n$ is the sum of a denumerable set of non-overlapping open intervals. Then we define an intermediate function $\psi_n$, continuous uniformly on $(a, b)$, equal to $\varphi$ on $E_n$, and linear on each interval of the complement of $E_n$. On account of the uniform continuity of $\psi_n$ we can obviously construct a step function $\varphi_n$ satisfying the inequality

$$|\varphi_n(x) - \psi_n(x)| < \frac{1}{n}, \qquad (a \leqq x \leqq b).$$

Then we have

$$|\varphi_n(x) - \varphi(x)| < \frac{1}{n}$$

except on a set of measure less than $1/n$.

11. The next two lemmas are fundamental for the main theorems. They replace Tonelli's fundamental lemma‡ and assist in simplifying the proofs.

* For a basis for the proof of this property, see, e. g., de la Vallée Poussin, *Intégrales de Lebesgue*, §§ 71–74.

† Cf. Lebesgue, Comptes Rendus, 137 (1903), p. 1228; Vitali, *Una proprietà delle funzioni misurabili*, Rend. del. R. Ist. Lombardo, (2) 38 (1905), p. 599; Tardini, *Sulle funzioni misurabili*, Giorn. di Mat. di Battaglini, 49 (1911), p. 23.

‡ I, p. 213.

LEMMA 4. *Suppose the function $g(x)$ is bounded and measurable on the interval $(a, b)$, and let $W$ be a class of equi-absolutely continuous functions $\varphi$ defined on the same interval $(a, b)$. Then for every $\varepsilon > 0$ there exists a $\gamma > 0$ such that, for every $\varphi$ in $W$ for which $|\varphi(x)| < \gamma$ we have*

$$\left|\int_a^b \varphi'(x)\, g(x)\, dx\right| < \varepsilon.$$

Suppose $M > |g(x)|$ on $(a, b)$, and let $\{g_n\}$ be a sequence of step functions converging approximately to $g$. (Lemma 3.) Assume $V_W(\lambda) \not\equiv 0$. (In the contrary case the conclusion is obvious.) Denote by $E_{n\varepsilon}$ the set of points of $(a, b)$ at which

$$|g(x) - g_n(x)| > \frac{\varepsilon}{6\, V_W(b-a)},$$

and fix an $n$ so great that

$$V_W(m\, E_{n\varepsilon}) < \frac{\varepsilon}{12\, M}.$$

Denote the intervals on which the function $g_n$ is constant by $(a_j, b_j)$, $(j = 1 \cdots m_n)$. Then we have, by lemma 2 and other familiar properties of Lebesgue integrals,

$$\left|\int_{E_{n\varepsilon}} \varphi'(g - g_n)\, dx\right| \leqq 2M \int_{E_{n\varepsilon}} |\varphi'|\, dx \leqq 2M \cdot 2 V_W(m\, E_{n\varepsilon}) < \frac{\varepsilon}{3},$$

$$\left|\int_{(a,b)-E_{n\varepsilon}} \varphi'(g - g_n)\, dx\right| \leqq \frac{\varepsilon}{6 V_W(b-a)} \int_a^b |\varphi'|\, dx \leqq \frac{\varepsilon}{3},$$

$$\left|\int_a^b \varphi' g_n\, dx\right| = \left|\sum_j \int_{a_j}^{b_j} \varphi' g_n\, dx\right| \leqq M \sum_j (|\varphi(b_j)| + |\varphi(a_j)|) < 2 m_n M \gamma.$$

For $\gamma$ sufficiently small this gives the desired inequality.

LEMMA 5. *If the integral $I_C$ satisfies the condition* (II), *and if $K$ is a bounded class of ordinary curves such that $I_C$ has an upper bound on $K$, then the curves of $K$ are equi-absolutely continuous.**

Let $\bar{A}$ be a bounded closed set in the domain $A$, containing all the curves of $K$, and let $M$ and $\alpha$ be constants corresponding to the set $\bar{A}$ according to condition (II). There exists a constant $N > 0$ satisfying the inequality $f(x, y, y') > -N$ for all $(x, y)$ in $\bar{A}$ and for all $y'$ with $\|y'\| \leqq M$, and hence for all $y'$, by condition (II). Let $D$ be the maximum difference of abscissas of points in $\bar{A}$. Then $I_C$ has a lower bound on the class $K$,

* This lemma is implicitly contained in Tonelli's proof of theorem 3. See II, p. 282 ff.

greater than or equal to $-ND$. Let $L$ be the upper bound of $I_C$ on $K$. Then for a curve

$$C: \quad y = y(x), \qquad (a, b),$$

of $K$, we have, if $E_1$ is the set of points where $\|y'\| > M$, and $E_2$ the set where $\|y'\| \leqq M$,

$$\int_a^b \|y'\|^\alpha \, dx \leqq M^\alpha D + \int_{E_1} f(x, y, y') \, dx$$

$$\leqq M^\alpha D + L - \int_{E_2} f(x, y, y') \, dx \leqq M^\alpha D + L + ND \equiv H.$$

We shall now show that the inequality

$$\int_a^b \|y'\|^\alpha \, dx \leqq H$$

holding for all the curves of $K$, is sufficient to guarantee the equi-absolute continuity of the curves of $K$. Let $\varepsilon$ be an arbitrary positive number. Then for every set $E$ of points in $(a, b)$ on which

$$\|y'(x)\| > \left(\frac{2H}{\varepsilon}\right)^{\frac{1}{\alpha-1}},$$

we have

$$H \geqq \int_E \|y'\| \, \|y'\|^{\alpha-1} \, dx \geqq \frac{2H}{\varepsilon} \int_E \|y'\| \, dx,$$

and hence

$$\int_E \|y'\| \, dx \leqq \frac{\varepsilon}{2}. \tag{2}$$

For a set $E$ on which

$$\|y'(x)\| \leqq \left(\frac{2H}{\varepsilon}\right)^{\frac{1}{\alpha-1}},$$

we have

$$\int_E \|y'\| \, dx \leqq \left(\frac{2H}{\varepsilon}\right)^{\frac{1}{\alpha-1}} m E. \tag{3}$$

Hence for every set $E$ of points of $(a, b)$ having

$$m E \leqq \frac{\varepsilon}{2} \left(\frac{\varepsilon}{2H}\right)^{\frac{1}{\alpha-1}},$$

we have by combining (2) and (3),

$$\int_E \|y'\| \, dx \leqq \varepsilon.$$

This yields the equi-absolute continuity of the curves of $K$, with $V_K(\lambda) \leqq 2(H\lambda^{\alpha-1})^{\frac{1}{\alpha}}$.

## III. Sufficient conditions for semi-continuity.

12. THEOREM 1. *If the integral $I_C$ satisfies the conditions* (I) *and* (II), *then $I_C$ has lower semi-continuity.**

We desire to show that, for every ordinary curve $C_0$ and every positive number $\varepsilon$ there exists a positive $\varrho$ such that, for every ordinary curve $C$ in $(C_0)_\varrho$ we have $I_C - I_{C_0} > -\varepsilon$. To do this we break up the intervals of definition of

$$C: \quad y = y(x), \qquad (a, b),$$
$$C: \quad y = y_0(x), \qquad (a_0, b_0),$$

into sets $E_1$, $E_2$, and $E_{20}$, $(a, b) = E_1 + E_2$, $(a_0, b_0) = E_1 + E_{20}$, such that $y_0'(x)$ exists and is bounded on $E_1$, while the measures of $E_2$ and $E_{20}$ are small.

Let $\bar{A}$ be the set composed of all points in the domain $A$ at a distance not greater than unity from some point of $C_0$. Let $K$ be the class of all ordinary curves lying in $\bar{A}$, for which $I_C < I_{C_0}$. Then the curves of $K$ (if such exist) are equi-absolutely continuous, by lemma 5 of the previous section.

Let $M$ and $\alpha$ correspond to the set $\bar{A}$, according to condition (II). As was shown in proving lemma 5, there is a constant $N > 0$ such that $f(x, y, y') > -N$ for all $(x, y)$ in $\bar{A}$ and all $y'$. If $E_P$ denotes the set of points in $(a_0, b_0)$ for which $\|y_0'\| > P$, then $mE_P$ approaches zero with $1/P$. Now there exists a positive constant $\delta < \varepsilon/16N$ and $< 1$, such that

$$\left| \int_E f_{C_0} \, dx \right| < \frac{\varepsilon}{16} \tag{4}$$

†whenever the set $E$ has $mE \leqq \delta$. Fix a number $P$ such that

$$P > M, \qquad mE_P < \delta. \tag{5}$$

We require $\varrho$ to be less than $\delta$, and such that all ordinary curves in $(C_0)_\varrho$ are in $\bar{A}$. Let $a' = a_0 + \delta$, $b' = b_0 - \delta$, $E_1 = (a', b') - Ep$, $E_2 = (a, b) - E_1$,

* Tonelli, I, pp. 400–406.

† The notation $f_{C_0}$, corresponding to $I_{C_0}$, denotes $f(x, y_0(x), y_0'(x))$.

$E_{20} = (a_0, b_0) - E_1$. Then if the curve $C$ of the class $K$ is in the neighborhood $(C_0)_\rho$, we have

$$\|y_0'\| \leqq P \text{ on } E_1, \qquad mE_2 < 5\delta, \qquad mE_{20} < 3\delta, \tag{6}$$

$$\begin{aligned} I_C - I_{C_0} &= \int_{E_1} (fc - fc_0)\,dx + \int_{E_2} fc\,dx - \int_{E_{20}} fc_0\,dx \\ &\geqq \int_{E_1} (fc - fc_0)\,dx - N\,mE_2 - \frac{3\epsilon}{16} \\ &> \int_{E_1} (fc - fc_0)\,dx - \frac{\epsilon}{2}. \end{aligned} \tag{7}$$

The integral remaining on the right is the sum of the following four parts:

$$i_1 = \int_{E_1} \mathcal{E}(x, y, y_0', y')\,dx,$$

$$i_2 = \int_{E_1} [f(x, y, y_0') - f(x, y_0, y_0')]\,dx,$$

$$i_3 = \int_{E_1} [y_i' - y_{0i}'][f_{y_i'}(x, y, y_0') - f_{y_i'}(x, y_0, y_0')]\,dx,$$

$$i_4 = \int_{E_1} [y_i' - y_{0i}'] f_{y_i'}(x, y_0, y_0')\,dx.$$

By condition (I), we have $i_1 \geqq 0$. Since $f$ is continuous and $\|y_0'(x)\| \leqq P$ on $E_1$, we have $|i_2| < \epsilon/8$ when $\rho$ is sufficiently small. Similarly we have, since the curve $C$ is in the class $K$,

$$\begin{aligned} |i_3| &\leqq \bar{\epsilon}\left[\int_{E_1} \|y'\|\,dx + \int_{E_1} \|y_0'\|\,dx\right] \\ &\leqq \bar{\epsilon}\left[2\,V_K(b_0 - a_0 + 2) + \int_{a_0}^{b_0} \|y_0'\|\,dx\right] < \frac{\epsilon}{8} \end{aligned}$$

whenever $\rho$ is sufficiently small. To show that $|i_4| < \epsilon/8$ when $\rho$ is sufficiently small, we apply lemma 4 to each term. For the $i$th term we set

$$\begin{aligned} g(x) &= f_{y_i'}(x, y_0, y_0') \quad \text{on} \quad E_1, \\ &= 0 \qquad\qquad\quad \text{on} \quad (a', b') - E_1, \\ \varphi(x) &= y_i(x) - y_{0i}(x). \end{aligned}$$

THEOREM 2. *Suppose the integral $I_C$ satisfies conditions* (I) *and* (II), *and let*

$$C_0: \quad y = y_0(x), \qquad (a_0, b_0)$$

*be an absolutely continuous curve lying in the domain $A$, for which the integral $I_C$ does not exist. Then for every positive number $\mu$ there exists a positive $\varrho$ such that, for every ordinary curve $C$ in $(C_0)_\varrho$ we have $I_C > \mu$.**

The proof deviates only slightly from the proof of theorem 1. For the class $K$ we take the class of all ordinary curves lying in $\bar{A}$, for which $I_C \leqq \mu$. The two sentences involving the inequalities (4) and (5) are to be replaced by the two following:

There exists a constant $P > M$ such that:

$$mE_P < \frac{1}{16N}, \qquad \int_{E_{1P}} f_{C_0}\, dx > \mu + 1,$$

where $E_{1P} = (a_0, b_0) - E_P$. Also there exists a positive number $\delta < 1/16N$ such that, for every subset $E$ of $E_{1P}$ having $mE \leqq \delta$ we have

$$\left| \int_E f_{C_0}\, dx \right| < \frac{1}{16}.$$

The inequalities (6) are to be replaced by

$$\|y_0'\| \leqq P \text{ on } E_1, \qquad mE_2 < \frac{5}{16N}, \qquad m(E_{1P} - E_1) \leqq 2\delta.$$

and (7) is to be replaced by

$$I_C = \int_{E_1} [f_C - f_{C_0}]\, dx + \int_{E_{1P}} f_{C_0}\, dx - \int_{(E_{1P} - E_1)} f_{C_0}\, dx + \int_{E_2} f_C\, dx$$

$$> \int_{E_1} [f_C - f_{C_0}]\, dx + (\mu + 1) - \frac{1}{8} - \frac{5}{16}.$$

The integral remaining is handled exactly as before. Hence $I_C > \mu$. Consequently there are no members of the class $K$ in $(C_0)_\varrho$ when $\varrho$ is sufficiently small.

## IV. Sufficient conditions for the existence of the absolute minimum of $I_C$.

13. We consider first the existence of a minimum in a *bounded* class $K$, and extend to unbounded classes in theorem 4. An additional hypothesis is needed to insure a minimum in unbounded classes, and cases in which this condition is fulfilled are considered in § 14.

* Cf. Tonelli I, p. 443. It is the proof of this theorem which requires our lemma 4 in place of Tonelli's lemma in I, p. 213.

THEOREM 3. *Suppose the integral $I_C$ satisfies the conditions* (I) *and* (II), *and let $K$ be a bounded closed class of ordinary curves. Then $I_C$ has an absolute minimum on $K$.**

As was shown in the proof of lemma 5, $I_C$ has a lower bound on the class $K$, and hence a greatest lower bound $i$. Let $\{C_n\}$ be a minimizing sequence† for $I_C$ in $K$, i. e., $\{C_n\}$ satisfies the condition

$$I_{C_n} \leqq i + \frac{1}{n}, \qquad (n = 1, 2, \cdots). \tag{8}$$

Then the curves of the sequence are equi-absolutely continuous, by lemma 5 (§ 11). Hence the sequence has a curve of accumulation $C_0$, by the corollary to Ascoli's theorem (§ 9). By lemma 1, $C_0$ is absolutely continuous. The curve $C_0$ lies in the domain $A$, since $A$ is closed in every finite portion of $(x, y)$ space. By theorem 2 (§ 12) and the condition (8), $C_0$ is an ordinary curve, and hence is contained in the closed class $K$. Therefore $I_{C_0} \geqq i$. Since by theorem 1 $I_C$ has lower semi-continuity, $I_{C_0} \leqq i$. Hence $I_{C_0} = i$.

For the statement of condition (III) and for later use, we introduce the notation $S_\varrho$ to denote the "sphere" (circle in case $k = 1$, hypersphere in case $k > 2$) having radius $\varrho$ and center at the origin.

(III) *For every bounded closed set $\bar{A}$ in the domain $A$ and every positive number $L$ there exists a positive $\varrho$ such that, if $C$ is an ordinary curve having at least one point in $\bar{A}$ and at least one point outside the sphere $S_\varrho$, then $I_C > L$.*

THEOREM 4. *Suppose the integral $I_C$ satisfies the conditions* (I), (II), *and* (III), *and let $K$ be a closed class of ordinary curves each of which has at least one point in a given bounded closed set $\bar{A}$ of points of the domain $A$. Then $I_C$ has an absolute minimum on $K$.*‡

Let $i$ be the greatest lower bound (finite or infinite) of $I_C$ on the class $K$. Then there exist curves $C$ in $K$ for which

$$I_C < i + 1 \equiv L, \qquad \text{if } i \text{ is finite},$$
$$I_C < 1 \equiv L, \qquad \text{if } i = -\infty.$$

Let $\varrho$ correspond to the set $\bar{A}$ and the number $L$ according to condition (III), and let $\bar{K}$ be the subclass of $K$ consisting of all those curves of $K$ which have no points outside the sphere $S_\varrho$. Then $\bar{K}$ is bounded and closed, and $I_C$ has a minimum $\bar{i} < L$ on $\bar{K}$. If $\bar{i}$ were greater than $i$, there would be curves in the class $K - \bar{K}$ making $I_C < \bar{i} < L$, which is a contradiction. Hence $\bar{i} = i =$ minimum of $I_C$ on $K$.

* Tonelli II, p. 282.
† Tonelli II, p. 281.
‡ Tonelli II, p. 307.

14. In applying theorem 4 to particular cases, it is useful to have some criteria for condition (III). We state five further conditions, for which with (III) the following relations hold:*

(IV), (V), and (VI) imply (III);

(IV) and (VII) imply (III);

(VIII) implies (VI).

(IV) *The maximum difference of abscissas of points in the domain $A$ is finite.*

(V) *There exists a constant $N>0$ such that $f(x, y, y')>-N$ in the domain $R$.*

(VI) *There exist constants $M>0$ and $\alpha \geqq 1$ and a function $\varphi(y)$ defined and continuous for all values of $y$, such that*

1) $\varphi(y)>0$ *whenever* $\|y\| \geqq M$;

2) $\lim_{\|y\|=\infty} \|y\|^{\alpha} \varphi(y) = \infty$;

3) $f(x, y, y') \geqq \|y'\|^{\alpha} \varphi(y)$ *for every* $(x, y, y')$ *in* $R$ *with* $\|y\| \geqq M$, $\|y'\| \geqq M$.

(VII) *There exist positive constants $c$, $d$, $\alpha$, $\beta$, such that we have, for all $(x, y, y')$ in $R$,*

$$f(x, y, y') = \varphi(x, y') - \psi(x, y),$$

$$\varphi(x, y') \geqq c\|y'\|^{1+\alpha+\beta} - d, \qquad \psi(x, y) \leqq d\|y\|^{1+\alpha} + d.$$

(VIII) *There exist positive constants $M$ and $c$ such that we have, for all $(x, y, y')$ in $R$ with $\|y'\| \geqq M$,*

$$f(x, y, y') \geqq c\|y'\|.$$

We shall show first that (IV), (V), and (VI) imply (III). Let $D$ be the maximum difference of abscissas of points in the domain $A$. Fix a constant $p$ greater than $M$ and greater than the maximum of $\|y\|$ in the bounded closed set $\bar{A}$. Consider an ordinary curve

$$C: \quad y = y(x), \qquad (a, b).$$

having at least one point in the set $\bar{A}$, and with maximum $\|y(x)\| \equiv Y \geqq 2p$. Then there exists an arc

$$\bar{C}: \quad y = y(x), \qquad (\bar{a}, \bar{b}),$$

of $C$, having

$\|y(x)\| = Y$ at one end-point,

$\|y(x)\| = p$ at the other end-point,

$p < \|y(x)\| < Y$ at interior points.

* Cf. Tonelli II, pp. 308, 311.

For every number $q \geqq p$, the function $\varphi(y)$ has a minimum $\varphi_q$ on the closed set of points in the $y$-space for which $p \leqq \| y \| \leqq q$. Also, $\varphi_q > 0$ since $p > M$. We have also the preliminary inequalities

$$\left[\int_{\bar{a}}^{\bar{b}} \| y' \| \, dx\right]^{\alpha} \leqq (\bar{b}-\bar{a})^{\alpha-1} \int_{\bar{a}}^{\bar{b}} \| y' \|^{\alpha} \, dx,$$

$$\frac{Y}{2} \leqq Y - p = \left| \| y(\bar{b}) \| - \| y(\bar{a}) \| \right| \leqq \| y(\bar{b}) - y(\bar{a}) \|$$

$$= \left\| \int_{\bar{a}}^{\bar{b}} y' \, dx \right\| \leqq \int_{\bar{a}}^{\bar{b}} \| y' \| \, dx,$$

$$\left(\frac{Y}{2}\right)^{\alpha} \leqq D^{\alpha-1} \int_{\bar{a}}^{\bar{b}} \| y' \|^{\alpha} \, dx. \tag{9}$$

The first is obvious if $\alpha = 1$, and follows from the Schwarz inequality if $\alpha > 1$. The second is readily verified, and (9) results from combining the first two.

Now let $E_1$ denote the set of points in $(\bar{a}, \bar{b})$ where $\| y'(x) \| \geqq M$, and $E_2 \equiv (\bar{a}, \bar{b}) - E_1$. Then

$$I_{\bar{C}} = \int_{E_1} f_{\bar{C}} \, dx + \int_{E_2} f_{\bar{C}} \, dx \geqq \varphi_Y \int_{E_1} \| y' \|^{\alpha} dx - N(\bar{b}-\bar{a})$$

$$\geqq \varphi_Y \left[\int_{\bar{a}}^{\bar{b}} \| y' \|^{\alpha} dx - M^{\alpha}(\bar{b}-\bar{a})\right] - N(\bar{b}-\bar{a})$$

$$\geqq \varphi_Y \left(\frac{Y}{2}\right)^{\alpha} \frac{1}{D^{\alpha-1}} - \varphi_p M^{\alpha} D - ND.$$

Hence

$$I_C = I_{\bar{C}} + \int_{(a,b)-(\bar{a},\bar{b})} f_C \, dx \geqq \varphi_Y Y^{\alpha} \frac{1}{2^{\alpha} D^{\alpha-1}} - \varphi_p M^{\alpha} D - 2ND. \tag{10}$$

It remains to examine the behavior of $\varphi_Y Y^{\alpha}$. From the condition $(VI_2)$ we have that for every positive number $H$ there exists a positive $q > p$ such that, whenever $\| y \| > q$ we have $\varphi(y) \| y \|^{\alpha} > H$. Now if a point $\bar{y}$ for which $\varphi_Y = \varphi(\bar{y})$ has $\| \bar{y} \| \leqq q$, then $\varphi_Y \geqq \varphi_q$. If on the other hand, $\| \bar{y} \| > q$, then $\varphi_Y Y^{\alpha} > H$. In either case, if $\bar{Y}$ is greater than $p$, $q$, and $H/\varphi_q$, then $\varphi_Y Y^{\alpha} > H$ whenever $Y > \bar{Y}$. Since the other terms on the right of (10) are fixed, we can make $I_C > L$ by taking $Y$ large enough.

To prove that (IV) and (VII) imply (III), let the symbols $D$, $p$ and $Y$ have the same significance as before. The inequality (9) is still valid if we replace $(\bar{a}, \bar{b})$ by $(a, b)$, and $\alpha$ by $(1 + \alpha + \beta)$. Then we have

$$I_C = \int_a^b \varphi(x, y')\, dx - \int_a^b \psi(x, y)\, dx$$
$$\geqq c \int_a^b \| y' \|^{1+\alpha+\beta}\, dx - d \int_a^b \| y \|^{1+\alpha}\, dx - 2\, dD$$
$$\geqq \frac{c}{D^{\alpha+\beta}} \left(\frac{Y}{2}\right)^{1+\alpha+\beta} - dD\, Y^{1+\alpha} - 2\, dD.$$

Since $\alpha$, $\beta$, and $c$ are positive, $I_C$ approaches $+\infty$ with $Y$.

That the condition (VIII) implies (VI) is obvious. Also it is simple to make a direct proof that (IV), (V), and (VIII) imply (III).

As examples in which some of these conditions are satisfied, Tonelli cites the following cases in the plane:

$$f = y'^2, \qquad f = \frac{y'^2}{1+y^2} + \sqrt{1+y'^2},$$
$$f = \frac{y'^4}{1+y^2}, \qquad f = y'^2 - y.$$

A case in the plane in which a minimum does not exist is the following: $f = y'^2 - y^2$, and $K$ is the class of all ordinary curves joining the points (0, 0) and (4, 0). A sequence $\{C_n\}$ composed of the equal sides of isosceles triangles on the base (0, 0), (4, 0), with increasing altitudes, makes $\lim I_{C_n} = -\infty$.

# THE MIXED MEAN FUNCTION.*

By C. A. Shook.

1. **Definition and existence.** In dealing with the problem of finding the potential of a circumference of radius $a$ at a point $P$ in its plane at a distance $\varrho$ from its center, where $\varrho < a$, Gauss employed the *arithmetico-geometrical mean* of two numbers. For brevity we shall refer to this as the *mixed mean*. This mean may be defined as follows. Let $a$ and $b$ be two positive numbers, $a > b$. Let $a_1 = (a+b)/2$ and $b_1 = \sqrt{ab}$. Similarly, let $a_n = (a_{n-1} + b_{n-1})/2$ and $b_n = \sqrt{a_{n-1}\, b_{n-1}}$. It will be shown that as $n$ increases without limit, $\lim a_n$ and $\lim b_n$ exist and moreover that these limits are equal. This common limit is called the mixed mean of $a$ and $b$. We shall denote this by $\mu(a, b)$. Gauss showed that if $M$ is the total mass of the circumference, then the Newtonian Potential, $V$, at $P$ is,

$$V = \frac{M}{\mu(a+\varrho,\ a-\varrho)}.$$

The proof of the existence of the common limit is as follows. Since $b < a$, $a + b < 2a$, or $\frac{a+b}{2} < a$, or $a_1 < a$; also, since $a > b$, $a + b > 2b$, or $\frac{a+b}{2} > b$, or $a_1 > b$. Hence,

$$b < a_1 < a.$$

Since $a > b$, $ab > b^2$, or $\sqrt{ab} > b$, or $b_1 > b$; also, since $a > b$, $a^2 > ab$, or $a > \sqrt{ab}$, or $b_1 < a$. Hence

$$b < b_1 < a.$$

Similarly, step by step, $b < a_n < a_{n-1}$ and $b_{n-1} < b_n < a$.

Since the $a_n$ form a decreasing sequence all terms of which are greater than $b$, $\lim a_n$ exists. Since the $b_n$ form an increasing sequence, all terms of which are less than $a$, $\lim b_n$ exists.

Now, since $b_1 > b$, $a_1 - b_1 < a_1 - b$, or $a_1 - b_1 < \frac{a-b}{2}$. Likewise, $a_2 - b_2 < \frac{a_1 - b_1}{2} < \frac{a-b}{2^2}$ and finally,

$$a_n - b_n < \frac{a-b}{2^n}.$$

* Received September 19, 1926.

Hence

$$\lim(a_n - b_n) = 0, \quad \text{or} \quad \lim a_n = \lim b_n.$$

2. **Mixed mean in terms of elliptic integrals.** To obtain an explicit formula for $\mu$ we shall start with the integral,

$$\varphi(u, v) = \int_0^{2\pi} \frac{d\omega}{\sqrt{u^2 \sin^2\omega + v^2 \cos^2\omega}}. \tag{2}$$

We note first that $\varphi(u, v) = \varphi(v, u)$. This results if we put $\omega = \omega' + \pi/2$ and notice that

$$\int_{-\pi/2}^{3\pi/2} = \int_0^{2\pi}$$

on account of the periodicity of the integrand. A second property of $\varphi$ is that $\varphi(u, u) = 2\pi/u$. A third property of $\varphi$ will appear from a change of variable of integration in (2). Let two sides of a triangle be $\frac{1}{2}(u-v)$ and $\frac{1}{2}(u+v)$ and let the angle included by these sides be $2\omega$. The triangle is uniquely determined by these parts. Now let $\pi - \psi$ be the angle opposite the side $\frac{1}{2}(u+v)$, thus defining a relation between $\omega$ and $\psi$. We shall change the variable of integration in (2) from $\omega$ to $\psi$. Let $r$ be the side opposite the angle $2\omega$ and $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$. Then

$$\begin{aligned} u^2 \sin^2\omega + v^2 \cos^2\omega &= (x+y)^2 \sin^2\omega + (x-y)^2 \cos^2\omega \\ &= x^2 + y^2 + 2xy(\sin^2\omega - \cos^2\omega) \\ &= x^2 + y^2 - 2xy\cos 2\omega \\ &= r^2. \end{aligned}$$

But $x^2 = r^2 + y^2 + 2ry\cos\psi$, so that $r = \sqrt{x^2 - y^2\sin^2\psi} - y\cos\psi$. The law of sines gives

$$\begin{aligned} y\sin\psi &= x\sin(\psi - 2\omega) \\ y\cos\psi\, d\psi &= x\cos(\psi - 2\omega)(d\psi - 2\,d\omega) \\ &= \sqrt{x^2 - y^2\sin^2\psi}\,(d\psi - 2\,d\omega) \\ d\omega &= \frac{\sqrt{x^2 - y^2\sin^2\psi} - y\cos\psi}{2\sqrt{x^2 - y^2\sin^2\psi}} \\ &= \frac{r}{2\sqrt{x^2 - y^2\sin^2\psi}}. \end{aligned}$$

Thus we have,

$$\begin{aligned}\varphi(u, v) &= \frac{1}{2}\int_0^{4\pi} \frac{d\psi}{\sqrt{x^2 - y^2\sin^2\psi}} \\ &= \int_0^{2\pi} \frac{d\psi}{\sqrt{(x^2 - y^2)\sin^2\psi + x^2\cos^2\psi}} \\ &= \int_0^{2\pi} \frac{d\psi}{\sqrt{uv\sin^2\psi + \tfrac{1}{4}(u+v)^2\cos^2\psi}} \\ &= \varphi\left(\sqrt{uv}, \frac{u+v}{2}\right) = \varphi\left(\frac{u+v}{2}, \sqrt{uv}\right).\end{aligned} \tag{3}$$

Here we have the most fundamental property of $\varphi$, namely,

$$\varphi(u, v) = \varphi(u_1, v_1) = \varphi(u_2, v_2) = \cdots$$

where,

$$u_{n+1} = \frac{u_n + v_n}{2}; \qquad v_{n+1} = \sqrt{u_n v_n}.$$

If $\mu$ is the mixed mean of $u$ and $v$, we have, on going to the limit,

$$\varphi(u, v) = \varphi(\mu, \mu) = \frac{2\pi}{\mu}. \tag{4}$$

But referring to (3),

$$\begin{aligned}\varphi(u, v) &= 4\int_0^{\pi/2} \frac{d\psi}{\sqrt{\tfrac{1}{4}(u+v)^2 - \tfrac{1}{4}(u-v)^2\sin^2\psi}} \\ &= \frac{8}{u+v}\int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \left(\frac{u-v}{u+v}\right)^2\sin^2\psi}} \\ &= \frac{8}{u+v} F\left(\frac{u-v}{u+v}, \frac{\pi}{2}\right)\end{aligned} \tag{5}$$

where $F$ is the elliptic integral of the first kind. (4) and (5) give,

$$\mu(u, v) = \frac{\pi(u+v)}{4F\left(\frac{u-v}{u+v}, \frac{\pi}{2}\right)} \tag{6}$$

which expresses the mixed mean in terms of a well known function. Putting $u = 4$, $v = 1$, for example and taking $F\left(\frac{3}{5}, \frac{\pi}{2}\right) = 1.752$ from a table we obtain from (6) $\mu(4, 1) = 2.243$. This value is easily verified by direct calculation. We obtain successively,

| | |
|---|---|
| 4 | 1 |
| 2.5 | 2 |
| 2.25 | 2.236 |
| 2.243 | 2.243 |

The rapid convergence to the common limit, making the computation of $\mu$ simple matter, suggests the reversal of (6). To do this note first that (6) gives,

$$\mu(u, v) = v \frac{\pi\left(\frac{u}{v}+1\right)}{4F\left(\frac{u/v-1}{u/v+1}, \frac{\pi}{2}\right)},$$

$$\mu(u, v) = v\mu\left(\frac{u}{v}, 1\right). \tag{7}$$

Hence, putting $\frac{u-v}{u+v} = k$, we get,

$$F\left(k, \frac{\pi}{2}\right) = \frac{\pi}{2\mu(1+k, 1-k)} \tag{8}$$

an equation which makes possible the rapid calculation of the complete integral of the first kind.

3. **The derivatives of $\mu(x)$.** By (7) the mixed mean of two numbers can always be made to depend on the mixed mean of their quotient and one. We shall therefore in what follows suppose the second and smaller argument to be one and shall write $\mu(x)$, $x>1$, and consider $\mu$ as a function of one argument only. Since we shall deal only with complete elliptic integrals, we shall write $F$ or $F(k)$ for $F(k, \pi/2)$ and $E$ or $E(k)$ for $E(k, \pi/2)$, the complete elliptic integral of the second kind. A prime will always denote differentiation with respect to $x$.

Dividing both sides of (6) by $v$ and putting $u/v = x$, $k = (x-1)/(x+1)$, we get,

$$\mu(x) = \frac{\pi}{4}\frac{x+1}{F(k)} \tag{9}$$

which may also be written,

$$F(k) = \frac{\pi}{2}\frac{1}{(1-k)\mu(x)} = \frac{\pi}{4}\frac{x+1}{\mu(x)}. \tag{10}$$

Putting $\Delta\varphi = \sqrt{1-k^2\sin^2\varphi}$,

$$F = \int_0^{\pi/2} \frac{d\varphi}{\Delta\varphi},$$

$$\frac{dF}{dk} = \int_0^{\pi/2} \frac{k\sin^2\varphi\, d\varphi}{(\Delta\varphi)^3},$$

$$= \frac{k}{1-k^2}\int_0^{\pi/2} \frac{\cos^2\varphi\, d\varphi}{\Delta\varphi},$$

$$= \frac{1}{k(1-k^2)}\int_0^{\pi/2} \Delta\varphi\, d\varphi - \frac{1}{k}\int_0^{\pi/2} \frac{d\varphi}{\Delta\varphi},$$

$$\frac{dF}{dk} = \frac{1}{k(1-k^2)}E - \frac{1}{k}F. \tag{11}$$

Since,

$$\frac{dk}{dx} = \frac{2}{(x+1)^2},$$

we have,

$$\frac{dF}{dx} = F' = \frac{1}{2x(x^2-1)}[(x+1)^2E - 4xF] \tag{12}$$

or in terms of $\mu$,

$$F' = \frac{1}{2x(x-1)\mu}[(x+1)\mu E - \pi x]. \tag{13}$$

We are now in position to find $\mu'$. Multiplying (9) by $F$ and differentiating with respect to $x$,

$$\mu' F + \mu F' = \frac{\pi}{4}.$$

whence, replacing $F'$ by its value,

$$\mu'(x) = \frac{\mu}{\pi x(x-1)}[\pi x - 2\mu E]. \tag{14}$$

This may also be written,

$$\mu'(x) = \frac{\pi}{4}\,\frac{(x+1)^2}{x(x-1)}\,\frac{1}{F}\left[\frac{x}{x+1} - \frac{E}{2F}\right]. \tag{15}$$

To find $\mu''$ it is perhaps easiest to start from (14), but we must first find $E'$. Now,

$$E = \int_0^{\pi/2} \Delta\varphi\, d\varphi,$$

so that

$$\frac{dE}{dk} = -k\int_0^{\pi/2} \frac{\sin^2\varphi\, d\varphi}{\Delta\varphi}$$

$$= \frac{1}{k}\int_0^{\pi/2} \left(\Delta\varphi - \frac{1}{\Delta\varphi}\right) d\varphi \tag{16}$$

$$= \frac{1}{k}(E-F).$$

From this we get

$$E' = \frac{2}{x^2-1}(E-F) \tag{17}$$

$$E' = \frac{1}{2\mu(x^2-1)}[4\mu E - \pi(x+1)]. \tag{18}$$

Equation (14) may be written

$$\pi x(x-1)\mu' = \pi\mu x - 2\mu^2 E,$$

whence, differentiating,

$$\pi x(x-1)\mu'' + \pi(2x-1)\mu' = \pi\mu' x + \pi\mu - 4\mu\mu' E - 2\mu^2 E'.$$

Replacing $E'$ by (18) and $E$ by (14), we get, after reduction,

$$x\mu\mu'' = 2x\mu'^2 + \mu\frac{x-\mu'(3x^2-1)}{x^2-1}. \tag{19}$$

We have so far considered $\mu(x)$ only for $x \geqq 1$. If $0 \leqq x \leqq 1$, obviously

$$\mu(x) = x\mu\left(\frac{1}{x}\right), \tag{20}$$

so that

$$\mu'(x) = \mu\left(\frac{1}{x}\right) - \frac{1}{x}\mu'\left(\frac{1}{x}\right) \tag{21}$$

and

$$\mu''(x) = \frac{1}{x^3}\mu''\left(\frac{1}{x}\right). \tag{22}$$

4. **Particular values.** We shall next discuss the values of $\mu$ and its derivatives for special values of $x$, namely 0, 1 and $\infty$. If we form the mixed mean of $x$ and 1, $b_1 = \sqrt{x}$. But $\mu > b_1$ and since by taking $x$ large enough, $\sqrt{x}$ can be made as large as we please, $\lim_{x=\infty} \mu(x) = \infty$.

It is also necessary to examine $\lim_{x=\infty} \mu/x^n$ for positive values of $n$. Since $\lim_{x=\infty} F = \infty$,

$$\lim_{x=\infty} \frac{\mu(x)}{x} = \lim_{x=\infty} \frac{\mu(x)}{x+1} = \lim_{x=\infty} \frac{\pi}{4F} = 0.$$

Next, if $0<n<1$,

$$\lim_{x=\infty} \frac{\mu(x)}{x^n} = \lim_{x=\infty} \frac{\mu(x)}{(x+1)^n}$$

$$= \frac{\pi}{4} \lim_{x=\infty} \frac{(x+1)^{1-n}}{F(k)}$$

$$= \frac{\pi(1-n)}{2} \lim_{x=\infty} \frac{x(x^2-1)}{(x+1)^3} \frac{1}{E - \frac{4xF}{(x+1)^2}} (x+1)^{1-n}$$

$$= \infty,$$

since, from (9), $\lim_{x=\infty} \frac{F}{x+1} = 0$ and $\lim_{x=\infty} E = 1$.

Hence we have

THEOREM: *As $x$ increases without limit,*

$$\frac{\mu(x)}{x^n} \quad \begin{cases} \textit{tends to zero,} & n \geqq 1, \\ \textit{increases without limit,} & n<1. \end{cases}$$

From (15) we have immediately,

$$\lim_{x=\infty} \mu'(x) = 0.$$

Since,

$$\frac{dE}{dk} = -k \int_0^{\pi/2} \frac{\sin^2\varphi \, d\varphi}{\Delta\varphi},$$

$$E' = \frac{-2(x-1)}{(x+1)^2} \int_0^{\pi/2} \frac{\sin^2\varphi \, d\varphi}{\Delta\varphi},$$

so that

$$\lim_{x=1} E' = 0.$$

But,

$$E' = \frac{2}{x^2-1}(E-F) = \frac{2}{x+1} \frac{E-F}{x-1},$$

so that,

$$\lim_{x=1} E' = \frac{2}{1+1} \lim_{x=1} \frac{E'-F'}{1}, \text{ or, } \lim_{x=1} F' = 0.$$

From (15),

$$\mu'(x) = \frac{\pi}{4} \frac{x+1}{2xF^2} \frac{2Fx-(x+1)E}{x-1}.$$

Since for $x = 1$, $F = \pi/2 = E$,

$$\lim_{x=1+} \mu'(x) = \frac{1}{\pi} \lim_{x=1+} [2F'x + 2F - E - (x+1)E']$$
$$= \frac{1}{2}.$$

For $x = 0$ we have from (20) and (21),

$$\mu(x) = \lim_{x=0} x\mu\left(\frac{1}{x}\right) = \lim_{x=\infty} \frac{1}{x}\mu(x) = 0.$$

$$\mu'(x) = \lim_{x=0} \left[\mu\left(\frac{1}{x}\right) - \frac{1}{x}\mu'\left(\frac{1}{x}\right)\right] = \lim_{x=\infty} [\mu(x) - x\mu'(x)]$$
$$= \lim_{x=\infty} \mu\left[-\frac{(\pi-1)x+1}{x-1} + \frac{2\mu E}{x-1}\right], \text{ by (14)}$$
$$= \infty.$$

The values of $\mu''(x)$ are obtained by limiting processes similar to those already employed. The following table exhibits the various results.

| $x$ | $\mu$ | $\mu'$ | $\mu''$ |
|---|---|---|---|
| 0 | 0 | $\infty$ | $-\infty$ |
| 1 | 1 | $\frac{1}{2}$ | $-\frac{1}{4}$ |
| $\infty$ | $\infty$ | 0 | 0 |

5. **Tables of $\mu(x)$.** We are appending tables giving the value of $\mu(x)$ to four significant figures for values of $x$ from 1.0 to 20.0 at intervals of 0.1 and from 0 to 100 at intervals of 1. The intervals were chosen to make accurate interpolation possible, although at some points in the table the intervals, for convenience, are smaller than would be necessary.

For $x < 1$ we can find $\mu(x)$ by (20). For $x > 100$ we may use a reduction formula obtained as follows.

Mixed mean of $x$ and 1 = mixed mean of $\frac{x+1}{2}$ and $\sqrt{x}$, by definition:

$= \sqrt{x}$ times mixed mean of $\frac{x+1}{2\sqrt{x}}$ and 1, by (7);

so that

$$\mu(x) = \sqrt{x}\,\mu\left(\frac{x+1}{2\sqrt{x}}\right). \tag{23}$$

This may also be proved by means of Landen's transformation. This formula gives $\mu(x)$ for values of $x$ up to about 40,000 in terms of $\mu(x)$ for values of $x$ less than 100. It may also be written,

$$\mu(x) = \sqrt{x}\,\mu\left(\cosh\frac{1}{2}\log x\right) \tag{24}$$

which may be found useful if adequate tables are available.

For large values of $x$, (23) becomes approximately,

$$\mu(x) = \sqrt{x}\,\mu\left(\frac{\sqrt{x}}{2}\right). \tag{25}$$

For $x = 1000$ the error is less than one percent and for $x > 10{,}000$ the error is less than one part in 1500.

## Table of values of the function $\mu(x)$.

| | .0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.000 | 1.050 | 1.098 | 1.145 | 1.192 | 1.238 | 1.283 | 1.327 | 1.371 | 1.414 | 1.457 |
| 2 | 1.457 | 1.499 | 1.541 | 1.583 | 1.624 | 1.664 | 1.705 | 1.745 | 1.785 | 1.825 | 1.864 |
| 3 | 1.864 | 1.903 | 1.942 | 1.980 | 2.019 | 2.057 | 2.094 | 2.132 | 2.169 | 2.206 | 2.243 |
| 4 | 2.243 | 2.280 | 2.316 | 2.353 | 2.389 | 2.425 | 2.461 | 2.497 | 2.532 | 2.568 | 2.604 |
| 5 | 2.604 | 2.639 | 2.674 | 2.710 | 2.745 | 2.780 | 2.814 | 2.849 | 2.883 | 2.918 | 2.952 |
| 6 | 2.952 | 2.986 | 3.020 | 3.054 | 3.088 | 3.122 | 3.155 | 3.189 | 3.222 | 3.256 | 3.289 |
| 7 | 3.289 | 3.322 | 3.355 | 3.388 | 3.421 | 3.454 | 3.487 | 3.520 | 3.552 | 3.585 | 3.617 |
| 8 | 3.617 | 3.649 | 3.681 | 3.714 | 3.746 | 3.778 | 3.810 | 3.842 | 3.873 | 3.905 | 3.937 |
| 9 | 3.937 | 3.969 | 4.000 | 4.032 | 4.063 | 4.095 | 4.126 | 4.158 | 4.189 | 4.220 | 4.251 |
| 10 | 4.251 | 4.282 | 4.313 | 4.344 | 4.375 | 4.406 | 4.437 | 4.468 | 4.498 | 4.529 | 4.560 |
| 11 | 4.560 | 4.591 | 4.621 | 4.652 | 4.682 | 4.713 | 4.743 | 4.773 | 4.804 | 4.834 | 4.864 |
| 12 | 4.864 | 4.894 | 4.924 | 4.955 | 4.985 | 5.015 | 5.045 | 5.075 | 5.104 | 5.134 | 5.164 |
| 13 | 5.164 | 5.194 | 5.224 | 5.253 | 5.283 | 5.313 | 5.342 | 5.372 | 5.401 | 5.431 | 5.460 |
| 14 | 5.460 | 5.489 | 5.519 | 5.548 | 5.578 | 5.607 | 5.636 | 5.665 | 5.694 | 5.723 | 5.752 |
| 15 | 5.752 | 5.781 | 5.810 | 5.839 | 5.868 | 5.897 | 5.925 | 5.954 | 5.983 | 6.011 | 6.040 |
| 16 | 6.040 | 6.068 | 6.097 | 6.126 | 6.154 | 6.183 | 6.211 | 6.240 | 6.268 | 6.297 | 6.325 |
| 17 | 6.325 | 6.353 | 6.382 | 6.410 | 6.439 | 6.467 | 6.495 | 6.523 | 6.552 | 6.580 | 6.608 |
| 18 | 6.608 | 6.636 | 6.664 | 6.693 | 6.721 | 6.749 | 6.777 | 6.805 | 6.833 | 6.861 | 6.889 |
| 19 | 6.889 | 6.917 | 6.945 | 6.972 | 7.000 | 7.028 | 7.056 | 7.084 | 7.111 | 7.139 | 7.167 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1.000 | 1.457 | 1.864 | 2.243 | 2.604 | 2.952 | 3.289 | 3.617 | 3.937 | 4.251 |
| 10 | 4.251 | 4.560 | 4.864 | 5.164 | 5.460 | 5.752 | 6.040 | 6.325 | 6.608 | 6.889 | 7.167 |
| 20 | 7.167 | 7.442 | 7.715 | 7.987 | 8.257 | 8.525 | 8.791 | 9.056 | 9.320 | 9.582 | 9.843 |
| 30 | 9.84 | 10.10 | 10.36 | 10.62 | 10.88 | 11.13 | 11.38 | 11.63 | 11.88 | 12.13 | 12.38 |
| 40 | 12.38 | 12.63 | 12.88 | 13.13 | 13.37 | 13.62 | 13.86 | 14.10 | 14.34 | 14.59 | 14.83 |
| 50 | 14 83 | 15.07 | 15.31 | 15.54 | 15.78 | 16.02 | 16.26 | 16.49 | 16.73 | 16.96 | 17.20 |
| 60 | 17.20 | 17.43 | 17.66 | 17.90 | 18.13 | 18 36 | 18.59 | 18.82 | 19.05 | 19.28 | 19.51 |
| 70 | 19.51 | 19.74 | 19.97 | 20.20 | 20.43 | 20.66 | 20.89 | 21.11 | 21.34 | 21.56 | 21.79 |
| 80 | 21.79 | 22.01 | 22.24 | 22.46 | 22.69 | 22.91 | 23.13 | 23.35 | 23.58 | 23.80 | 24.02 |
| 90 | 24.02 | 24.24 | 24.46 | 24.69 | 24.91 | 25.13 | 25.35 | 25.57 | 25.79 | 26.01 | 26.23 |

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# THE FOCAL POINT FOR THE PROBLEM OF LAGRANGE WITH ONE VARIABLE END POINT.*

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**Introduction.** In the Problem of Lagrange of the Calculus of Variations, we seek to determine among the sets of functions

$$y_1(x),\ y_2(x),\ \cdots,\ y_n(x) \tag{1}$$

satisfying the conditions

$$y_i(x_1) = y_{i1}, \qquad y_i(x_2) = y_{i2} \tag{2}$$

and the system of $m$ differential equations

$$\varphi_\alpha(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \quad (\alpha = 1, \cdots, m < n) \tag{3}$$

one which shall render

$$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx \tag{4}$$

a minimum value. We shall suppose $x_1 < x_2$. Hereafter the set $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ will be indicated by $(x, y, y')$; thus the integral $I$ is denoted by

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

and the differential equations (3) become $\varphi_\alpha(x, y, y') = 0$. The ranges of $i$ and $\alpha$ will be from 1 to $n$ and 1 to $m$ respectively. We shall find it convenient to employ the notation of tensor analysis. The sum $y_1 A_1 + y_2 A_2 + \cdots + y_n A_n$, for example can thus be represented by the single term, $y_i A_i$. When the index $i$ occurs twice in the same term, it is understood to represent the sum of $n$ terms of the same type.

In this paper, we shall consider the problem in which the second set of conditions in (2) are replaced by the following ones,

$$y_i(x_2) = Y_i(t), \qquad x_2 = X(t), \tag{5}$$

or, considered as a problem in geometry, one for which the end point 1 of the curve $E_{12}$ defined by $y_i = y_i(x)$, shall be regarded as fixed while

* Received September 21, 1926.

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the end point 2 shall be required to lie on a curve $L$, defined by the equations

$$y_i = Y_i(t), \qquad x = X(t). \tag{6}$$

The arc $E_{12}$ must satisfy certain well known conditions which have been derived for the case where both end points are regarded as fixed. We shall begin our discussion with the treatment of the first necessary condition, the well known Euler-Lagrange multiplier rule. As we proceed, it is found necessary to consider admissible variations of the arc $E_{12}$ and admissible families of variation arcs. We are lead to a set of necessary and sufficient conditions for the existence of a family of admissible arcs.

As the result of computing the first variation of the integral $I$, a further necessary condition, which the arc $E_{12}$ must satisfy in order that it may minimize $I$, is obtained. This is the transversality condition, which must be satisfied at the point 2.

The second variation is expressed as an integral in $x$ and the variation functions $\eta_i(x)$. By treating the problem of minimizing this integral as a problem of Lagrange in $x\eta$-space with one variable end point, the focal point condition, analogous to the condition of Jacobi for the plane problem, is obtained. A determinant whose zeros give the abscissa of the focal point of the curve $L$ on $E_{12}$ is easily derived.

The treatment of the second variation of the integral $I$ shows that the abscissa $x_3$ of the focal point of $L$ on $E_{12}$, must not lie between $x_1$ and $x_2$ if $E_{12}$ is to render $I$ a minimum value. By a geometric argument, we are not only able to arrive at the same conclusion but we may also exclude the case where $x_3$ coincides with $x_1$, provided the envelope of the one parameter family of extremals containing $E_{12}$ and cutting $L$ transversally, does not have a cusp for $x = x_3$.

1. **First necessary conditions.** The following hypotheses are useful in the work that follows:

(a) The set of functions $y_i(x)$ defining the arc $E_{12}$ whose minimizing properties are to be investigated, are continuous on $x_1\, x_2$, and this interval can be subdivided into a finite number of parts on each of which the functions have continuous first derivatives.

(b) For values of $(x, y, y')$ in the neighborhood $R$ of $E_{12}$, $f$ and $\varphi_\alpha$ have continuous derivatives of the first four orders.

(c) The matrix

$$\left\| \begin{matrix} \varphi_{1y_1} & \cdots & \varphi_{1y'_n} \\ \cdot & \cdots & \cdot \\ \varphi_{my'_1} & \cdots & \varphi_{my'_n} \end{matrix} \right\|$$

is of rank $m$ along $E_{12}$.

The functions $X(t)$, $Y(t)$ have continuous first and second derivatives for values of $t$ sufficiently near $t_2$, where $t_2$ determines the point 2 on $L$ at which it intersects $E_{12}$.

Under the hypotheses just made, Bliss* has proven the following theorem:

*For every minimizing arc $E_{12}$ there exists a set of constants $c_i (i = 1, \cdots, n)$ and a function*

$$F(x, y, y', \lambda) = \lambda_0 f + \lambda_1 \varphi_1 + \cdots + \lambda_\alpha \varphi_m \tag{7}$$

*such that the equations*

$$F_{y_i'} = \int_{x_1}^{x} F_{y_i}\, dx + c_i \tag{8}$$

*are satisfied at every point of $E_{12}$. The constants $\lambda_0$ and the functions $\lambda_\alpha(x)$ $(\alpha = 1, \cdots, m)$ are not all identically zero on $x_1 x_2$ and are continuous except possibly at values of $x$ defining corners of $E_{12}$.*

Equations (8) and (3) serve to determine the functions $y_i(x)$ and $\lambda_\alpha(x)$. Let us assume the determinant

$$R(x, y, y', \lambda) = \begin{vmatrix} F_{y_i' y_k'} & \varphi_{\alpha y_i'} \\ \varphi_{\alpha y_k'} & 0 \end{vmatrix} \tag{9}$$

does not vanish along $E_{12}$. The system (8) may be replaced by the set

$$F_{y_i} - \frac{dF}{dx} y_i' = 0. \tag{10}$$

Subsequent considerations will exclude the possibility of $\lambda_0$ having the value zero. We may without loss of generality let $\lambda_0$ have the value 1.

The system of equations

$$v_i = F_{y_i'}(x, y, y', \lambda), \qquad 0 = \varphi_\alpha(x, y, y') \tag{11}$$

has solutions for $y_1', \lambda_\alpha$ near the values $x, y_i, y_i', \lambda_\alpha, v_i$ belonging to $E_{12}$ since $R(x, y, y', \lambda)$, the functional determinant with respect to $y_i', \lambda_\alpha$, does not vanish. Let the solutions be

$$y_i' = \psi_i(x, y, v), \qquad \lambda_\alpha = \pi_\alpha(x, y, v). \tag{12}$$

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* Bliss, Trans. Amer. Math. Soc., vol. 19 (1918), pp. 305, 314. See also *The Problem of Lagrange*, lectures by Bliss, Summer quarter, 1925, University of Chicago, mimeographed by O. E. Brown of Northwestern University. This will hereafter be referred to as "Lectures".

Equations (10) may now be replaced by the equivalent form used by Bolza*

$$\begin{aligned} \frac{dy_i}{dx} &= \psi_1(x, y, v), \\ \frac{dv_i}{dx} &= F_{y_i'}(x, y, \psi(x, y, v), \pi(x, y, v)). \end{aligned} \tag{13}$$

This system of equations has for solutions the set of functions

$$\begin{aligned} y_i &= y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n) = y_i(x, a, b), \\ v_i &= v_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n) = v_i(x, a, b) \end{aligned} \tag{14}$$

containing $E_{12}$ for special values of the parameters $a_{i0} = y_i(x_0) = y_{i0}$ $b_{i0} = v_i(x_0) = v_{i0}$ which are the initial values of $y_i$, $v_i$ at $x = x_0$. Substituting the functions (14) in the last $m$ of equations (12), determines a set of multipliers $\lambda_\alpha(x, a, b)$. The functions $y_i$, $y_i'$, $v_i$, $v_i'$, $\lambda_\alpha$ have continuous partial derivative of the first three orders for all sets of values $(x, a, b) = (x_0, y_{i0}, v_{i0})$ in the neighborhood of those defining $E_{12}$. Such solutions are called extremals. The extremal $E_{12}$ is therefore imbedded in a $2n$ parameter family of curves each of which is a solution of equations (10).

2. **Admissible families of curves and variations.** Let $y_i = y_i(x, b)$ be a one parameter family of arcs having the continuity property $(a)$, containing the arc $E_{12}$ for $b = b_0$, satisfying equations (3), passing through the point 1 and intersecting the curve $L$. The functions $y_{ib}(x, b_0)$ satisfy the $m$ equations

$$\varphi_{\alpha y_i} y_{ib}(x, b_0) + \varphi_{\alpha y_i'} y_{ib}'(x, b_0) = 0.$$

The functions $y_{ib}(x, b_0)$ are called the variations of $E_{12}$. Any set of solutions $\eta_i(x)$ satisfying the $m$ differential equations

$$\varphi_{\alpha y_i} \eta_i(x) + \varphi_{\alpha y_i'} \eta_i'(x) = 0 \tag{15}$$

is called an *admissible set of variations for* $E_{12}$.

If we have given the matrix

$$\left\| \begin{matrix} \eta_{1,1}(x) \cdots \cdots \eta_{1,p}(x) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \eta_{n,1}(x) \cdots \cdots \eta_{n,p}(x) \end{matrix} \right\|$$

* Bolza, *Vorlesungen über Variationsrechnung*, pp. 592–594.

whose columns constitute $p$ sets of admissible variations, it may be shown that there exists a $p$-parameter family of admissible arcs

$$y_i = y_i(x, b_1, \cdots, b_p) \tag{16}$$

containing $E_{12}$ for $b_1 = b_2 = , \cdots, b_p = 0$ and having the functions $\eta_{1,s}(x)$ as variations with respect to $b_s$ $(s = 1, \cdots, p)$ along $E_{12}$.*

Let there exist $2n$ sets of admissible variations $\eta_{i,1}(x), \cdots, \eta_{i,2n}(x)$, such that

$$\begin{vmatrix} \eta_{1,1}(x_1) & \cdots\cdots & \eta_{1,2n}(x_1) \\ \cdot & \cdots\cdots & \cdot \\ \eta_{n,1}(x_1) & \cdots\cdots & \eta_{n,2n}(x_1) \\ \eta_{1,1}(x_2) & \cdots\cdots & \eta_{1,2n}(x_2) \\ \cdot & \cdots\cdots & \cdot \\ \eta_{n,1}(x_2) & \cdots\cdots & \eta_{n,2n}(x_2) \end{vmatrix} \tag{17}$$

is different from zero. The extremal $E_{12}$ is said to be normal on the interval $x_1 x_2$. For a normal arc $\lambda_0$ does not vanish and may be taken equal to 1.

The statement that the family $y_i = y_i(x, b)$ contains $E_{12}$ for $b = 0$ and intersects $L$, means analytically that there is a function $t(b)$ such that $t(0) = t_2$ and

$$y_i(x_1, b) = y_i, \qquad y_i\big(X(t(b)), b\big) = Y_i(t(b)).$$

Differentiating these $2n$ eqnations with respect to $b$ and letting $b$ become zero, we find the equations

$$y_{ib}(x_1, 0) = 0, \quad y_i'(x_2, 0)\, X'(t_2)\, t_b(0) + y_{ib}(x_2, 0) = Y_i'(t_2)\, t_b(0),$$

which, written in another form, are

$$\eta_i(x_1) = 0, \quad \eta_i(x_2) = \tau(Y' - y' X')^i, \tag{18}$$

where $\tau = t_b(0)$. Hence, *for every one parameter family of admissible arcs* $y_i = y_i(x, b)$, *containing* $E_{12}$ *for* $b = 0$, *the variations* $\eta_i(x)$ *must satisfy equations* (18).

Consider now a set of variations satisfying equations (18). It is to be found that every such set is a set of variations for an admissible family

* Bliss, *Lectures*, pp. 5–6.

$y_i = y_i(x, b)$ containing $E_{12}$ for $b = 0$. It is possible first of all to construct a $2n+1$ parameter family of arcs

$$y_i = y_i(x, b, b_1, \cdots, b_{2n}), \tag{19}$$

containing $E_{12}$ for $b = b_1 = b_2 = \cdots = b_{2n} = 0$ and having $\eta_i(x)$, $\eta_{i,1}(x), \cdots, \eta_{i,2n}(x)$ as its variations with respect to $b, b_1, \cdots, b_{2n}$ respectively, where the last $2n$ sets of $\eta$'s are those of the determinant (17).* Now select a function $t(b)$ such that $t(0) = t_2$ and $t_b(0) = \tau$. The $2n$ equations

$$\begin{aligned} y_i(x_1, b, b_1, b_2, \cdots, b_{2n}) - y_{i1} &= 0, \\ y_i\big(X(t(b)), b, b_1, b_2, \cdots, b_{2n}\big) - Y(t(b)) &= 0 \end{aligned} \tag{20}$$

are satisfied for $(b, b_1, \cdots, b_{2n}) = (0, 0, \cdots, 0)$ and have for their functional determinant with respect to $b_1, b_2, \cdots, b_{2n}$ the determinant (17). Since this determinant does not vanish, there exist $2n$ functions $b_1 = B_1(b)$, $b_2 = B_2(b), \cdots, b_{2n} = B_{2n}(b)$ having continuous first derivatives and vanishing for $b = 0$. Replacing $b_1, b_2, \cdots, b_{2n}$ in the set (19), we find a one parameter family of admissible arcs

$$y_i = y_i(x, b, B_1(b), \cdots, B_{2n}(b)) = y_i(x, b), \tag{21}$$

containing $E_{12}$ for $b = 0$. The following $2n$ equations are true

$$\begin{aligned} y_i(x_1, b, B_1(b), \cdots, B_{2n}(b)) - y_{i1} &= 0, \\ y_i\big(X(t(b)), b, B_1(b), \cdots, B_{2n}(b)\big) - Y_i(t(b)) &= 0. \end{aligned}$$

The differentiation of these with respect to $b$ gives, for the value $b = 0$,

$$\begin{aligned} \eta_i(x_1) + \eta_{i,1}(x_1) B_1'(0) + \cdots \eta_{i,2n}(x_1) B_{2n}'(0) &= 0, \\ y'(x_2) X'(t_2)\tau + \eta_i(x_2) + \eta_{i,2n}(x_2) B_1'(0) + \cdots \eta_{i,2n}(x_2) B_{2n}'(0) - Y'(t_2)\tau &= 0. \end{aligned}$$

By virtue of equations (18), the first term in each of the first $n$ of the above equations vanishes while the first two terms of the remaining $n$ are cancelled by their last terms. The resulting equations are linear and homogeneous in $B_1'(0), \cdots, B_{2n}'(0)$ and have (17) for the determinant of the coefficients. Since (17) does not vanish, we have $B_1'(0) = \cdots = B_{2n}'(0) = 0$. Hence the variations of the family (21) are

$$\begin{aligned} y_{ib}(x, 0) &= \eta_i(x) + y_{ib_r} B_r'(0) \qquad (r = 1, \cdots, 2n) \\ &= \eta_i(x). \end{aligned}$$

* Bliss, *Lectures*, p. 6.

We have proved the following theorem:

*For every one parameter family of admissible arcs* $y_i = y_i(x, b)$, *containing* $E_{12}$ *for* $b = 0$, *the variations* $\eta_i(x) = y_{ib}(x, 0)$ *satisfy the differential equations* (3) *and the end conditions* (18) *at the points* 1 *and* 2. *Conversely, for every set of functions* $\eta_i(x)$ *satisfying these conditions, there exists an admissible family* $y_i = y_i(x, b)$, *containing* $E_{12}$ *for* $b = 0$ *and having the functions* $\eta_i(x)$ *as its variations along* $E_{12}$.

3. **The variations of the integral** $I$. If the integral $I$ is taken over an admissible set of arcs containing $E_{12}$ for $b = 0$ for which the derivative $t_b(0)$ of the previous section is different from zero, then $I$ becomes a function of $b$. The upper limit $x_2$ becomes $x_2 = X(t) = X(t(b)) = x_2(b)$. The integral $I$ may then be written

$$I(b) = \int_{x_1}^{x_2(b)} f(x, y(x, b), y'(x, b))\, dx.$$

In order that $E_{12}$ shall minimize $I$, it is necessary that the function $I(b)$ shall have a minimum value for $b = 0$; $I_b(0)$ must vanish and $I_{bb}(0)$ must be positive or zero. Differentiating $I(b)$ with respect to $b$ gives

$$I_b(b) = f(x, y(x, b), y'(x, b))\, x_b \Big|^{x_2(b)} + \int_{x_1}^{x_2(b)} (f_{y_i} y_{ib} + f_{y_i'} y_{ib}')\, dx.$$

The expression

$$\lambda_\alpha \varphi_\alpha + \int_{x_1}^{x_2(b)} \lambda_\alpha (\varphi_{\alpha y_i} y_{ib} + \varphi_{\alpha y_i'} y_{ib}')\, dx$$

has the value zero along an admissible arc. Let this be added to the above expression for $I_b(b)$.

$$I_b(b) = F(x, y(x, b), y'(x, b), \lambda(x, b))\, x_b \Big|^{x_2(b)} + \int_{x_1}^{x_2(b)} (F_{y_i} y_{ib} + F_{y_i'} y_{ib}')\, dx.$$

If the terms containing $y_{ib}'$ are integrated by parts and $b'$ set equal to zero, we have

$$I_b(0) = F(x, y, y', \lambda)\, x_b + y_{ib} F_{y_i'} \Big|^2 + \int_{x_1}^{x_2} y_{ib} \left(F_{y_i} - \frac{dF}{dx} y_i'\right) dx$$
$$= F(x, y, y', \lambda)\, x_b + y_{ib} F_{y_i'} \Big|^2$$

since $E_{12}$ is an extremal. This last expression must vanish if $I_b(0)$ is to be zero. But $x_b(0) = X' t_b\big|^2$, $y_{ib}\big|^2 = t_b(Y_i' - y_i' X'\big|^2$. Since $t_b(0)$ is not zero, we must have

$$F(x, y, y', \lambda) X' + F_{y_i'}(Y_i' - y_i' X')\big|^2 = 0$$

if $I_b(0)$ is to vanish. This gives us the Transversality Condition.

*If the arc $E_{12}$ is to minimize the integral I, then, at the end point 2, the condition*

$$(F - y_i' F_{y_i'}) X' + F_{y_i'} Y_i'\big|^2 = 0 \tag{22}$$

*must be satisfied.*

Differentiating $I(b)$ a second time and letting $b = 0$, gives

$$I''(0) = F x_{bb} + F' x_b^2 + 2 F_{y_i} y_{ib} x_b + 2 F_{y_i'} y_{ib}' x_b\big|^2 + \int_{x_1}^{x_2} (2\omega + F_{y_i} y_{ibb} + F_{y_i'} y_{ibb}')\, dx$$

where

$$2\omega = F_{y_i y_j} \eta_i \eta_j + 2 F_{y_i y_j'} \eta_i \eta_j' + F_{y_i' y_j'} \eta_i' \eta_j',$$

Integration by parts reduces the terms following $2\omega$ in the integrand to

$$y_{ibb} F_{y_i'}\big|^2.$$

Hence

$$I''(0) = F x_{bb} + F' x_b^2 + 2 F_{y_i} y_{ib} x_b + F_{y_i'} (2 x_b y_{ib}' + y_{ibb})\big|^2 + \int_{x_1}^{x_2} 2\omega\, dx. \tag{23}$$

The expression for $I''(0)$ may be further simplified by use of the $n+1$ equations

$$x_2(b) = X(t(b)), \quad y_{i2}(b) = y_i(x_2(b), b) = Y_i(t(b)).$$

From these we may obtain the equations

$$\begin{aligned} &\frac{dx_2}{db}\Big|^2 = X' t_b\big|^2, \quad \frac{d^2 x_2}{db^2}\Big|^2 = X' t_{bb} + X'' t_b^2\big|^2, \\ &\frac{dy_{i2}}{db}\Big|^2 = y_i' x_b + y_{ib}\big|^2 = Y' t_b\big|^2, \\ &\frac{d^2 y_{i2}}{db^2}\Big| = y_i'' x_b^2 + y_i' x_{bb} + 2 y_{ib}' x_b + y_{ibb}\big|^2 \\ &\qquad = Y_i' t_{bb} + Y'' t_b^2\big|^2. \end{aligned} \tag{24}$$

If $F'$ in (23) be expressed as

$$F' = F_x + y_i' F_{y_i} + y_i'' F_{y_i'}$$

and, if $y_i'' x_b^2$ is obtained from equations (24) in the form

$$y_i'' x_b^2 = \frac{d^2 y_{i2}}{db^2} - y_i' x_{bb} - 2 y_{ib}' x_b - y_{ibb},$$

then (23) becomes

$$I''(0) = (F - y_i' F_{y_i'}) x_{bb} + (F_x + y_i' F_{y_i'}) x_b^2 + F_{y_i'} \frac{d^2 y_{i2}}{db^2} + 2 F_{y_i} y_{ib} \Big|^2 + \int_{x_1}^{x_2} 2\omega \, dx.$$

Now by making use of the equations in the first, third and last rows of (24), the Transversality Condition (22) and equations (18), one obtains

$$I''(0) = [(F - y_i' F_{y_i'}) X'' + F_{y_i'} Y_i'' + (F_x - y_i' F_{y_i}) X'^2 + 2 F_{y_i} X' Y_i'] t_b^2 \Big|^2 + \int_{x_1}^{x_2} 2\omega \, dx$$

$$= A t_{bb}^2 \Big|^2 + \int_{x_2}^{x_1} 2\omega \, dx \tag{25}$$

where $A$ has the value

$$(F - y_i' F_{y_i'}) X'' + F_{y_i'} Y'' + (F_x - y_i' F_{y_i}) X'^2 + 2 F_{y_i} X' Y' \Big|^2.$$

4. **The analogue of the Jacobi condition.** The expression for $I_{bb}(0)$, derived in the last section may be transformed into an integral in $\eta_i(x)$ without terms outside of the integral sign. For brevity, $Y_i' - y_i' X' \Big|^2$ will be written $\varrho_i$ so that from (18) we have $\varrho_i \tau = \eta_i(x_2)$. From these equations, we have

$$t_b^2(0) = \tau^2 = \eta_i(x_2) \cdot \eta_i(x_2)/\varrho^2$$

where $\varrho^2 = \varrho_1^2 + \cdots + \varrho_n^2$. The customary assumption is made here that $E_{12}$ and $L$ are not tangent at 2. Consequently $\varrho^2 \neq 0$. Equation (25) may then be given the form

$$I''(0) = \int_{x_1}^{x_2} \left( \frac{d}{dx} L \eta_i \, \eta_i + 2\omega \right) dx \tag{26}$$

where $L$ is a function taking on the value $A/\varrho^2$ for $x = x_2$.

We shall now consider the problem of determining the set $\eta_i(x)$ satisfying the differential equations (15) and the initial conditions (18) which

shall render the integral (26) a minimum value. This is a calculus of variations problem in $x\eta$-space with the endpoint 1 fixed while the point 2 is restricted to lie on the straight line

$$x = x_2, \qquad \eta_i = \tau \varrho_i. \tag{27}$$

It will be assumed that $E_{12}$ is normal on every subinterval of $x_1 x_2$. For our $x\eta$ problem, the equations of variation are identical with those of the original $xy$ problem. Hence the new problem in $x\eta$-space is a normal one.

If the sum

$$2 \sum_{\alpha=1}^{m} \mu_\alpha (\varphi_{\alpha y_i} \eta_i + \varphi_{\alpha y_i'} \eta_i') = 0$$

is added to the integrand of (26), then

$$I''(0) = \int_{x_1}^{x_2} \left( \frac{d}{dx} L\eta_i(x)\, \eta_i(x) + 2\Omega \right) dx \tag{28}$$

where

$$\Omega = \omega + \sum_{\alpha=1}^{m} \mu_\alpha (\varphi_{\alpha y_i} \eta_i + \varphi_{\alpha y_i'} \eta_i').$$

The integrand of (28) will be designated by $2\Phi(x, \eta, \eta')$. The Euler equations for the integral (28) are readily shown to be independent of the term $d(L\eta_i(x) \cdot \eta_i(x))/dx$. They are

$$\Omega_{\eta_i} - \frac{d}{dx} \Omega_{\eta_i'} = 0. \tag{29}$$

These are called the *accessory differential equations.** They have $2n$ independent sets of solutions

$$y_{1a_i}, \cdots, y_{na_i}, \qquad \lambda_{1a_i}, \cdots, \lambda_{ma_i},$$

$$y_{1b_i}, \cdots, y_{nb_i}, \qquad \lambda_{1b_i}, \cdots, \lambda_{mb_i},$$

and every set of solutions must be of the form

$$\begin{aligned}
\eta_1(x) &= c_k y_{1a_k}(x) + d_k y_{1b_k}(x), \\
&\cdots\cdots\cdots\cdots \\
\eta_n(x) &= c_k y_{na_k}(x) + d_k y_{nb_k}(x), \\
\mu_1(x) &= c_k y_{1a_k}(x) + d_k \lambda_{1b_k}(x), \\
&\cdots\cdots\cdots\cdots \\
\mu_m(x) &= c_k \lambda_{ma_k}(x) + d_k \lambda_{mb_k}(x),
\end{aligned} \tag{30}$$

where the range of $k$ is 1 to $n$.

* von Escherich, *Wiener Berichte,* vol. 107, p. 1236.

The transversality condition at the point 2 in $x\eta$-space is

$$(\Phi - \eta_i' \Phi_{\eta_i'}) x_\tau + \eta_{i\tau} \Phi_{\eta_i'} \Big|^2 = 0$$

or since along the line (27) we have $x_\tau = 0$, $\eta_{i\tau} = \varrho_i$, this condition becomes

$$\varrho_i \Phi_{\eta_i'} \Big|^2 = 0. \tag{31}$$

Remembering that $2\Phi = 2\Omega + L'\eta_i \eta_i + 2 L \eta_i \eta_i'$, we find

$$\Phi_{\eta_i'} \Big|^2 = \Omega_{\eta_i'} + L\eta_i \Big|^2$$

and (31) reduces to

$$\varrho_i (\Omega_{\eta_i'} + L\eta_i) \Big|^2 = 0. \tag{32}$$

The function $2\Omega$ may be expressed in the form $\eta_i \Omega_{\eta_i} + \eta' \Omega_{\eta_i'} + \mu_\alpha \Omega_{\mu_\alpha}$. The second variation then becomes

$$I''(0) = L\eta_i \eta_i \Big|^2 + \int_{x_1}^{x_2} (\eta_i \Omega_{\eta_i} + \eta_i' \Omega_{\eta_i} + \mu_\alpha \Omega_{\mu_\alpha}) \, dx$$

from which we obtain

$$I''(0) = L\eta_i \eta_i + \eta_i \Omega_{\eta_i'} \Big|^2 + \int_{x_1}^{x_2} \eta_i \left( \Omega_{\eta_i} - \frac{d}{dx} \Omega_{\eta_i'} \right) dx \tag{33}$$

after performing an integration by parts and making use of the fact that $\Omega_{\mu_\alpha}$ vanishes along $E_{12}$. The term of (33) free of the integral sign differs only by the factor $\tau$ from the left hand member of (32). Thus

$$I''(0) = \tau [\varrho_i (L\eta_i + \Omega_{\eta_i'})] \Big|^2 + \int_{x_1}^{x_2} \eta_i \left( \Omega_{\eta_i} - \frac{d}{dx} \Omega_{\eta_i'} \right) dx.$$

Let there exist a set of solutions $\eta_i(x)$, $\mu_\alpha(x)$ of equations (29), not all identically zero in $x_1 x_2$, satisfying the Transversality Condition (32) and having the $\eta_i(x)$ vanish simultaneously for some value $x_3$, where $x_1 < x_3 < x_2$. Using these functions $\eta_i(x)$, $\mu_\alpha(x)$ on the interval $x_3 x_2$ and setting $\eta_i \equiv 0$, $\mu_\alpha \equiv 0$ on $x_1 x_3$, it is clear that $I''(0)$ takes on the value zero. If there exists such a set of solutions $\eta_i(x)$, $\mu_\alpha(x)$ of the accessory equations, we shall see that it is possible to make $I''(0)$ take on negative values.

At $x = x_3$ the functions $\eta_i(x)$ just defined are continuous but the $\eta_i'(x)$, $\mu_\alpha(x)$ are not necessarily so. If these functions minimize $I''(0)$, the corner condition*

* D. M. Smith, Trans. Amer. Math. Soc., vol. 17, p. 466.

$$\Phi_{\eta_i'}\big|_{x_3-0} = \Phi_{\eta_i'}\big|_{x_3+0}$$

would have to be satisfied. Since $\Phi_{\eta_i'} = \Omega_{\eta_i'}$, when $\eta_i = 0$, the above condition becomes

$$\Omega_{\eta_i'}\big|_{x_3-0} = \Omega_{\eta_i}\big|_{x_3+0}$$

if we form these $n$ equations and remember that $\eta_i'$ vanishes on the interval $x_1 x_3$ and that the $m$ equations (15) hold, we find

$$\left.\begin{aligned} F_{y_i' y_k'}\,\eta_k' + \varphi_{\alpha y_i'}\,\mu_\alpha\Big|^3 &= 0, \\ \varphi_{\alpha y_k'}\,\eta_k'\Big|^3 &= 0. \end{aligned}\right.$$

We may regard these as $n+m$ linear homogeneous equations in $\eta_i'(x_3)$, $\mu_\alpha(x_3)$. We see that the determinant of the coefficients is $R(x, y, y', \lambda)$. Since this determinant does not vanish along $E_{12}$, the quantities $\eta_i'(x_3+0)$, $\mu_\alpha(x_3+0)$ necessarily vanish. The solutions of equations (29) are uniquely determined by the values of $\eta_i$, $\eta_i'$ and $\mu_\alpha$ for $x = x_3$. The only set for which $\eta_i = \eta_i' = \mu_\alpha = 0$ is the set $\eta_i \equiv 0$, $\mu_\alpha \equiv 0$, which contradicts the assumption that the $\eta_i(x)$, $\mu_\alpha(x)$ did not all vanish identically of $x_3 x_2$. If there exists a set of solutions of the accessory equations, vanishing simultaneously for $x = x_3$, where $x_1 < x_3 < x_2$, satisfying (32) and not identically zero on $x_3 x_2$, then they cannot satisfy the corner condition and therefore do not minimize $I''(0)$. There must, therefore, exist $\eta_i(x)$ which furnish a negative value for $I''(0)$.

A point 3 on $E_{12}$ for which there exists a set of solutions $\eta_i(x)$, $\mu_\alpha(x)$ of the accessory equations, satisfying the end conditions (18), and the condition of transversality in $x\eta$-space at the point 2 and having all the functions $\eta_i(x)$ vanish for $x = x_3$, is called a *Focal Point for $L$ on $E_{12}$*.

In order to characterize the point $x_3$ more specifically the transversality condition (32) may first be written

$$\begin{aligned} \varrho_i\,(L\,\eta_i + \Omega_{\eta_i'}) &= \tau\,L\,\varrho_i\,\varrho_i + \varrho_i\,(\eta_j\,F_{y_i' y_j} + \eta_j'\,F_{y_i' y_j} + \mu_\alpha\,\varphi_\alpha\,y_i') \\ &= \tau\,[A + \varrho_i\,\varrho_j\,F_{y_i' y_j}] + c_k\,A_k + d_k\,B_k = 0, \end{aligned} \tag{35}$$

where

$$\begin{aligned} A &= (F - y_i'\,F_{y_i'})\,X'' + F_{y_i}\,Y_i'' + (F_x - y_i'\,F_{y_i})\,X'^2 + 2\,F_{y_i}\,X'\,Y'\Big|^2, \\ A_k &= F_{y_i' y_j'}\,\varrho_j\,y_{ia_k}' + \varphi_{\alpha y_j'}\,\varrho_j\,\lambda_{\alpha a_k}\Big|^2, \\ B_k &= F_{y_i' y_j'}\,\varrho_j\,y_{ib_k}' + \varphi_{\alpha y_j'}\,\varrho_j\,\lambda_{\alpha b_k}\Big|^2, \quad (i, j = 1, \cdots, n;\ \alpha = 1, \cdots, m). \end{aligned}$$

The solutions (30) of the accessory equations, which satisfy the equation (32 and have all their functions $\eta_i(x)$ zero at a particular value $x$ must have constants $c_k$, $d_k$ satisfying the equations

$$0 = c_k y_{ia_k}(x) + d_k y_{ib_k}(x),$$
$$\varrho_i \tau = c_k y_{ia_k}(x_2) + d_k y_{ib_k}(x_2),$$

where $\tau$ is also a constant to be determined. Evidently these equations have solutions not all zero if and only if the determinant

$$H(x, x_2) = \begin{vmatrix} A + \varrho_i \varrho_j F_{y_i' y_j'} & A_k & B_k \\ -\varrho_i & y_{ia_k}(x_2) & y_{ib_k}(x_2) \\ 0 & y_{ia_k}(x) & y_{ib_k}(x) \end{vmatrix}$$

vanishes. The zeros of the function $H(x, x_2)$ are therefore the abscissas of the focal points of $L$ on $E_{12}$.

THEOREM. *In order that an arc $E_{12}$, normal on every subinterval on $x_1, x_2$, shall render the integral $I$ a minimum value, it is necessary that the determinant $H(x, x_2)$ shall not vanish for $x_1 < x < x_2$, or in other words, that there shall be no focal point 3 of $L$ between 1 and 2 on $E_{12}$.*

For $n = 1$, $m = 0$, the function $H(x, x_2)$ is identical with the expression given by Bolza* for the determination of the focal point of a curve for the case of one variable end point in the plane.

If the end point 2 is fixed the expressions $X'$, $Y_i'$ vanish and the same is true of $A$, $A_k$, $B_k$. To determine the *conjugate* point of 2, the $2n$ constants $c_k$, $d_k$ may be eliminated from the $2n$ equations

$$c_k y_{ia_k}(x_2) + d_k y_{ib_k}(x_2) = 0,$$
$$c_k y_{ia_k}(x) + d_k y_{ib_k}(x) = 0,$$

which gives

$$\begin{vmatrix} y_{ia_k}(x_2) & y_{ib_k}(x_2) \\ y_{ia_k}(x) & y_{ib_k}(x) \end{vmatrix} = 0$$

for the determination of the *Conjugate Point.*

5. **The Envelope Theorem.** As shown in § 1, the extremal $E_{12}$ may be embedded in a $2n$-parameter family of extremals

$$(36) \qquad y_i = y_i(x, a, b), \qquad \lambda_\alpha = \lambda_\alpha(x, a, b),$$

* Bolza, *Lectures on the Calculus of Variations*, University of Chicago Press, 1904, p. 107.

containing $E_{12}$ for the set of values $(a_i, b_i) = (a_{i0}, b_{i0})$, where $a_{i0} = Y_i(t_2)$. Now suppose the $a_i$ in the set (36) are replaced by the functions $Y_i(t)$. We thus obtain an $(n+1)$-parameter family of extremals

$$(37) \quad y_i = y_i(x, Y(t), b) = y_i(x, t, b), \quad \lambda_\alpha = \lambda_\alpha(x, Y(t), b) = \lambda_\alpha(x, t, b),$$

each of which intersects the curve $L$ and containing the curve $E_{12}$ for the set of values $(t, b_i) = (t_2, b_{i0})$. Let the members of the family (37) be substituted in the equation

$$(38) \qquad T \equiv (F - y_i' v_i) X' + Y_i' v_i = 0.$$

If $E_{12}$ satisfies the Transversality Condition, this equation has a set of solutions $(t, b_i) = (t_2, b_{i0})$ and can easily be shown to have for its partial derivative with respect to $b_k$ at the point $x = x_2$, the value $\varrho_k$. Since by hypothesis at least one of the quantities $\varrho_1, \varrho_2, \cdots, \varrho_n$ is different from zero, it follows from the theorems on implicit functions that $T = 0$ may be solved for some $b$, say $b_s$ as a function of the remaining $b$'s and $t$, in the neighborhood of the set $(t_2, b_{i0})$. This function is continuous and has continuous first derivatives in the neighborhood of this set of values.

Inserting the function just found, in the set of equatien (37), we thus obtain an $n$ parameter family of extremals cutting $L$ transversally and containing $E_{12}$ for special value of the parameter. Let us write this family in the form

$$(39) \qquad y_i = y_i(x, c_1, c_2, \cdots, c_n)$$

where $c_1 = t$ and $c_2, \cdots, c_n$ are the remaining $b$'s exclusive of $b_s$.

Let $x_3$ be a value of $x$ less than $x_2$ such that the functional determinant of the family (39) with respect to the $c$'s for the values $c_{i0}$ defining $E_{12}$,

$$\Delta(x, c_{i0}) = \begin{vmatrix} y_{1c_1} & \cdots & y_{1c_n} \\ \cdot & \cdot & \cdot \\ y_{nc_1} & \cdots & y_{nc_n} \end{vmatrix}$$

vanishes and is different from zero for $x_3 < x < x_2$. Furthermore suppose that $\Delta_x(x_3, c_{i0}) \neq 0$. By the method used by Bolza* for an $n$ parameter family of extremals through a fixed point in the Problem of Lagrange with fixed end points, the following theorem may easily be established:

THEOREM. *For the extremal $E_{12}$, along which $R(x, y, y', \lambda) \neq 0$, let there be a point $x_3$ defined as above and for which $\Delta_x(x_3, c_{i0}) \neq 0$. Then it is possible to determine $c_1, \cdots, c_n$ as functions of a parameter $\gamma$ having continuous first derivatives, such that the one parameter family of extremals*

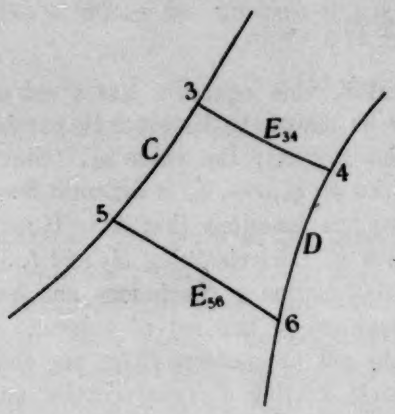
* Bolza, *Vorlesungen über Variationsrechnung*, p. 615.

$$y_i = y_i(x, c(\gamma)) = y_i(x, \gamma) \tag{40}$$

*contains $E_{12}$ for some $\gamma$, say $\gamma_0$ and such that the members of the above family have an enveloping curve $d$ touched by $E_{12}$ for $x = x_3$.*

The proof of this theorem is a mere repetition of the argument of Bolza and is therefore omitted. Without difficulty one may identify the point $x = x_3$ on $E_{12}$ with the focal point of the preceding section.

We may now apply the following theorem due to Bliss.*




*Let $y_i = y_i(x, e)$ be a set of extremal arcs with multipliers $\lambda_0 = 1$, $\lambda_\alpha = \lambda_\alpha(x, e)$. The values of $I$ on two arcs $E_{34}$ and $E_{56}$ of the family satisfy the equation*

$$I(E_{56}) - I(E_{34}) = I^*(D_{46}) - I^*(C_{35})$$

*with the value of the integral*

$$I^* = \int F\,dx + (dy_i - y_i'\,dx)\,F_{y_i'}$$

*along the corresponding segments $C_{35}$ and $D_{46}$ as shown in the figure.*

In $I^*$ the differentials refer to the path of integration and $y_i'$ to the variable extremal.

The family of extremals (40) satisfies the conditions of the theorem. For, the determinant (17) is not zero by hypothesis for values of $x, y, y', \lambda$ in the neighborhood of those defining $E_{12}$. Owing to the continuity properties of the functions in the differential equations defining the variations $\eta_i(x)$, it is possible to determine a positive constant $\delta$, such that for $|\gamma - \gamma_0| < \delta$, the determinant (17) will remain different from zero. The members of the one parameter family of extremals (40) for values of $\gamma$ in this neighborhood of $\gamma_0$ are therefore normal. There exists then for values of $\gamma$ sufficiently near $\gamma_0$ a set of multipliers $\lambda_0 = 1$, $\lambda_\alpha = \lambda_\alpha(x, \gamma)$ of the type required in the theorem.

In order to apply the above theorem, we let the curve $L$ correspond to the arc $C_{35}$ and the envelope $d$ of the family (40) to the arc $D_{45}$. We shall assume that the curve $d$ has a branch projecting from 5 as shown in the figure. If the point 4 is selected sufficiently near 2 we shall have

$$I(E_{32}) = I(E_{54}) + I^*(d_{35}) - I^*(L_{24}).$$

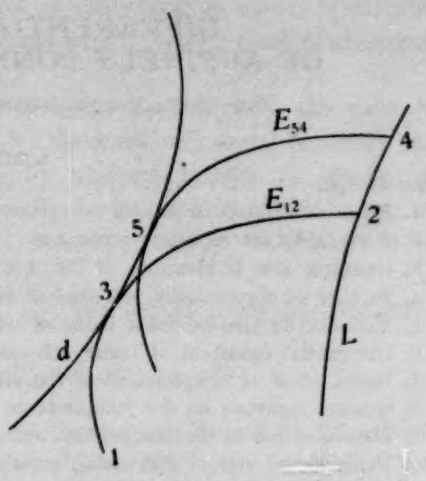
* Bliss, *Lectures*, p. 48.

The term $I^*(L_{24})$ vanishes since every member of the family (40) cuts $L$ transversally and the vanishing of the integrand of $I^*(L_{24})$ expresses the Transversallity Condition. Since $d$ is tangent to every member of (40) the expressions $dy_i - y_i' dx$ all vanish along $d$ and we have $I^*(d_{35}) = I(d_{35})$. Hence

$$I(E_{32}) = I(d_{35}) + I(E_{54})$$

or by adding $I(E_{13})$ to each member of this last equality, one obtains

$$I(E_{132}) = I(E_{13}) + I(d_{35}) + I(E_{54}).$$




We may show however, that the arc $K_{14} = E_{13} + d_{35} + E_{54}$ is not an extremal. If it were an extremal, it would possess a unique set of continuous multipliers $\lambda_0 = 1$, $\lambda_\alpha = \lambda_\alpha(x)$ since it contains no corners. These multipliers would be identical with those of the extremal $E_{12}$ along $E_{13}$. However, the values of $y_i$, $y_i'$, $v_i$ or rather those of $y_i$, $y_i'$, $\lambda_\alpha$ at the point 3 determine the extremal $E_{12}$ uniquely. Hence $K_{14}$ would coincide with $E_{12}$ which is not the case. There must exist, then, arcs joining 1 and 4 which furnish a smaller value for the integral $I$ than the arc $E_{12}$ does. The argument holds when 0 coincides with 3.

We may summarize our results in the following theorem:

THEOREM. *Let there exist a curve $d$ which is the envelope of a one parameter family of extremals cutting $L$ transversally and containing $E_{12}$ for a special value of the parameter. Suppose furthermore that $d$ does not possess a cusp for the value $x = x_3$ where $E_{12}$ touches it. Then in order that $E_{12}$ shall render the integral $I$ a minimum value with respect to its value along other curves joining* 1 *and the curve $L$, it is necessary that the value $x_3$ defining the point of contact of $E_{12}$ with the envelope $d$ shall not lie on the intervall $x_{12}$.*

The analytic proof of § 4 excluded the focal point from lying between $x_1$ and $x_2$ but places no restrictions upon the form of the envelope at the point with abscissa $x_3$. By making a special assumption with respect to the form of the envelope $d$ for $x = x_3$, the argument of this section shows that in order that $E_{12}$ shall render $I$ a minimum value, the point of contact 3 of $E_{12}$ with $d$ must not only be excluded from lying between 1 and 2 but must also not coincide with 1.

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# DIFFERENTIAL INVARIANTS OF AFFINELY CONNECTED MANIFOLDS.*

By Tracy Yerkes Thomas and Aristotle D. Michal.

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**Introduction.** The ordinary treatment of the differential invariants of affinely connected manifolds involving the actual construction of the invariants by processes of algebraic elimination is applicable only to the simplest type of linear invariant. Evidently there is room for a qualitative treatment of invariants offering the possibility of study of whole classes of invariants without their actual determination. In the present paper we have undertaken this on the basis of the differential equations which determine *all* the invariants of the manifold. The differential equations of the absolute scalar invariants are studied in detail; in this connection we show that they constitute a complete set of invariants of the manifold. This common idea of a complete set of invariants is distinct from the notion of a fundamental set of invariants, which is introduced in § 10 and which does not seem to have been clearly perceived by writers in the theory of differential invariants.

It is shown in § 12 that the general invariant theory at which we arrive is formally equivalent to the invariant theory of a certain continuous group of $n^2$ parameters.

* Received September 28, 1926.

† National Research Fellow in Mathematics.

At the end of the paper an extension of our work is made to include the treatment of affine differential invariants which involve a set of arbitrary functions and their derivatives.

In a later paper we shall apply the methods used here to the case of the relative quadratic differential form. This work will stand in somewhat closer analogy to that already done by various writers on quadratic differential forms, among whom is especially to be mentioned the name of C. N. Haskins.* It is on this account that a criticism of the work of others will be deferred to our second paper.

1. **Affine differential invariants.** Let the set of analytic symmetric functions†

$$\Gamma^i_{\alpha\beta}(x^1, x^2, \cdots, x^n) = \Gamma^i_{\beta\alpha}(x^1, x^2, \cdots, x^n)$$

determine the affine connection of an $n$-dimensional manifold $\mathfrak{M}$ whose points are represented by coördinates $(x^1, x^2, \cdots, x^n)$.

We shall denote by the symbol $\mathfrak{G}$ any transformation

$$x^i = f^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$$

of the group of arbitrary analytic transformations with non-vanishing Jacobians. As is well known the functions $\Gamma^i_{\alpha\beta}$ have the transformation:

$$\bar{\Gamma}^\gamma_{\alpha\beta} \frac{\partial x^i}{\partial \bar{x}^\gamma} = \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} + \Gamma^i_{\lambda\mu} \frac{\partial x^\lambda}{\partial \bar{x}^\alpha} \frac{\partial x^\mu}{\partial \bar{x}^\beta} \tag{1.1}$$

under a transformation $\mathfrak{G}$ of the $(x)$ coördinates. It is the invariant theory of the transformations (1.1) which constitutes the theory of the affinely connected manifold $\mathfrak{M}$.

*A set of functions of the $\Gamma$'s and their derivatives of the form*

$$T^{\mu\nu\cdots\varepsilon}_{\sigma\tau\cdots\eta}\left(\Gamma^i_{\alpha\beta}; \frac{\partial \Gamma^i_{\alpha\beta}}{\partial x^\gamma}; \cdots; \frac{\partial^r \Gamma^i_{\alpha\beta}}{\partial x^\gamma \cdots \partial x^\delta}\right) \tag{1.2}$$

*will be called an affine differential invariant if each function of the set retains its form as a function of the $\Gamma$'s and their derivatives under the transformation $\mathfrak{G}$ of the $(x)$ coördinates and the transformation of the*

* Haskins, Trans. Amer. Math. Soc., vol. 3 (1902) pp. 71–91; ibid., vol. 5 (1904), pp. 167–192. For other references see R. Weitzenböck, Enc. d. Math. Wiss., Bd. III$_3$, Heft 6, pp. 60–65.

† Unless otherwise stated subscripts and superscripts can take on positive integer values $(1, 2, \cdots, n)$. We shall also employ the convention throughout our paper of letting a repetition of an index in a term stand for a summation over the values $(1, 2, \cdots, n)$ with respect to this index.

*functions $T$ which is determined by the transformation $\mathfrak{G}$ according to the law of transformation of the system of functions $T$.** In this paper, however, we shall restrict the term invariant to scalars or tensors and more generally to relative scalars or tensors of arbitrary weight.†

By means of a system of normal coördinates $(y^1, y^2, \cdots, y^n)$ we can immediately express the differential invariant (1.2) as an algebraic invariant (in the sense that all derivatives are explicitly eliminated) depending on a set of variables $A$. These normal coördinates $(y)$ are completely characterized by the identities

$$C^i_{\alpha\beta}(y)\, y^\alpha\, y^\beta = 0$$

where $C^i_{\alpha\beta}$ denotes the affine connection in the $(y)$ coördinate system, and a set of initial conditions

$$y^i = 0, \quad \frac{\partial x^i}{\partial y^\alpha} = \delta^i_\alpha \text{ when } x^i = q^i.$$

Under a transformation $\mathfrak{G}$ of the $(x)$ coördinates, the normal coördinates $(y)$ which are defined by the $(x)$ coördinates and a point $(q)$ suffer a linear homogeneous transformation:

$$y^i = a^i_\alpha\, \overline{y}^\alpha$$

where

$$a^i_\alpha = \left(\frac{\partial x^i}{\partial \overline{x}^\alpha}\right)_{x=q}.$$

The process of "extending" a tensor $R$ is as follows.‡ Let the components $R^{\alpha\cdots\beta}_{\mu\cdots\nu}$ in the $(x)$ coördinate system become $W^{\alpha\cdots\beta}_{\mu\cdots\nu}$ in the $(y)$ coördinate system. Then the equations

$$R^{\alpha\cdots\beta}_{\mu\cdots\nu,\,\rho\cdots q} = \left(\frac{\partial^m W^{\alpha\cdots\beta}_{\mu\cdots\nu}}{\partial y^\rho \cdots \partial y^q}\right)_{y=0}$$

define a set of functions

$$R^{\alpha\cdots\beta}_{\mu\cdots\nu,\,\rho\cdots q}$$

of the $(x)$ coördinates which constitute the components of a tensor $\mathfrak{R}$. The tensor $\mathfrak{R}$ is called the $m$th extension of the tensor $R$ provided that

* For a more extensive discussion of the definition of an invariant see a forthcoming Cambridge tract by O. Veblen, *Invariants of Quadratic Differential Forms.*

† Cf. O. Veblen and T. Y. Thomas, Trans. Amer. Math. Soc., vol. 26 (1924), p. 373.

‡ O. Veblen and T. Y. Thomas, Trans. Amer. Math. Soc., vol. 25 (1923), § 11. For brevity we shall refer to this paper as V & T.

there are $m$ subscripts in the set $(p, \cdots, q)$. In a similar manner a tensor $A$ is defined by the equations

$$A^i_{\alpha\beta\gamma\cdots\eta} = \left(\frac{\partial^t C^i_{\alpha\beta}}{\partial y^\gamma \cdots \partial y^\eta}\right)_{y=0}.$$

The tensors $A$ are called *normal tensors* on account of their definition by means of normal coördinates.*

Let us now transform the affine differential invariant $T$ to a system of normal coördinates $(y)$ with origin at an arbitrary point $(q)$ of the affinely connected manifold and evaluate at the origin of this system; the $\Gamma$'s become zero and their derivatives go over into the corresponding normal tensors $A$. Thus $T$ becomes

$$T^{\mu\nu\cdots\varepsilon}_{\sigma\tau\cdots\eta}(0; A^i_{\alpha\beta\gamma}; \cdots; A^i_{\alpha\beta\gamma\cdots\vartheta}) \tag{1.3}$$

and we may immediately state the following *replacement theorem* for affine differential invariants.†

THEOREM I. *Any affine differential invariant* (1.2) *can be put into the form* (1.3) *by replacing the $\Gamma$'s by zero and their derivatives by the corresponding normal tensors $A$.*

For the particular case in which the invariant $T$ is assumed to depend on the functions $\Gamma$ alone the replacement theorem shows that $T$ reduces to a set of constants and consequently would not be spoken of as an affine differential invariant. Hence there exists no affine differential invariant which depends on the functions $\Gamma$ alone.

The ease with which we have proved the above replacement theorem is but a single illustration of the many advantages of the use of normal coördinates.

*Definition.* An *affine differential invariant* (1.2) will be said to be of *order* $r$ if at least one $r$th order derivative of a component of the $\Gamma$'s occurs explicitly in the invariant.

The idea of an affine differential invariant may be extended to the case of a set of functions of the $\Gamma$'s and their derivatives which depend also on a set of auxiliary arbitrary scalar functions $F^{(k)}(x)$ and their derivatives. Such an invariant will have the form‡

* O. Veblen, Proc. Nat. Acad. Sci., vol. 8 (1922), p. 192. See also V. & T., § 9.

† See T. Y. Thomas, Math. Zeit., vol. 25 (1926), p. 729. Regarding the so-called "Reduktion" theorems, see R. Weitzenböck, *Invariantentheorie* (1923), pp. 349–462.

‡ We need not assume that $s = r + 1$, as is usually the case in the corresponding differential parameters of the Riemannian geometry.

$$(1.4)\quad T^{\mu\nu\cdots\varepsilon}_{\sigma\tau\cdots\eta}\left(\Gamma^i_{\alpha\beta};\ \frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^\gamma};\ \cdots;\ \frac{\partial^r\Gamma^i_{\alpha\beta}}{\partial x^\gamma\cdots\partial x^\delta};\ F^{(k)};\ \frac{\partial F^{(k)}}{\partial x^\pi};\ \cdots;\ \frac{\partial^s F^{(k)}}{\partial x^\pi\cdots\partial x^\varrho}\right)$$

where the index $k$ may take on values $(1, 2, \cdots, w)$.

If we let $F^{(k)}_\pi, \cdots, F^{(k)}_{\pi\cdots\varrho}$ stand for the first to the $s$th extension of $F^{(k)}$ respectively, we can, by a process analogous to that by means of which we arrived at Theorem I, state the following replacement theorem.

THEOREM II. *Any affine differential invariant* (1.4) *can be put into the form*

$$(1.5)\quad T^{\mu\nu\cdots\varepsilon}_{\sigma\tau\cdots\eta}(0;\ A^i_{\alpha\beta\gamma};\ \cdots;\ A^i_{\alpha\beta\gamma\cdots\delta};\ F^{(k)};\ F^{(k)}_\pi;\ \cdots;\ F^{(k)}_{\pi\cdots\varrho})$$

*by replacing the $\Gamma$'s by zero, their derivatives by the corresponding normal tensors $A$, and the derivatives of the $F^{(k)}$ by the corresponding extensions $F^{(k)}_\pi, \cdots, F^{(k)}_{\pi\cdots\varrho}$.*

We observe that the functions $F^{(k)}$ themselves remained unchanged in the replacement process of Theorem II.

It is obvious that the set of scalars $F^{(k)}$ can be replaced by a set of tensors in the above replacement theorem.

For the particular case in which the invariant (1.4) is assumed not to depend on the derivatives of the $\Gamma$'s the replacement Theorem II shows that $T$ reduces to a set of functions depending on $F^{(k)}$ and its extensions and consequently would not be spoken of as an affine differential invariant. Hence there exists no affine differential invariant which is dependent only on the $\Gamma^i_{\alpha\beta}$, the $F^{(k)}$ and the derivatives of the $F^{(k)}$.

2. **A complete set of tensor invariants.** In the general problem of the equivalence of two affinely connected manifolds $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ determined by two given sets of functions $\Gamma^i_{\alpha\beta}$ and $\overline{\Gamma}^i_{\alpha\beta}$ respectively, we inquire if there exists a transformation

$$(2.1)\quad \mathfrak{G}:\ x^i = f^i(\bar{x}^1, \cdots, \bar{x}^n), \qquad \left|\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right| \neq 0$$

by means of which the $\Gamma$'s can be transformed into the $\overline{\Gamma}$'s. *If this is possible we say that the two manifolds $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ are equivalent.*

Connected with the equivalence problem is the concept of a *complete set of differential invariants.* A set of affine differential invariants of an affinely connected manifold $\mathfrak{M}$ is complete if algebraic necessary and sufficient conditions for the equivalence of $\mathfrak{M}$ with any other affinely connected manifold $\overline{\mathfrak{M}}$ can be expressed in terms of the members of this set.

It is clear that each normal tensor $A$ is an affine differential invariant.* We can therefore proceed immediately to demonstrate the following theorem.

* For a method of deducing expressions giving the normal tensors in terms of the $\Gamma$'s and their derivatives see V. & T., pp. 568–569.

THEOREM III. *The totality of normal tensors A constitute a complete set of affine differential invariants.*

To prove this theorem we first write down the following infinite sequence

$$
\begin{aligned}
&\overline{A}^{\sigma}_{\alpha\beta\gamma}\ u^{i}_{\sigma} = A^{i}_{\mu\nu\sigma}\, u^{\mu}_{\alpha}\, u^{\nu}_{\beta}\, u^{\sigma}_{\gamma} \\
&\overline{A}^{\sigma}_{\alpha\beta\gamma\delta}\, u^{i}_{\sigma} = A^{i}_{\mu\nu\sigma\tau}\, u^{\mu}_{\alpha}\, u^{\nu}_{\beta}\, u^{\sigma}_{\gamma}\, u^{\tau}_{\delta} \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots
\end{aligned}
\tag{2.2}
$$

Obviously a necessary condition for the equivalence of two manifolds $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ determined by $\Gamma$'s and $\overline{\Gamma}$'s respectively is that the above infinite sequence possesses a numerical solution

$$
x^{i} = q^{i}, \quad \overline{x}^{i} = \overline{q}^{i}, \quad u^{i}_{j} = h^{i}_{j}. \tag{2.3}
$$

We shall prove our theorem if we show that this condition is also sufficient. To do this we erect normal coördinates $(y)$ and $(\overline{y})$ with origins at the points $x = q$, $\overline{x} = \overline{q}$ respectively. Connect these normal coördinates by the relations

$$
y^{i} = u^{i}_{\alpha}\, \overline{y}^{\alpha} \tag{2.4}
$$

where $u^{i}_{\alpha}$ is the set of constants $h^{i}_{\alpha}$ in (2.3). Let us denote the affine connection in the normal coördinates $(y)$ and $(\overline{y})$ by $C^{i}_{\alpha\beta}$ and $\overline{C}^{i}_{\alpha\beta}$ respectively. We can then write the following familiar expansions*

$$
\begin{aligned}
C^{i}_{\alpha\beta} &= A^{i}_{\alpha\beta\gamma}(q)\, y^{\gamma} + \frac{1}{2!}\, A^{i}_{\alpha\beta\gamma\delta}(q)\, y^{\gamma}\, y^{\delta} + \cdots + \cdots \\
\overline{C}^{i}_{\alpha\beta} &= \overline{A}^{i}_{\alpha\beta\gamma}(\overline{q})\, \overline{y}^{\gamma} + \frac{1}{2!}\, \overline{A}^{i}_{\alpha\beta\gamma\delta}(\overline{q})\, \overline{y}^{\gamma}\, \overline{y}^{\delta} + \cdots + \cdots
\end{aligned}
\tag{2.5}
$$

Since by hypothesis the relations (2.2) hold between the normal tensors $A(q)$ and $\overline{A}(\overline{q})$ it follows that the affine connections $C$ and $\overline{C}$ have the law of transformation

$$
\overline{C}^{\sigma}_{\alpha\beta}\, u^{i}_{\sigma} = C^{i}_{\mu\nu}\, u^{\mu}_{\alpha}\, u^{\nu}_{\beta} \tag{2.6}
$$

under the relation (2.4). This assertion is shown readily by calculation on employing the expansions (2.5). The relation (2.4) in conjunction with the relation between the coördinates $(x)$, $(y)$ and the coördinates $(\overline{x})$, $(\overline{y})$ defines a relation

$$
x^{i} = f^{i}(\overline{x}^{1}, \cdots, \overline{x}^{n}) \tag{2.7}
$$

* Cf. V. & T., p. 566.

between the coördinates $(x)$ and the coördinates $(\bar{x})$. The relation (2.7) is such that $x = q$ when $\bar{x} = \bar{q}$. Furthermore $\partial x^i/\partial \bar{x}^\alpha = h^i_\alpha$ when $\bar{x} = \bar{q}$. This last statement may readily be verified by using the fact that at the origin of normal coördinates $\partial x^i/\partial y^\alpha = \delta^i_\alpha$. In fact

$$\left(\frac{\partial x^i}{\partial \bar{x}^\alpha}\right)_{\bar{q}} = \left(\frac{\partial x^i}{\partial y^j}\,\frac{\partial y^j}{\partial \bar{y}^k}\,\frac{\partial \bar{y}^k}{\partial \bar{x}^\alpha}\right)_{\bar{q}} = \delta^i_j\, h^j_k\, \delta^k_\alpha = h^i_\alpha.$$

It follows therefore that under (2.7) the affine connection $\Gamma$ is transformed into the affine connection $\bar{\Gamma}$, i. e., the relations (1.1) are satisfied by (2.7). This concludes the proof of our theorem.

3. **Complete sets of identities of the $A$-invariants.** In our subsequent discussions we shall find it necessary to consider the identities satisfied by the normal tensors $A$. It is known* that the normal tensors $A$ satisfy the following identities

$$(3.1)\quad A^i_{\alpha\beta\gamma\delta\ldots\tau} = A^i_{\beta\alpha\gamma\delta\ldots\tau};\quad A^i_{\alpha\beta\gamma\delta\ldots\tau} = A^i_{\alpha\beta\mu\nu\ldots\sigma};\quad S(A^i_{\alpha\beta\gamma\delta\ldots\tau}) = 0,$$

where $(\mu\nu\ldots\sigma)$ is any permutation of $(\gamma\delta\ldots\tau)$ and $S$ denotes the sum of the terms obtainable from the one in the parenthesis which are not identical because of the two other sets of identities. For example,

$$(3.2)\qquad A^i_{jk\alpha} + A^i_{k\alpha j} + A^i_{\alpha jk} = 0,\quad A^i_{\alpha\beta\gamma} = A^i_{\beta\alpha\gamma}.$$

It has been shown by one of us† that the identities (3.1) constitute a *complete set of identities* of the normal tensor $A^i_{\alpha\beta\gamma\delta\ldots\tau}$. By this it is meant that every identity satisfied by the invariant $A^i_{\alpha\beta\gamma\delta\ldots\tau}$ can be deduced from the set of algebraically independent identities (3.1) by algebraic processes.

4. **Number of algebraically independent components of the $A$-invariants.** Since the identities (3.1) constitute a complete set, it follows that the enumeration of the independent components of any normal tensor $A$ from (3.1) will yield the total number of algebraically independent components of the normal tensor $A$. Let $N(n, p)$ stand for the number of algebraically independent components of $A^i_{\alpha\beta\gamma\ldots\tau}$ with $p+2$ subscripts $(\alpha\beta\gamma\ldots\tau)$ in the $n$-dimensional manifold $\mathfrak{M}$. If then $K(\mu, \lambda)$ denotes the number of combinations with repetitions of $\mu$ things taken $\lambda$ at a time, we shall have the following formula:

* V & T, p. 568.

† Cf. T. Y. Thomas, Math. Zeit., vol. 25 (1926) p. 714. The proof depends essentially on the fact that the normal tensors $A$ can take on preassigned values consistent with the identities (3.1) at a point of the affinely connected manifold for a suitable choice of the functions $\Gamma^i_{jk}$.

(4.1) $$N(n, p) = n(K(n, 2)\, K(n, p) - K(n, p+2))$$

as a consequence of (3.1). Since $K(\mu, \lambda) = \mu(\mu+1)\cdots(\mu+\lambda-1)/\lambda!$ we find

$$N(n, p) = \frac{1}{2}\frac{(n+p-1)!\, n}{(n-1)!\,(p+2)!}[n(n+1)(p+1)(p+2) - 2(n+p)(n+p+1)].$$

On further reduction we have

(4.2) $$N(n, p) = \frac{1}{2}\frac{(n+p-1)!\, np}{(n-2)!\,(p+2)!}[(n+2)(p+1)+2n].$$

**5. Variations of tensors under infinitesimal transformations.** Our subsequent work requires the determination of the expression

(5.1) $$\left[\frac{d\,\bar{T}^{lm\cdots np}_{ab\cdots cd}(\bar{x})}{d\varepsilon}\right]_{\varepsilon=0}$$

under the infinitesimal deformations

(5.2) $$x^i = \bar{x}^i + \varepsilon\,\xi^i(\bar{x}).$$

The transformation law of the tensor $T$ in (5.1) can be written in the form

(5.3) $$\bar{T}^{\lambda\mu\cdots\nu\pi}_{ab\cdots cd}(\bar{x})\frac{\partial x^l}{\partial \bar{x}^\lambda}\frac{\partial x^m}{\partial \bar{x}^\mu}\cdots\frac{\partial x^n}{\partial \bar{x}^\nu}\frac{\partial x^p}{\partial \bar{x}^\pi} = T^{lm\cdots np}_{\alpha\beta\cdots\gamma\delta}\frac{\partial x^\alpha}{\partial \bar{x}^a}\frac{\partial x^\beta}{\partial \bar{x}^b}\cdots\frac{\partial x^\gamma}{\partial \bar{x}^c}\frac{\partial x^\delta}{\partial \bar{x}^d}.$$

From (5.2) we get

(5.4) $$\frac{\partial x^i}{\partial \bar{x}^j} = \delta^i_j + \varepsilon\frac{\partial \xi^i(\bar{x})}{\partial \bar{x}^j}.$$

Substituting the expressions (5.4) in (5.3) we get

$$\begin{aligned}
&\bar{T}^{lm\cdots np}_{ab\cdots cd} + \varepsilon\left(\bar{T}^{\lambda m\cdots np}_{ab\cdots cd}\frac{\partial \xi^l(\bar{x})}{\partial \bar{x}^\lambda} + \bar{T}^{l\lambda\cdots np}_{ab\cdots cd}\frac{\partial \xi^m}{\partial \bar{x}^\lambda}\right.\\
&\qquad\left. + \cdots + \bar{T}^{lm\cdots\lambda p}_{ab\cdots cd}\frac{\partial \xi^n}{\partial \bar{x}^\lambda} + \bar{T}^{lm\cdots n\lambda}_{ab\cdots cd}\frac{\partial \xi^p}{\partial \bar{x}^\lambda}\right) + H_1(\varepsilon)\\
&= T^{lm\cdots np}_{ab\cdots cd} + \varepsilon\left(T^{lm\cdots np}_{\lambda b\cdots cd}\frac{\partial \xi^\lambda}{\partial \bar{x}^a} + T^{lm\cdots np}_{a\lambda\cdots cd}\frac{\partial \xi^\lambda}{\partial \bar{x}^b}\right.\\
&\qquad\left. + \cdots + T^{lm\cdots np}_{ab\cdots\lambda d}\frac{\partial \xi^\lambda}{\partial \bar{x}^c} + T^{lm\cdots np}_{ab\cdots c\lambda}\frac{\partial \xi^\lambda}{\partial \bar{x}^d}\right) + H_2(\varepsilon)
\end{aligned} \tag{5.5}$$

where $H_1(\varepsilon)$ and $H_2(\varepsilon)$ denote expressions in higher powers of $\varepsilon$. As a consequence therefore of (5.2) and (5.5) we have

$$\left[\frac{d\,\overline{T}^{lm\cdots np}_{ab\cdots cd}(\overline{x})}{d\varepsilon}\right]_{\varepsilon=0} = T^{lm\cdots np}_{\lambda b\cdots cd}(x)\frac{\partial\,\xi^{\lambda}(x)}{\partial\,x^{a}} + T^{lm\cdots np}_{a\lambda\cdots cd}\frac{\partial\,\xi^{\lambda}}{\partial\,x^{b}} + \cdots + T^{lm\cdots np}_{ab\cdots c\lambda}\frac{\partial\,\xi^{\lambda}}{\partial\,x^{d}} - T^{\lambda m\cdots np}_{ab\cdots cd}\frac{\partial\,\xi^{l}}{\partial\,x^{\lambda}} - T^{l\lambda\cdots np}_{ab\cdots cd}\frac{\partial\,\xi^{m}}{\partial\,x^{\lambda}} - \cdots - T^{lm\cdots n\lambda}_{ab\cdots cd}\frac{\partial\,\xi^{p}}{\partial\,x^{\lambda}}. \tag{5.6}$$

6. **Differential equations of scalar differential invariants.** If $f^i(\overline{x})$ is an arbitrarily given set of functions of $(\overline{x}^1, \cdots, \overline{x}^n)$, then the transformation

$$\mathfrak{G}\colon\ x^i = f^i(\overline{x}^1, \cdots, \overline{x}^n) \tag{6.1}$$

defines a transformation which can be included in a family of transformations of the form

$$x^i = \overline{x}^i + \varepsilon\,\xi^i(\overline{x}). \tag{6.2}$$

In fact on taking $\varepsilon = \delta \neq 0$ and $\xi^i(\overline{x}) = (f^i(\overline{x}) - \overline{x}^i)/\delta$ in (6.2) these equations go over into (6.1). Obviously each member of the family (6.2) is a member of the group of all analytic transformations of the form (6.1). Furthermore, the identical transformation is included in the family (6.2), e. g., for $\varepsilon = 0$.

Consider a function of the form $G(A^i_{\alpha\beta\gamma})$. By definition $G$ will be an affine scalar differential invariant if and only if

$$G(\overline{A}^i_{\alpha\beta\gamma}) = G(A^i_{\alpha\beta\gamma}) \tag{6.3}$$

for each transformation $\mathfrak{G}$. The condition (6.3) under an arbitrary one parameter family of transformations (6.2) is a sufficient condition for the invariance of $G$.

Expanding $G(\overline{A})$ about $\varepsilon = 0$, we get*

$$G(\overline{A}) = G(A) + \varepsilon\left[\frac{dG(\overline{A})}{d\varepsilon}\right]_{\varepsilon=0} + \frac{\varepsilon^2}{2!}\left[\frac{d^2G(\overline{A})}{d\varepsilon^2}\right]_{\varepsilon=0} + \cdots + \cdots. \tag{6.4}$$

It is clear that

$$\left[\frac{dG(\overline{A})}{d\varepsilon}\right]_{\varepsilon=0} = \frac{\partial\,G(A)}{\partial\,A^i_{\alpha\beta\gamma}}\left[\frac{d\,\overline{A}^i_{\alpha\beta\gamma}}{d\varepsilon}\right]_{\varepsilon=0}. \tag{6.5}$$

By means of (5.6) and (6.2) we can write (6.5) as

$$\left[\frac{dG(\overline{A})}{d\varepsilon}\right]_{\varepsilon=0} = \frac{\partial\,G}{\partial\,A^i_{\alpha\beta\gamma}}\left(A^i_{\alpha\beta\gamma}\frac{\partial\,\xi^{\sigma}}{\partial\,x^{\alpha}} + A^i_{\alpha\sigma\gamma}\frac{\partial\,\xi^{\sigma}}{\partial\,x^{\beta}} + A^i_{\alpha\beta\sigma}\frac{\partial\,\xi^{\sigma}}{\partial\,x^{\gamma}} - A^{\sigma}_{\alpha\beta\gamma}\frac{\partial\,\xi^{i}}{\partial\,x^{\sigma}}\right). \tag{6.6}$$

* We assume that $G(\overline{A})$ is analytic as a function of $\varepsilon$ within a circle of radius $\eta \neq 0$ which may be as small as we please.

Defining the linear operator $X$ by

$$XG = \frac{\partial G}{\partial A^i_{\alpha\beta\gamma}}\left(A^i_{\sigma\beta\gamma}\frac{\partial \xi^\sigma}{\partial x^\alpha} + A^i_{\alpha\sigma\gamma}\frac{\partial \xi^\sigma}{\partial x^\beta} + A^i_{\alpha\beta\sigma}\frac{\partial \xi^\sigma}{\partial x^\gamma} - A^\sigma_{\alpha\beta\gamma}\frac{\partial \xi^i}{\partial x^\sigma}\right),$$

we can rewrite (6.4) in the form

$$G(\bar{A}) = G(A) + \sum_{i=1}^{\infty}\frac{\epsilon^i}{i!}X^i G. \tag{6.7}$$

Hence a necessary and sufficient condition that $G(\bar{A}) = G(A)$ is that

$$XG = 0 \tag{6.8}$$

This follows from the fact that the equations $X^i G = 0$ when $i$ has values $2, 3, \cdots$, are consequences of (6.8).

We can write (6.8) in the convenient form

$$\frac{\partial \xi^\sigma}{\partial x^\tau}\begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{pmatrix}\frac{\partial G}{\partial A^i_{\alpha\beta\gamma}} = 0, \tag{6.9}$$

where

$$\begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{pmatrix} = A^i_{\sigma\beta\gamma}\,\delta^\tau_\alpha + A^i_{\alpha\sigma\gamma}\,\delta^\tau_\beta + A^i_{\alpha\beta\sigma}\,\delta^\tau_\gamma - A^\tau_{\alpha\beta\gamma}\,\delta^i_\sigma. \tag{6.10}$$

Owing to the tensor character of $A^i_{\alpha\beta\gamma}$ and $\delta^\tau_\sigma$ we observe that $\begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{pmatrix}$ is a mixed tensor which is covariant in the indices $(\alpha\,\beta\,\gamma\,\sigma)$ and contravariant in the indices $(i\,\tau)$. Since the vector $\xi^i$ in (6. 9) is arbitrary, we can state immediately the following theorem.

THEOREM IV. *A necessary and sufficient condition that $G(A)$ be an affine scalar differential invariant is that it satisfy the following system of differential equations*:

$$\begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{pmatrix}\frac{\partial G}{\partial A^i_{\alpha\beta\gamma}} = 0. \tag{6.11}$$

By a direct extension of the methods used in proving Theorem IV it is easy to show that the following theorem holds.

THEOREM V. *A necessary and sufficient condition that $G(A^i_{\alpha\beta\gamma}; A^i_{\alpha\beta\gamma\delta}; \cdots \cdots; A^i_{\alpha\beta\gamma\delta\cdots\nu})$ be an affine scalar differential invariant is that it satisfy the following system of differential equations*:

$$\begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{pmatrix}\frac{\partial G}{\partial A^i_{\alpha\beta\gamma}} + \begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\delta\,\sigma \end{pmatrix}\frac{\partial G}{\partial A^i_{\alpha\beta\gamma\delta}} + \cdots + \begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\delta\cdots\nu\,\sigma \end{pmatrix}\frac{\partial G}{\partial A^i_{\alpha\beta\gamma\delta\cdots\nu}} = 0, \tag{6.12}$$

*where*

$$\left(\begin{matrix} i\tau \\ \alpha\beta\gamma\delta\ldots\lambda\sigma \end{matrix}\right) = A^i_{\alpha\beta\gamma\delta\ldots\lambda}\,\delta^\tau_\alpha + A^i_{\alpha\sigma\gamma\delta\ldots\lambda}\,\delta^\tau_\beta + \cdots + A^i_{\alpha\beta\gamma\delta\ldots\sigma}\,\delta^\tau_\lambda - A^\tau_{\alpha\beta\gamma\delta\ldots\lambda}\,\delta^i_\sigma. \tag{6.13}$$

It is easily verified by the above method of constructing the differential equations for $G$ that $G$ can not involve the variables $(x^1, \cdots, x^n)$ explicitly and it is for this reason that we have omitted the possible occurrence of these variables in the definition of the invariant $G$. A corresponding remark may be made for affine differential invariants in general.

**7. Direct proof of completeness of the differential equations.** We shall demonstrate now that the system of partial differential equations (6.11) is complete.

By definition

$$Z^\varkappa_\mu(1)\,G = \left(\begin{matrix} i\varkappa \\ \alpha\beta\gamma\mu \end{matrix}\right)\frac{\partial G}{\partial A^i_{\alpha\beta\gamma}} \tag{7.1}$$

and

$$(Z^\varkappa_\mu(1),\, Z^p_q(1))\,G = Z^\varkappa_\mu(1)\,(Z^p_q(1)\,G) - Z^p_q(1)\,(Z^\varkappa_\mu(1)\,G). \tag{7.2}$$

Our task is to show that the alternants $(Z^\varkappa_\mu(1),\, Z^p_q(1))\,G$ can be expressed as linear combinations of the $Z^\xi_\eta(1)\,G$. By calculation we have

$$(Z^\varkappa_\mu(1),\, Z^p_q(1))\,G = \left[\left(\begin{matrix} i\varkappa \\ \alpha\beta\gamma\mu \end{matrix}\right)\frac{\partial\left(\begin{matrix} tp \\ uvwq \end{matrix}\right)}{\partial A^i_{\alpha\beta\gamma}} - \left(\begin{matrix} ip \\ \alpha\beta\gamma q \end{matrix}\right)\frac{\partial\left(\begin{matrix} t\varkappa \\ uvw\mu \end{matrix}\right)}{\partial A^i_{\alpha\beta\gamma}}\right]\frac{\partial G}{\partial A^t_{uvw}}. \tag{7.3}$$

Employing the Kronecker tensor $\delta^\lambda_\mu$ to advantage in the differentiations in (7.3), we get

$$\begin{aligned}(Z^\varkappa_\mu(1),\, Z^p_q(1))\,G = \Big[&\left(\begin{matrix} t\varkappa \\ qvw\mu \end{matrix}\right)\delta^p_u + \left(\begin{matrix} t\varkappa \\ uqw\mu \end{matrix}\right)\delta^p_v + \left(\begin{matrix} t\varkappa \\ uvq\mu \end{matrix}\right)\delta^p_w - \left(\begin{matrix} p\varkappa \\ uvw\mu \end{matrix}\right)\delta^t_q \\ &- \left(\begin{matrix} tp \\ \mu vwq \end{matrix}\right)\delta^\varkappa_u - \left(\begin{matrix} tp \\ u\mu wq \end{matrix}\right)\delta^\varkappa_v - \left(\begin{matrix} tp \\ uv\mu q \end{matrix}\right)\delta^\varkappa_w + \left(\begin{matrix} \varkappa p \\ uvwq \end{matrix}\right)\delta^t_\mu\Big]\frac{\partial G}{\partial A^t_{uvw}}.\end{aligned} \tag{7.4}$$

Carrying out the cancellations in (7.4) we get

$$\begin{aligned}(Z^\varkappa_\mu(1),\, Z^p_q(1))\,G &= \left[\delta^\varkappa_q\left(\begin{matrix} tp \\ uvw\mu \end{matrix}\right) - \delta^p_\mu\left(\begin{matrix} t\varkappa \\ uvwq \end{matrix}\right)\right]\frac{\partial G}{\partial A^t_{uvw}} \\ &= \delta^\varkappa_q\,Z^p_\mu(1)\,G - \delta^p_\mu\,Z^\varkappa_q(1)\,G\end{aligned} \tag{7.5}$$

and hence the system of equations

$$Z^{\varkappa}_{\mu}(1)\, G = 0 \tag{7.6}$$

is complete.

We shall now proceed with the proof of the completeness of the differential equations (6.12). We define the linear operator $Z^{\tau}_{\sigma}(r)$ by

$$Z^{\tau}_{\sigma}(r)\, G = \sum_{u=3}^{k} \begin{pmatrix} i\,r \\ \alpha_1\,\alpha_2\,\alpha_3 \cdots \alpha_u\,\sigma \end{pmatrix} \frac{\partial G}{\partial A^{i}_{\alpha_1\alpha_2\alpha_3\cdots\alpha_u}} \tag{7.7}$$

where $r = k-2$ is the order of the differential invariant $G$. We define the alternant

$$(Z^{\varkappa}_{\mu}(r),\ Z^{p}_{q}(r))\, G = Z^{\varkappa}_{\mu}(r)\,(Z^{p}_{q}(r)\, G) - Z^{p}_{q}(r)\,(Z^{\varkappa}_{\mu}(r)\, G). \tag{7.8}$$

By calculation we have therefore

$$\begin{aligned}
(Z^{\varkappa}_{\mu}(r),\ Z^{p}_{q}(r))\, G &= \sum_{u=3}^{k}\sum_{v=3}^{k}\Bigg[\begin{pmatrix} i\varkappa \\ \alpha_1\,\alpha_2\,\alpha_3\cdots\alpha_u\,\mu \end{pmatrix} \frac{\partial \begin{pmatrix} tp \\ \beta_1\,\beta_2\,\beta_3\cdots\beta_v\,q \end{pmatrix}}{\partial A^{i}_{\alpha_1\alpha_2\alpha_3\cdots\alpha_u}} \\
&\qquad - \begin{pmatrix} ip \\ \alpha_1\,\alpha_2\,\alpha_3\cdots\alpha_u\,q \end{pmatrix} \frac{\partial \begin{pmatrix} t\varkappa \\ \beta_1\,\beta_2\,\beta_3\cdots\beta_v\,\mu \end{pmatrix}}{\partial A^{i}_{\alpha_1\alpha_2\alpha_3\cdots\alpha_u}}\Bigg] \frac{\partial G}{\partial A^{t}_{\beta_1\beta_2\beta_3\cdots\beta_v}} \\
&= \sum_{u=3}^{k}\Bigg[\begin{pmatrix} i\varkappa \\ \alpha_1\,\alpha_2\cdots\alpha_u\,\mu \end{pmatrix} \frac{\partial \begin{pmatrix} tp \\ \beta_1\,\beta_2\cdots\beta_u\,q \end{pmatrix}}{\partial A^{i}_{\alpha_1\alpha_2\cdots\alpha_u}} \\
&\qquad - \begin{pmatrix} ip \\ \alpha_1\,\alpha_2\cdots\alpha_u\,q \end{pmatrix} \frac{\partial \begin{pmatrix} t\varkappa \\ \beta_1\,\beta_2\cdots\beta_u\,\mu \end{pmatrix}}{\partial A^{i}_{\alpha_1\alpha_2\cdots\alpha_u}}\Bigg] \frac{\partial G}{\partial A^{t}_{\beta_1\beta_2\cdots\beta_u}} \qquad (7.9) \\
&= \sum_{u=3}^{k}\left[\delta^{\varkappa}_{q}\begin{pmatrix} tp \\ \beta_1\,\beta_2\cdots\beta_u\,\mu \end{pmatrix} - \delta^{p}_{\mu}\begin{pmatrix} t\varkappa \\ \beta_1\,\beta_2\cdots\beta_u\,q \end{pmatrix}\right] \frac{\partial G}{\partial A^{t}_{\beta_1\beta_2\cdots\beta_u}} \\
&= \delta^{\varkappa}_{q}\, Z^{p}_{\mu}(r)\, G - \delta^{p}_{\mu}\, Z^{\varkappa}_{q}(r)\, G.
\end{aligned}$$

Hence we have the theorem.

THEOREM VI. *The system of differential equations for the determination of affine scalar differential invariants of order $r$:*

$$Z^{\tau}_{\sigma}(r)\, G = 0 \tag{7.10}$$

*is a complete system.*

In particular the equations (7.10) will be complete when referred to a set of independent variables $A$ as determined by the complete sets of identities (3.1). In treating the independence of the differential equations (7.10) these equations will be referred to independent variables $A$. A similar remark applies to the differential equations of invariants involving arbitrary functions (§ 16).

8. **General theorems on the independence of the differential equations.** In the theory of complete systems of linear partial differential equations there is a theorem which states that the number of functionally independent solutions of a complete system is equal to the number of variables in the system minus the number of linearly independent equations of the system. The knowledge of the number of independent equations in the system (6.12) is necessary for the determination of the number of functionally independent affine scalar differential invariants of order $r$. We proceed therefore to demonstrate the following two general theorems.

THEOREM VII. *If the differential equations for the determination of the absolute scalar differential invariants of order $p$ for the general affinely connected manifold are independent, those for the determination of the invariants of order $p+1$ are also independent.*

THEOREM VIII. *If the differential equations for the determination of the affine scalar differential invariants of order* 1 *are dependent for every affinely connected manifold of $n$ dimensions, they are dependent for every affinely connected manifold of $n-1$ dimensions.*

To prove the first theorem we observe that by hypothesis the differential equations

$$\begin{pmatrix} i\tau \\ \alpha\beta\gamma_1\sigma \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1}} + \begin{pmatrix} i\tau \\ \alpha\beta\gamma_1\gamma_2\sigma \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1\gamma_2}} + \cdots + \begin{pmatrix} i\tau \\ \alpha\beta\gamma_1\gamma_2\cdots\gamma_p\sigma \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_p}} = 0 \tag{8.1}$$

are independent. But the system

$$\begin{pmatrix} i\tau \\ \alpha\beta\gamma_1\sigma \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1}} + \cdots + \begin{pmatrix} i\tau \\ \alpha\beta\gamma_1\gamma_2\cdots\gamma_p\sigma \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_p}} + \begin{pmatrix} i\tau \\ \alpha\beta\gamma_1\gamma_2\cdots\gamma_p\gamma_{p+1}\sigma \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_p\gamma_{p+1}}} = 0 \tag{8.2}$$

has the same number of equations as (8.1) and can be formed from (8.1) merely by adding

$$\begin{pmatrix} i\tau \\ \alpha\beta\gamma_1\gamma_2\cdots\gamma_p\gamma_{p+1} \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_p\gamma_{p+1}}} \tag{8.3}$$

to the left hand members of the corresponding equations (8.1). Our theorem therefore follows immediately.*

We proceed now with the proof of Theorem VIII. By hypothesis the equations

$$\binom{i\alpha}{jkl\sigma}\frac{\partial G}{\partial A^i_{jkl}} = 0 \tag{8.5}$$

are dependent. Hence the linear homogeneous equations

$$\binom{i\alpha}{jkl\sigma}\eta^\sigma_\alpha = 0 \tag{8.5}$$

admit solutions $\eta^\sigma_\alpha$ not all identically zero. Owing to the remark about the independent variables $A$ made at the end of § 7 the indices $(i, j, k, l)$ in (8.5) are to be considered to take on values $(1, 2, \cdots, n)$ corresponding to those of the indices $(i, j, k, l)$ in the set of independent variables $A^i_{jkl}$. However, the coefficients of $\eta^\sigma_\alpha$ in (8.5) satisfy identities

$$\binom{i\alpha}{jkl\sigma} = \binom{i\alpha}{kjl\sigma}, \qquad \binom{i\alpha}{jkl\sigma} + \binom{i\alpha}{klj\sigma} + \binom{i\alpha}{ljk\sigma} = 0,$$

which correspond exactly to the identities satisfied by the components $A^i_{jkl}$. Consequently equations (8.5) hold for *all* values of the indices $(i, j, k, l)$.

We can write (8.5) explicitly in the form

$$A^\alpha_{jkl}\eta^i_\alpha - A^i_{\alpha kl}\eta^\alpha_j - A^i_{j\alpha l}\eta^\alpha_k - A^i_{jk\alpha}\eta^\alpha_l = 0. \tag{8.6}$$

We choose a particular affinely connected manifold of $n$ dimensions in a manner determined by the following scheme

$$\begin{aligned}
&\Gamma^n_{jk} = 0 && (j, k = 1, 2, \cdots, n),\\
&\Gamma^i_{nk} = 0 && (i, k = 1, 2, \cdots, n-1),\\
&\Gamma^i_{nn} = \frac{3}{2}x^i && (i = 1, 2, \cdots, n-1),\\
&\Gamma^i_{jk} = \psi^i_{jk}(x^1, x^2, \cdots, x^{n-1}) && (i, j, k = 1, 2, \cdots, n-1),
\end{aligned} \tag{8.7}$$

where any $\psi^i_{jk}$ for $i, j, k = 1, 2, \cdots, n-1$ is an arbitrary function of the variables $(x^1, x^2, \cdots, x^{n-1})$. In the proof of our theorem we shall need the result of the following lemma.

* In Lie's theory of differential invariants the *number of equations* for the determination of the differential invariants *increases* with the *order* of the invariants. C. Haskins using Lie's methods proved an analogous theorem for the Riemannian geometry. His theorem however, is not immediate as in our work. Cf. Haskins, Trans. Amer. Math. Soc., vol. 3 (1902), p. 76.

LEMMA. *The particular components*

$$A^i_{jkl} \qquad (i, j, k, l = 1, 2, \cdots, n-1) \tag{8.8}$$

*of the normal tensor A of the special n-dimensional affinely connected manifold determined by the scheme* (8.7) *constitute the totality of the components for the normal tensor A of the general affinely connected manifold of* $n-1$ *dimensions.*

The expression for $A^i_{jkl}$ in terms of the $\Gamma^i_{jk}$ and their derivatives is given by the formula*

$$A^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{1}{3} P\left(\frac{\partial \Gamma^i_{jk}}{\partial x^l} - 2\,\Gamma^i_{\alpha j}\,\Gamma^\alpha_{kl}\right) - \Gamma^i_{\beta k}\,\Gamma^\beta_{jl} - \Gamma^i_{j\beta}\,\Gamma^\beta_{kl} \tag{8.9}$$

where the symbol $P$ denotes the sum of the terms obtainable from the ones inside the parenthesis by permuting the set of *subscripts* $(j, k, l)$ cyclically. By means of (8.7) and (8.9) we get

$$A^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{1}{3} P\left(\frac{\partial \Gamma^i_{jk}}{\partial x^l} - 2\sum_{\alpha=1}^{n-1} \Gamma^i_{\alpha j}\,\Gamma^\alpha_{kl}\right) - \sum_{\beta=1}^{n-1} (\Gamma^i_{\beta k}\,\Gamma^\beta_{jl} - \Gamma^i_{j\beta}\,\Gamma^\beta_{kl}), \tag{8.10}$$

where $i, j, k, l = 1, 2, \cdots, n-1$. The lemma now follows immediately on observing that the components of the $\Gamma^i_{jk}$ in (8.10) depend only on $(x^1, x^2, \cdots, x^{n-1})$ as is seen by a reference to the last assumption in (8.7).

From (8.9) we have

$$A^n_{jkl} = \frac{\partial \Gamma^n_{jk}}{\partial x^l} - \frac{1}{3} P\left(\frac{\partial \Gamma^n_{jk}}{\partial x^l} - 2\,\Gamma^n_{\alpha j}\,\Gamma^\alpha_{kl}\right) - \Gamma^n_{\beta k}\,\Gamma^\beta_{jl} - \Gamma^n_{j\beta}\,\Gamma^\beta_{kl}.$$

But by (8.7), $\Gamma^n_{jk} = 0$ and hence

$$A^n_{jkl} = 0. \tag{8.11}$$

By (8.9) we have also

$$A^i_{nkl} = \frac{\partial \Gamma^i_{nk}}{\partial x^l} - \frac{1}{3} P\left(\frac{\partial \Gamma^i_{nk}}{\partial x^l} - 2\,\Gamma^i_{\alpha n}\,\Gamma^\alpha_{kl}\right) - \Gamma^i_{\beta k}\,\Gamma^\beta_{nl} - \Gamma^i_{n\beta}\,\Gamma^\beta_{kl}. \tag{8.12}$$

Hence by (8.7) the relations (8.12) yield

$$A^i_{nkl} = 0, \qquad (i, k, l = 1, 2, \cdots, n-1) \tag{8.13}$$

Employing the complete set of identities (3.2) and relations (8.13) we get

$$A^i_{knl} = 0, \quad A^i_{kln} = 0, \qquad (i, k, l = 1, 2, \cdots, n-1). \tag{8.14}$$

* Cf. V. & T., formulae (6.4) and (9.13). We remind the reader again that subscripts and superscripts can take on values $(1, 2, \cdots, n)$ unless otherwise stated explicitly.

From relations (8.11), (8.13) and (8.14) it follows that equations (8.6) reduce to the equations

$$(8.15)\quad \sum_{\alpha=1}^{n-1}(A^{\alpha}_{jkl}\eta^{i}_{\alpha}-A^{i}_{\alpha kl}\eta^{\alpha}_{j}-A^{i}_{jal}\eta^{\alpha}_{k}-A^{i}_{jk\alpha}\eta^{\alpha}_{l})=0,\quad (i,j,k,l=1,2,\cdots,n-1).$$

It is clear that if $\eta^{i}_{j}$ for $i,j=1,2,\cdots,n-1$ are not all identically zero that the differential equations for the determination of the differential invariants for every $(n-1)$-dimensional affinely connected manifold are dependent. We shall give now a proof by reductio ad absurdum that $\eta^{i}_{j}\neq 0$ for $i,j=1,2,\cdots,n-1$. Suppose then that

$$(8.16)\qquad \eta^{i}_{j}=0,\qquad (i,j=1,2,\cdots,n-1).$$

Consider equations (8.6) for $j,k=n$; $i,l=1,2,\cdots,n-1$ getting

$$(8.17)\quad A^{\alpha}_{nnl}\eta^{i}_{\alpha}-A^{i}_{\alpha nl}\eta^{\alpha}_{n}-A^{i}_{n\alpha l}\eta^{\alpha}_{n}-A^{i}_{nn\alpha}\eta^{\alpha}_{l}=0,\quad (i,l=1,2,\cdots,n-1).$$

By (3.2), (8.11), (8.16), (8.13) the relations (8.17) yield

$$(8.18)\qquad A^{i}_{nnl}\eta^{n}_{n}=0,\qquad (i,l=1,2,\cdots,n-1).$$

On employing relations (8.7), (8.9) we get by calculation

$$(8.19)\qquad A^{i}_{nnl}=\delta^{i}_{l}+\sum_{\alpha=1}^{n-1}x^{\alpha}\Gamma^{i}_{\alpha l},\quad (i,l=1,2,\cdots,n-1).$$

It is clear from (8.7) that $A^{i}_{nnl}$ for $i,l=1,2,\cdots,n-1$ depends only on the points of the $(n-1)$-dimensional affinely connected manifold. If $x^{\alpha}=0$ for $\alpha=1,2,\cdots,n-1$ then the determinant of $n-1$ rows

$$(8.20)\qquad |A^{i}_{nnl}|=1.$$

Hence in the neighborhood of the point $x^{\alpha}=0$ for $\alpha=1,2,\cdots,n-1$ we have

$$(8.21)\qquad |A^{i}_{nnl}|\neq 0,\qquad (i,l=1,2,\cdots,n-1).$$

We have also

$$(8.22)\qquad A^{i}_{nnl}\neq 0 \text{ when } i=l,\quad (i,l=1,2,\cdots,n-1).$$

From (8.18) and (8.22) we get

$$(8.23)\qquad \eta^{n}_{n}=0.$$

Putting $i,j,k=n$ in (8.6) and employing conditions (8.11) we see that

$$(8.24)\qquad \sum_{\alpha=1}^{n-1}A^{\alpha}_{nnl}\eta^{n}_{\alpha}=0,\qquad (l=1,2,\cdots,n-1).$$

By (8.21) it follows that

$$\eta^n_\alpha = 0, \qquad (\alpha = 1, 2, \cdots, n-1). \tag{8.25}$$

Now consider equations (8.6) for $i, j, k = 1, 2, \cdots, n-1$ and $l = n$. Then by means of (8.11), (8.13), (8.14), (8.16), (8.23) and (8.25) we have

$$\sum_{\alpha=1}^{n-1} A^i_{jk\alpha}\, \eta^\alpha_n = 0, \qquad (i, j, k = 1, 2, \cdots, n-1). \tag{8.26}$$

In general there exists an $(n-1)$-rowed determinant*

$$D \neq 0 \tag{8.27}$$

which is formed from the coefficients $A$ in (8.26) and hence in general

$$\eta^\alpha_n = 0 \qquad (\alpha = 1, 2, \cdots, n-1). \tag{8.28}$$

Hence contrary to our hypothesis about (8.5) we have

$$\eta^\sigma_\alpha = 0 \tag{8.29}$$

under condition (8.27). Hence under (8.27) equations (8.15) have solutions $\eta \neq 0$. This means that under (8.27) there exists a number $R$ of determinants of the matrix of (8.15) which vanish in the variables $A$. Now these $R$ determinants and the determinant $D$ are polynomials in the variables $A$. Furthermore these $R$ determinants vanish for all values of the variables $A$ for which $D$ does not vanish and hence by a theorem in algebra† the $R$ determinants vanish identically irrespective of whether $D$ vanishes or not. Hence (8.15) in all cases possesses solutions $\eta \neq 0$ and thus our theorem has been proved.

9. **Determination of the independent differential equations.** We turn now to the demonstration of some theorems for the general three dimensional and two dimensional affinely connected manifolds. By means of these theorems and the one proved in the preceding paragraph we shall be enabled to give definite information as regards the independence of the differential equations for the determination of the invariants connected with a general $n$-dimensional affinely connected manifold.

* The existence of such a determinant $D$ may easily be inferred from the fact that the $I^i_{jk}$ with indices $(1, 2, \cdots, n-1)$ may be chosen so that the $A$'s at any point have arbitrary values subject only to the complete set of identities for the $A$. Cf. T. Y. Thomas, Math. Zeit., vol. 25 (1926), p. 717.

† Cf. Bôcher, *Introduction to Higher Algebra*, p. 8.

Consider therefore the proof of the following theorem.

THEOREM IX. *The differential equations for the determination of the absolute scalar differential invariants of order* 1 *for the general affinely connected manifold of three dimensions are independent.**

The differential equations in question are

$$\begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma}} = 0 \tag{9.1}$$

From formula (4.2) we see that the number of algebraically independent components of the normal tensor $A^i_{\alpha\beta\gamma}$ is given by

$$N(3,1) = 24. \tag{9.2}$$

By means of the complete set of identities (3.2) it is easily seen that the following twenty-four components of $A^i_{\alpha\beta\gamma}$ are algebraically independent:

$$\begin{aligned} &A^i_{121} = a^i, \quad A^i_{131} = b^i, \quad A^i_{212} = c^i, \quad A^i_{232} = d^i, \\ &A^i_{313} = e^i, \quad A^i_{323} = f^i, \quad A^i_{231} = g^i, \quad A^i_{312} = h^i. \end{aligned} \tag{9.3}$$

The non-vanishing dependent components not equal to those in (9.3) by the second set of identities (3.2) are

$$\begin{aligned} &A^i_{112} = -2\,a^i; \quad A^i_{113} = -2\,b^i; \quad A^i_{221} = -2\,c^i; \quad A^i_{223} = -2\,d^i, \\ &A^i_{331} = -2\,e^i; \quad A^i_{332} = -2\,f^i, \quad A^i_{123} = A^i_{213} = -(g^i + h^i). \end{aligned} \tag{9.4}$$

Let us now express the invariant $G$ in (9.1) as well as the quantities

$$\begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{pmatrix} \tag{9.5}$$

in terms of the independent components $A^i_{jkl}$ in (9.3) alone. Then all the derivatives of $G$ with respect to the dependent components $A^i_{jkl}$ vanish and the equations (9.1) assume a form after some calculation which can be represented by the matrix given in Table I.

Putting

$$a^2 = d^3 = e^1 = 1; \quad \text{other components} = 0 \tag{9.6}$$

in the matrix and selecting the rows of the matrix corresponding to the coefficients of

* In the proof of this theorem we shall understand that all subscripts and superscripts can take on values (1, 2, 3).

| $(\tau,\sigma)=$ | (1,1) | (2,1) | (3,1) | (1,2) | (2,2) | (3,2) | (1,3) | (2,3) | (3,3) |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{\partial G}{\partial a^1}$ | $a^1$ | $-a^2$ | $-a^3$ | $-c^1$ | $a^1$ | $0$ | $-h^1$ | $b^1$ | $0$ |
| $\frac{\partial G}{\partial a^2}$ | $2a^2$ | $0$ | $0$ | $-(c^2+a^1)$ | $0$ | $-a^3$ | $-h^2$ | $b^2$ | $0$ |
| $\frac{\partial G}{\partial a^3}$ | $2a^3$ | $0$ | $0$ | $-c^3$ | $a^3$ | $0$ | $-(h^3+a^1)$ | $(b^3-a^2)$ | $-a^3$ |
| $\frac{\partial G}{\partial b^1}$ | $b^1$ | $-b^2$ | $-b^3$ | $(g^1+h^1)$ | $0$ | $a^1$ | $-e^1$ | $0$ | $b^1$ |
| $\frac{\partial G}{\partial b^2}$ | $2b^2$ | $0$ | $0$ | $(g^2+h^2-b^1)$ | $-b^2$ | $(a^2-b^3)$ | $-e^2$ | $0$ | $b^2$ |
| $\frac{\partial G}{\partial b^3}$ | $2b^3$ | $0$ | $0$ | $(g^3+h^3)$ | $0$ | $a^3$ | $-(e^3+b^1)$ | $-b^2$ | $0$ |
| $\frac{\partial G}{\partial c^1}$ | $0$ | $-(a^1+c^2)$ | $-c^3$ | $0$ | $2c^1$ | $0$ | $d^1$ | $-g^1$ | $0$ |
| $\frac{\partial G}{\partial c^2}$ | $c^2$ | $-a^2$ | $0$ | $-c^1$ | $c^2$ | $-c^3$ | $d^2$ | $-g^2$ | $0$ |
| $\frac{\partial G}{\partial c^3}$ | $c^3$ | $-a^3$ | $0$ | $0$ | $2c^3$ | $0$ | $(d^3-c^1)$ | $-(g^3+c^2)$ | $-c^3$ |
| $\frac{\partial G}{\partial d^1}$ | $-d^1$ | $(g^1+h^1-d^2)$ | $(c^1-d^3)$ | $0$ | $2d^1$ | $0$ | $0$ | $-f^1$ | $d^1$ |
| $\frac{\partial G}{\partial d^2}$ | $0$ | $g^3+h^2$ | $c^2$ | $-d^1$ | $d^2$ | $-d^3$ | $0$ | $-f^2$ | $d^2$ |
| $\frac{\partial G}{\partial d^3}$ | $0$ | $g^3+h^3$ | $c^3$ | $0$ | $2d^3$ | $0$ | $-d^1$ | $-(f^3+d^2)$ | $0$ |
| $\frac{\partial G}{\partial e^1}$ | $0$ | $-e^2$ | $-(b^1+e^3)$ | $f^1$ | $0$ | $-g^1$ | $0$ | $0$ | $2e^1$ |
| $\frac{\partial G}{\partial e^2}$ | $e^2$ | $0$ | $-b^2$ | $(f^2-e^1)$ | $-e^2$ | $-(g^2+e^3)$ | $0$ | $0$ | $2e^2$ |
| $\frac{\partial G}{\partial e^3}$ | $e^3$ | $0$ | $-b^3$ | $f^3$ | $0$ | $-g^3$ | $-e^1$ | $-e^2$ | $e^3$ |
| $\frac{\partial G}{\partial f^1}$ | $-f^1$ | $(e^1-f^2)$ | $-(h^1+f^3)$ | $0$ | $f^1$ | $-d^1$ | $0$ | $0$ | $2f^1$ |
| $\frac{\partial G}{\partial f^2}$ | $0$ | $e^2$ | $-h^2$ | $-f^1$ | $0$ | $-(d^2+f^3)$ | $0$ | $0$ | $2f^2$ |
| $\frac{\partial G}{\partial f^3}$ | $0$ | $e^3$ | $-h^3$ | $0$ | $f^3$ | $-d^3$ | $-f^1$ | $-f^2$ | $f^3$ |
| $\frac{\partial G}{\partial g^1}$ | $0$ | $(b^1-g^2)$ | $(a^1-g^3)$ | $d^1$ | $g^1$ | $-2c^1$ | $f^1$ | $-2e^1$ | $g^1$ |
| $\frac{\partial G}{\partial g^2}$ | $g^2$ | $b^2$ | $a^2$ | $(d^2-g^1)$ | $0$ | $-(2c^2+g^3)$ | $f^2$ | $-2e^2$ | $g^2$ |
| $\frac{\partial G}{\partial g^3}$ | $g^3$ | $b^3$ | $a^3$ | $d^3$ | $g^3$ | $-2c^3$ | $(f^3-g^1)$ | $-(2e^3+g^2)$ | $0$ |
| $\frac{\partial G}{\partial h^1}$ | $0$ | $(b^1-h^2)$ | $-(2a^1+h^3)$ | $d^1$ | $h^1$ | $c^1$ | $-2f^1$ | $e^1$ | $h^1$ |
| $\frac{\partial G}{\partial h^2}$ | $h^2$ | $b^2$ | $-2a^2$ | $(d^2-h^1)$ | $0$ | $(c^2-h^3)$ | $-2f^2$ | $e^2$ | $h^2$ |
| $\frac{\partial G}{\partial h^3}$ | $h^3$ | $b^3$ | $-2a^3$ | $d^3$ | $h^3$ | $c^3$ | $(-2f^2+h^1)$ | $(e^3-h^2)$ | $0$ |

Table I.

$$\left(\frac{\partial G}{\partial a^1}, \frac{\partial G}{\partial a^2}, \frac{\partial G}{\partial a^3}, \frac{\partial G}{\partial b^2}, \frac{\partial G}{\partial c^3}, \frac{\partial G}{\partial d^3}, \frac{\partial G}{\partial e^1}, \frac{\partial G}{\partial g^2}, \frac{\partial G}{\partial g^3}\right)$$

we get the determinant

$$(9.7) \qquad \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 8.$$

Thus the rank of the matrix for the general affinely connected manifold of three dimensions is 9 and hence Theorem IX is proved.

The following theorem is a consequence of Theorems VII, VIII, and IX.

THEOREM X. *The differential equations for the determination of the absolute scalar differential invariants of order* $\mu \geqq 1$ *for the general affinely connected manifold of* $n \geqq 3$ *dimensions are independent.*

The differential equations for the determination of the differential invariants of order 1 for the general affinely connected manifold of two dimensions turn out to be dependent.* We can demonstrate, however, the following theorem for the general affinely connected manifold of two dimensions.

THEOREM XI. *The differential equations for the determination of the absolute scalar differential invariants of order* $\mu \geqq 2$ *are independent for the general affinely connected manifold of two dimensions.*

To prove this theorem completely it is sufficient to show that the equations for the invariants of order $\mu = 2$ are independent, as is evident by a reference to Theorem VII.

From (6.12) the differential equations in question are†

$$(9.8) \qquad \begin{Bmatrix} i\tau \\ \alpha\beta\gamma\sigma \end{Bmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma}} + \begin{Bmatrix} i\tau \\ \alpha\beta\gamma\delta\sigma \end{Bmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma\delta}} = 0.$$

By calculation we see that formula (4.2) gives the number of algebraically independent components of the normal tensors $A^i_{\alpha\beta\gamma}$ and $A^i_{\alpha\beta\gamma\delta}$ for the two dimensional affinely connected manifold as

$$N(2, 1) = 4 \text{ and } N(2, 2) = 8$$

* Cf. the example in § 14.

† In the proof of this theorem we shall understand that subscripts and superscripts take on only the values (1,2).

respectively. These twelve independent components can be chosen as

$$
\begin{aligned}
&A^1_{211} = \alpha, \quad A^2_{122} = \beta, \quad A^1_{122} = \gamma, \quad A^2_{211} = \delta,\\
&A^1_{1122} = a, \quad A^2_{1122} = b, \quad A^1_{1212} = c, \quad A^2_{1212} = d,\\
&A^1_{1112} = e, \quad A^2_{1112} = f, \quad A^1_{2212} = g, \quad A^2_{2212} = h,
\end{aligned} \tag{9.9}
$$

and by calculation from the identities

$$
\begin{aligned}
A^i_{jk\alpha} + A^i_{k\alpha j} + A^i_{\alpha jk} &= 0,\\
A^i_{jk\alpha\beta} + A^i_{j\alpha k\beta} + A^i_{j\beta k\alpha} + A^i_{k\alpha j\beta} + A^i_{k\beta j\alpha} + A^i_{\alpha\beta jk} &= 0,
\end{aligned} \tag{9.10}
$$

we get

$$
\begin{aligned}
&A^1_{112} = -2\alpha, \quad A^2_{221} = -2\beta, \quad A^1_{221} = -2\gamma, \quad A^2_{112} = -2\delta,\\
&A^1_{2211} = -(a+4c), \quad A^2_{2211} = -(b+4d), \quad A^1_{1211} = -e,\\
&A^2_{1211} = -f, \qquad A^1_{1222} = -g, \qquad A^2_{1222} = -h.
\end{aligned} \tag{9.11}
$$

Expressing the differential equations (9.8) in terms of the independent components (9.9) after the manner of constructing Table I, we obtain a system of four differential equations with the matrix given in table II.

| $(\tau, \sigma) =$ | (1,1) | (2,1) | (1,2) | (2.2) |
|---|---|---|---|---|
| $\partial G/\partial\alpha$ | $\alpha$ | $-\delta$ | $-\gamma$ | $\alpha$ |
| $\partial G/\partial\beta$ | $\beta$ | $-\delta$ | $-\gamma$ | $\beta$ |
| $\partial G/\partial\gamma$ | 0 | $-(\alpha+\beta)$ | 0 | 0 |
| $\partial G/\partial\delta$ | $2\delta$ | 0 | $-(\alpha+\beta)$ | $2\gamma$ |
| $\partial G/\partial a$ | $a$ | $(2c-b)$ | $-2g$ | $2a$ |
| $\partial G/\partial b$ | $2b$ | $2f$ | $-(a+2h)$ | $b$ |
| $\partial G/\partial c$ | $c$ | $-d$ | 0 | $2c$ |
| $\partial G/\partial d$ | $2d$ | 0 | $-c$ | $d$ |
| $\partial G/\partial e$ | $2e$ | $-f$ | $(a+2c)$ | $e$ |
| $\partial G/\partial f$ | $3f$ | 0 | $(b+2d-e)$ | 0 |
| $\partial G/\partial g$ | 0 | $-(a+2c+h)$ | 0 | $3g$ |
| $\partial G/\partial h$ | $h$ | $-(b+2d)$ | $-g$ | $2h$ |

Table II.

The determinant formed from the first four rows of this matrix does not vanish identically and hence our theorem is proved.

10. **Fundamental sets of differential invariants.** *Definition.* A fundamental set of affine scalar differential invariants of order $l$ is a set of functionally independent affine scalar differential invariants each of which is of order $l$ such that any other affine scalar differential invariant of order $l$ is expressible as a function of the members of the set.

Let $\mathfrak{N}(n, p)$ stand for the number of functionally independent solutions of the complete system of $n^2$ differential equations

$$\begin{Bmatrix} i\tau \\ \alpha\beta\gamma_1\sigma \end{Bmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1}} + \begin{Bmatrix} i\tau \\ \alpha\beta\gamma_1\gamma_2\sigma \end{Bmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1\gamma_2}} + \cdots + \begin{Bmatrix} i\tau \\ \alpha\beta\gamma_1\gamma_2\cdots\gamma_p\sigma \end{Bmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_p}} = 0. \tag{10.1}$$

From the theory of complete systems of differential equations and by means of Theorems X and XI we have

$$\mathfrak{N}(n, p) = \sum_{\alpha=1}^{p} N(n, \alpha) - n^2 \quad (n \geqq 2,\ p \geqq 1;\ n \neq 2 \text{ if } p = 1) \tag{10.2}$$

where $N(n, \alpha)$ represents the number of algebraically independent components of the normal tensor $A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_\alpha}$ of the general $n$-dimensional affinely connected manifold as given by formula (4.2). *This number* $\mathfrak{N}(n, p)$ *in* (10.2) *is precisely the number of affine scalar differential invariants in a fundamental set of affine scalar differential invariants of order* $p$. This statement can be seen clearly in the following manner. There must be at least one component of the normal tensor $A$ with $p+2$ subscripts in at least one of the functionally independent solutions of the partial differential equations (10.1); for, if not, the equations

$$\begin{Bmatrix} i\tau \\ \alpha\beta\gamma_1\sigma \end{Bmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1}} + \cdots + \begin{Bmatrix} i\tau \\ \alpha\beta\gamma_1\gamma_2\cdots\gamma_{p-1}\sigma \end{Bmatrix} \frac{\partial G}{\partial A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_{p-1}}} = 0 \tag{10.3}$$

for the determination of the scalar differential invariants of order $p-1$ will have the same number of functionally independent solutions as equations (10.1), which is impossible as

$$\mathfrak{N}(n, p) > \mathfrak{N}(n, p-1). \tag{10.4}$$

We can thus pick $\mathfrak{N}(n, p)$ suitable independent functions each of which actually involves at least one component of the normal tensor $A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_p}$.

Taking cognizance of the fact* that $\mathfrak{N}(2, 1) = 1$, we can tabulate the following values of $\mathfrak{N}(n, p)$ as calculated from (10.2):

(10.5)

| $n$ \ $p$ | 1 | 2 | 3 |
|---|---|---|---|
| 2 | 1 | 8 | 20 |
| 3 | 15 | 78 | 195 |
| 4 | 64 | 324 | 900 |

* Cf. § 14.

11. **A complete set of absolute scalar invariants.** In § 2 we saw that the totality of normal tensors $A$ constitute a complete set of affine differential invariants. We are now however, in a position to prove the following theorem.

THEOREM XII. *The totality of the absolute scalar affine differential invariants of an affinely connected manifold $\mathfrak{M}$ constitute a complete set of affine differential invariants for this manifold.*

Let

$$I_i(A), \qquad (i = 1, 2, \cdots, \mathfrak{N}(n, p)) \tag{11.1}$$

be a fundamental set of scalar differential invariants of order $p$ of an $n$-dimensional manifold $\mathfrak{M}$. Form the relations

$$I_i(\bar{A}) = I_i(A), \qquad (i = 1, 2, \cdots, \mathfrak{N}(n, p)) \tag{11.2}$$

where $I_i(\bar{A})$ are the corresponding invariants of a manifold $\bar{\mathfrak{M}}$. Since the $I_i(A)$ are functionally independent as functions of the normal tensors $A$, there must exist a Jacobian determinant of order $\mathfrak{N}(n, p)$ in the matrix

$$\left\| \frac{\partial I_i(\bar{A})}{\partial \bar{A}} \right\| \tag{11.3}$$

which does not vanish identically. Hence there exists a set $\{\bar{A}\}$ of $\mathfrak{N}(n, p)$ components of the normal tensors $\bar{A}$ in the relations (11.2) which are uniquely determined (within a sufficiently small neighborhood of a point for which the above Jacobian does not vanish) by the components $A$ and a set $R(\bar{A})$ of components $\bar{A}$ in (11.2) not contained in $\{\bar{A}\}$. Thus we may write

$$\{\bar{A}\} = \Phi(A, R(\bar{A})). \tag{11.4}$$

Now let $r(\bar{A})$ stand for the set of $n^2$ independent components of the $\bar{A}$'s which are not in the set $\{\bar{A}\}$ and form the tensor equations of transformation by which the components $A$ are transformed into the components of the set $r(\bar{A})$. These equations may be represented by

$$r(\bar{A}) = \varphi(A, u); \qquad u^i_\alpha = \partial x^i / \partial \bar{x}^\alpha. \tag{11.5}$$

It is easily seen that all of the $n^2$ quantities $u^i_\alpha$ actually occur in the equations (11.5). Elimination of the members of the set $R(\bar{A})$ from (11.4) by means of (11.5) will give a set of equations which together with (11.5) will constitute a set of relations between all of the components $\bar{A}$, the components $A$ in (11.4) and (11.5) and the quantities $u$ in (11.5). Hence if all of the $u$'s did not occur in (11.5) we should have a relation between

the above quantities $A$, $\bar{A}$ and $u$ which is not identical with the tensor relation whereas in fact the tensor relationship is the only one which can exist in general between them. The $n^2$ equations represented by (11.5) may therefore be solved uniquely* (within a sufficiently small region) for the $n^2$ quantities $u^i_\alpha$ as is easily seen if we regard these equations as equations between the $n^2$ *independent* variables $\bar{A}$ and the $n^2$ *independent* variables $u^i_\alpha$. Thus

$$u = \psi(A, r(\bar{A})). \tag{11.6}$$

Let us now assume that the equations (11.2) are algebraically consistent considered as equations between the independent variables $(x^1, \cdots, x^n)$ and $(\bar{x}^1, \cdots, \bar{x}^n)$, which is obviously a necessary condition for the equivalence of the manifolds $\mathfrak{M}$ and $\bar{\mathfrak{M}}$. Take a solution of (11.2) to be

$$x^i = f^i(\bar{x}^1, \cdots, \bar{x}^n). \tag{11.7}$$

Since (11.4) is only another form of (11.2) the solution (11.7) likewise satisfies (11.4). When we eliminate the variables $(x^1, \cdots, x^n)$ from the right hand members of (11.6) by means of (11.7) we obtain

$$u^i_\alpha = \theta^i_\alpha(\bar{x}^1, \cdots, \bar{x}^n). \tag{11.8}$$

Hence relations (11.7) and (11.8) together constitute an algebraic solution of the equations

$$\begin{aligned}
&\bar{A}^\sigma_{\alpha\beta\gamma_1}\, u^i_\sigma = A^i_{\mu\nu\sigma_1}\, u^\mu_\alpha\, u^\nu_\beta\, u^{\sigma_1}_{\gamma_1} \\
&\bar{A}^\sigma_{\alpha\beta\gamma_1\gamma_2}\, u^i_\sigma = A^i_{\mu\nu\sigma_1\sigma_2}\, u^\mu_\alpha\, u^\nu_\beta\, u^{\sigma_1}_{\gamma_1}\, u^{\sigma_2}_{\gamma_2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\bar{A}^\sigma_{\alpha\beta\gamma_1\cdots\gamma_p}\, u^i_\sigma = A^i_{\mu\nu\sigma_1\cdots\sigma_p}\, u^\mu_\alpha\, u^\nu_\beta\, u^{\sigma_1}_{\gamma_1} \cdots u^{\sigma_p}_{\gamma_p}
\end{aligned} \tag{11.9}$$

for, as we have seen above, the equations (11.9) are equivalent to the two sets of equations (11.4) and (11.5). Since by Theorem III the totality of normal tensors $A$ constitute a complete set of invariants our present theorem follows by allowing $p$ to increase without limit.

12. **The $r$-parameter continuous group.** We shall proceed now to connect our theory with a certain $n^2$-parameter continuous group of transformations.† From (7.5) it is clear that we can write

* For $n = 2$, $p = 1$ there are three components in $r(\bar{A})$ so that $u^i$ are not uniquely determined by (11.5). This does not interfere with the argument here or in § 18.

† It is evident that the group in question is in all cases (i. e. $n \geqq 2$, $p \geqq 1$) the group of transformations of the form (12.12) in the variables $A$ and $\bar{A}$ and the parameters $u$ but we wish to establish this by connecting it in a formal manner with our preceding work.

$$(Z_\mu^\varkappa(1),\ Z_q^\rho(1))\,G = C_{\mu q\sigma}^{\varkappa\rho\tau}\,Z_\tau^\sigma(1)\,G, \tag{12.1}$$

where

$$C_{\mu q\sigma}^{\varkappa\rho\tau} = \delta_q^\varkappa\,\delta_\sigma^\rho\,\delta_\mu^\tau - \delta_\mu^\rho\,\delta_\sigma^\varkappa\,\delta_q^\tau. \tag{12.2}$$

Since the constants $C_{\mu q\sigma}^{\varkappa\rho\tau}$ defined by (12.2) satisfy the identities

$$\begin{aligned} C_{\mu q\sigma}^{\varkappa\rho\tau} + C_{q\mu\sigma}^{\rho\varkappa\tau} &= 0, \\ C_{\mu q\sigma}^{\varkappa\rho\tau}\,C_{\tau\beta\delta}^{\sigma\alpha\gamma} + C_{q\beta\sigma}^{\rho\alpha\tau}\,C_{\tau\mu\delta}^{\sigma\varkappa\gamma} + C_{\beta\mu\sigma}^{\alpha\varkappa\tau}\,C_{\tau q\delta}^{\sigma\rho\gamma} &= 0, \end{aligned} \tag{12.3}$$

it follows by Lie's third fundamental theorem* that they are the constants of composition for some $n^2$-parameter continuous group of transformations. From Theorem X the operators $Z_\mu^\varkappa(1)$ are linearly independent for $n \geqq 3$. This shows that

$$Z_\mu^\varkappa(1)\,G \qquad (\varkappa, \mu = 1, 2, \cdots, n) \qquad (n \geqq 3) \tag{12.3}$$

are the $n^2$ independent infinitesimal transformations of an $n^2$ parameter continuous group of transformations in the variables $A_{\alpha\beta\gamma}^i$ whose structural constants are the $C_{\mu q\sigma}^{\varkappa\rho\tau}$ defined in (12.2).

Consider the transformations in the $n^2$ parameters $u_\mu^\lambda$:

$$\overline{A}_{\alpha\beta\gamma}^\varkappa\,u_\varkappa^i = A_{pqr}^i\,u_\alpha^p\,u_\beta^q\,u_\gamma^r, \tag{12.4}$$

by which the variables $A$ are transformed into the variables $\overline{A}$, since we assume that the determinant

$$|u_\mu^\lambda| \neq 0. \tag{12.5}$$

We can write equations (12.4) in the form

$$\overline{A}_{\alpha\beta\gamma}^i = A_{pqr}^\varkappa\,u_\alpha^p\,u_\beta^q\,u_\gamma^r\,\overline{u}_\varkappa^i, \tag{12.6}$$

where $\overline{u}_\varkappa^i$ is the cofactor of the element $u_i^\varkappa$ in the determinant $|u_\mu^\lambda|$ divided by $|u_\mu^\lambda|$. It is clear that the transformations (12.6) form a group with inverse transformations with the $n^2$ essential parameters $u$ as the parameters of the group in the variables $A$. The identical transformation of this group corresponds to the values of the parameters

$$u_\mu^\lambda = \delta_\mu^\lambda. \tag{12.7}$$

* Lie & Scheffers, *Kontinuierliche Gruppen*, p. 396.

To form the infinitesimal transformations of the group (12.6) we put

$$\bar{u}^{\lambda}_{\mu} = \delta^{\lambda}_{\mu} + \varepsilon e^{\lambda}_{\mu} \tag{12.8}$$

where $\varepsilon$ is infinitesimal. We find that

$$\bar{u}^{i}_{x} = \delta^{i}_{x} - \varepsilon e^{i}_{x} \tag{12.9}$$

when we neglect second and higher powers of $\varepsilon$. Substituting (12.8) and (12.9) in (12.6) we see that within infinitesimals of the second order we have

$$\bar{A}^{i}_{\alpha\beta\gamma} - A^{i}_{\alpha\beta\gamma} = (e^{p}_{\alpha} A^{i}_{p\beta\gamma} + e^{q}_{\beta} A^{i}_{\alpha q\gamma} + e^{r}_{\gamma} A^{i}_{\alpha\beta r} - e^{i}_{x} A^{x}_{\alpha\beta\gamma})\,\varepsilon,$$

or

$$\bar{A}^{i}_{\alpha\beta\gamma} - A^{i}_{\alpha\beta\gamma} = \varepsilon e^{\sigma}_{\tau} \begin{Bmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{Bmatrix}. \tag{12.10}$$

Since the $e^{\sigma}_{\tau}$ are arbitrary constants, it follows that the symbols for the $n^2$ infinitesimal transformations are given by

$$Z^{\tau}_{\sigma}(1)\,G = \begin{Bmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{Bmatrix} \frac{\partial G}{\partial A^{i}_{\alpha\beta\gamma}}. \tag{12.11}$$

Consequently our invariant theory for affine differential invariants of order 1 is formally equivalent to the invariant theory of the $n^2$ parameter linear homogeneous continuous group (12.6). On following out the above methods one can easily see now that our invariant theory for affine differential invariants of order $p$ is formally equivalent to the invariant theory of the $n^2$ parameter continuous group

$$\begin{aligned} \bar{A}^{i}_{\alpha\beta\gamma} &= A^{x}_{pqr}\, u^{p}_{\alpha}\, u^{q}_{\beta}\, u^{r}_{\gamma}\, \bar{u}^{i}_{x} \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ \bar{A}^{i}_{\alpha\beta\gamma_1\cdots\gamma_p} &= A^{x}_{pqr_1\ldots r_p}\, u^{p}_{\alpha}\, u^{q}_{\beta}\, u^{r_1}_{\gamma_1} \cdots u^{r_p}_{\gamma_p}\, \bar{u}^{i}_{x}. \end{aligned} \tag{12.12}$$

The symbols of the $n^2$ independent infinitesimal transformations of the group (12.12) are given by (7.7), and the structural constants are given by (12.2). Up to this point in this paragraph we have restricted ourselves to the dimensional number $n \geqq 3$ and $p \geqq 1$, the order of the invariants. A similar discussion can evidently be made for the case $n = 2$, $p \geqq 2$. *The singular case* $n = 2$, $p = 1$ *will be treated in* § 14.

13. **Differential equations of relative tensor invariants.** Up to this point we have been dealing with the differential equations for the determination of absolute affine scalar differential invariants. We proceed

now to deduce the corresponding differential equations for the determination of relative scalar differential invariants and relative tensor differential invariants.

A relative scalar affine differential invariant satisfies the condition

$$f(\bar{A}^i_{\alpha_1\alpha_2\alpha_3}, \cdots, \bar{A}^i_{\alpha_1\alpha_2\cdots\alpha_{r+2}}) = (x\bar{x})^K f(A^i_{\alpha_1\alpha_2\alpha_3}, \cdots, A^i_{\alpha_1\alpha_2\cdots\alpha_{r+2}}) \tag{13.1}$$

where

$$(x\bar{x}) = \left|\frac{\partial x^i}{\partial \bar{x}^k}\right| \tag{13.2}$$

and $K$ is an arbitrarily given constant, called the weight of the scalar invariant $f(A)$. We can rewrite equation (13.1) in the form

$$f(\bar{A})\,(\bar{x}x)^K = f(A). \tag{13.3}$$

Using the methods of § 6 we can expand $f(\bar{A})\,(\bar{x}x)^K$ about $\varepsilon = 0$. This gives

$$f(\bar{A})\,(\bar{x}x)^K = f(A) + \varepsilon\left[\frac{d\{f(\bar{A})\,(\bar{x}x)^K\}}{d\varepsilon}\right]_{\varepsilon=0} + \cdots. \tag{13.4}$$

Hence as in § 6 a necessary and sufficient condition that (13.1) hold under the group of analytic transformations $\mathfrak{G}$ is that

$$\left[\frac{d[f(\bar{A})\,(\bar{x}x)^K]}{d\varepsilon}\right]_{\varepsilon=0} = 0. \tag{13.5}$$

By calculation we have

$$\left[\frac{d(\bar{x}x)}{d\varepsilon}\right]_{\varepsilon=0} = -\sum_{\alpha=1}^{n}\frac{\partial\,\xi^\alpha(x)}{\partial\,x^\alpha}. \tag{13.6}$$

Hence condition (13.5) yields the system of differential equations

$$\frac{\partial\,\xi^\mu}{\partial\,x^\varkappa}(Z^\varkappa_\mu(r) - K\delta^\varkappa_\mu)f = 0. \tag{13.7}$$

Finally since the $\xi$'s are arbitrary we have

$$Z^\varkappa_\mu(r)f = K\delta^\varkappa_\mu f \tag{13.8}$$

where

$$Z^\varkappa_\mu(r)f = \sum_{u=3}^{r+2}\begin{pmatrix} i\,\varkappa \\ \alpha_1\,\alpha_2\cdots\alpha_u\,\mu\end{pmatrix}\frac{\partial f}{\partial A^i_{\alpha_1\alpha_2\cdots\alpha_u}} \tag{13.9}$$

as defined by (7.7). Any relative scalar differential invariant $f(A)$ of weight $K$ and order $r$ satisfies equations (13.8) and conversely any expression

$$f(A^i_{\alpha_1\alpha_2\alpha_3};\cdots;A^i_{\alpha_1\alpha_2\alpha_3\cdots\alpha_{r+2}})$$

which satisfies (13.8) is a relative scalar differential invariant of order $r$ and weight $K$.

More generally we can derive the differential equations for the determination of relative tensor differential invariants. The invariance condition is

$$(13.10)\quad T^{ij\cdots s}_{\alpha\beta\cdots\lambda}(\bar{A})\frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\cdots\frac{\partial x^t}{\partial \bar{x}^s}=(x\bar{x})^K T^{pq\cdots t}_{\mu\nu\cdots\varphi}(A)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\cdots\frac{\partial x^\varphi}{\partial \bar{x}^\lambda}$$

or

$$(13.11)\quad (\bar{x}x)^K T^{ij\cdots s}_{\alpha\beta\cdots\lambda}(\bar{A})\frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\cdots\frac{\partial x^t}{\partial \bar{x}^s}\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu}\cdots\frac{\partial \bar{x}^\lambda}{\partial x^\varphi}=T^{pq\cdots t}_{\mu\nu\cdots\varphi}(A).$$

Expanding the left member of (13.11) about $\varepsilon=0$ we obtain

$$(13.12)\quad T^{pq\cdots t}_{\mu\nu\cdots\varphi}(A)+\varepsilon\left\{\frac{d}{d\varepsilon}\left[(\bar{x}x)^K T^{ij\cdots s}_{\alpha\beta\cdots\lambda}(\bar{A})\frac{\partial x^p}{\partial \bar{x}^i}\cdots\frac{\partial \bar{x}^\lambda}{\partial x^\varphi}\right]\right\}_{\varepsilon=0}+\cdots+\cdots.$$

Hence as before necessary and sufficient conditions that (13.10) hold are given by

$$(13.13)\qquad \left\{\frac{d}{d\varepsilon}\left[(\bar{x}x)^K T^{ij\cdots s}_{\alpha\beta\cdots\lambda}(\bar{A})\frac{\partial x^p}{\partial \bar{x}^i}\cdots\frac{\partial \bar{x}^\lambda}{\partial x^\varphi}\right]\right\}_{\varepsilon=0}=0.$$

Using the equations

$$(13.14)\quad \left[\frac{d}{d\varepsilon}\frac{\partial x^i}{\partial \bar{x}^j}\right]_{\varepsilon=0}=\frac{\partial \xi^i(x)}{\partial x^j},\qquad \left[\frac{d}{d\varepsilon}\frac{\partial \bar{x}^i}{\partial x^j}\right]_{\varepsilon=0}=-\frac{\partial \xi^i(x)}{\partial x^j},$$

$$\left[\frac{d\bar{A}^i_{\alpha\beta\gamma\cdots\zeta}}{d\varepsilon}\right]_{\varepsilon=0}=\binom{i\tau}{\alpha\beta\gamma\cdots\zeta\sigma}\frac{\partial \xi^\sigma}{\partial x^\tau},\qquad \left[\frac{d(\bar{x}x)^K}{d\varepsilon}\right]_{\varepsilon=0}=-K\sum_{\alpha=1}^{n}\frac{\partial \xi^\alpha(x)}{\partial x^\alpha},$$

which we have from (5.4), (5.6), (6.13) and (13.6) the equations (13.13) become

$$(13.15)\quad \frac{\partial \xi^\sigma}{\partial x^\tau}\Big[Z^\tau_\sigma(r)\,T^{pq\cdots t}_{\mu\nu\cdots\varphi}(A)-K\delta^\tau_\sigma T^{pq\cdots t}_{\mu\nu\cdots\varphi}+\delta^p_\sigma T^{\tau q\cdots t}_{\mu\nu\cdots\varphi}+\cdots+\delta^t_\sigma T^{pq\cdots s\tau}_{\mu\nu\cdots\varphi}$$
$$-\delta^\tau_\mu T^{pq\cdots t}_{\sigma\nu\cdots\varphi}-\cdots-\delta^\tau_\varphi T^{pq\cdots t}_{\mu\nu\cdots\sigma}\Big]=0.$$

Hence from the arbitrariness of the $\xi$'s we obtain

$$(13.16)\quad Z^\tau_\sigma(r)\,T^{pq\cdots t}_{\mu\nu\cdots\varphi}=K\delta^\tau_\sigma T^{pq\cdots t}_{\mu\nu\cdots\varphi}+\delta^\tau_\mu T^{pq\cdots t}_{\sigma\nu\cdots\varphi}+\delta^\tau_\nu T^{pq\cdots t}_{\mu\sigma\cdots\varphi}$$
$$+\cdots+\delta^\tau_\varphi T^{pq\cdots t}_{\mu\nu\cdots\sigma}-\delta^p_\sigma T^{\tau q\cdots t}_{\mu\nu\cdots\varphi}-\delta^q_\sigma T^{p\tau\cdots t}_{\mu\nu\cdots\varphi}-\cdots-\delta^t_\sigma T^{pq\cdots\tau}_{\mu\nu\cdots\varphi}.$$

We have therefore demonstrated the following theorem.

THEOREM XIII. *Necessary and sufficient conditions that the quantities*

$$T^{pq\cdots t}_{\mu\nu\cdots q}(A)$$

*be the components of a relative tensor affine differential invariant of order r and weight K are that they satisfy the system of linear partial differential equations* (13.16).

14. **Examples of invariants.** Let us consider the problem of the determination of all absolute scalar affine differential invariants $G$ in the two dimensional manifold which depend on the $\Gamma$'s and their first derivatives. In other words we wish to determine all functions $G(\Gamma; \partial\Gamma/\partial x)$ which satisfy the equations

$$\binom{i\tau}{\alpha\beta\gamma\sigma}\frac{\partial G}{\partial A^i_{\alpha\beta\gamma}} = 0, \quad n = 2. \tag{14.1}$$

From (4.2) the number $N(2, 1)$ of algebraically independent components of the normal tensor $A^i_{\alpha\beta\gamma}$ is equal to 4. Let us select

$$\alpha = A^1_{211}, \quad \beta = A^2_{122}, \quad \gamma = A^1_{122}, \quad \delta = A^2_{211} \tag{14.2}$$

as the four independent $A$'s. Then

$$A^1_{112} = -2\alpha, \quad A^2_{221} = -2\beta, \quad A^1_{221} = -2\gamma, \quad A^2_{112} = -2\delta \tag{14.3}$$

and all other $A$'s not equal to those in (14.2) by (3.2) are zero. We may now consider $G$ and the quantities $\binom{i\tau}{\alpha\beta\gamma\sigma}$ in (14.1) as functions of the variables $(\alpha, \beta, \gamma, \delta)$ alone. When this has been done the equations (14.1) become

$$\begin{aligned}
&\alpha\frac{\partial G}{\partial \alpha} + \beta\frac{\partial G}{\partial \beta} + \Big(\qquad\Big) + 2\delta\frac{\partial G}{\partial \delta} = 0,\\
&\delta\frac{\partial G}{\partial \alpha} + \delta\frac{\partial G}{\partial \beta} + (\alpha+\beta)\frac{\partial G}{\partial \gamma} + \Big(\qquad\Big) = 0,\\
&\gamma\frac{\partial G}{\partial \alpha} + \gamma\frac{\partial G}{\partial \beta} + \Big(\qquad\Big) + (\alpha+\beta)\frac{\partial G}{\partial \delta} = 0,\\
&\alpha\frac{\partial G}{\partial \alpha} + \beta\frac{\partial G}{\partial \beta} + 2\gamma\frac{\partial G}{\partial \gamma} + \Big(\qquad\Big) = 0.
\end{aligned} \tag{14.4}$$

From Theorem VI it follows that the above differential equations form a complete system. The determinant, however, of equations (14.4) is found to vanish identically so that the equations are not all independent; but,

the first, second and fourth equations are independent. Hence from the theory of complete systems of linear partial differential equations we know that there exists a solution $G$ of equations (14.4) such that any function of $G$ is a solution of (14.4) and any solution of (14.4) is a function of $G$. Solving the equations (14.4) we find that this solution has the form

$$G(\alpha, \beta, \gamma, \delta) = \frac{(\alpha - \beta)^2}{\alpha\beta - \gamma\delta}. \tag{14.5}$$

Thus the invariant $G(\alpha, \beta, \gamma, \delta)$ is a fundamental set of absolute scalar invariants of order 1 for the two dimensional manifold. The invariant property of this scalar $G(\alpha, \beta, \gamma, \delta)$ does not appear to be evident on inspecting its form. There is interest in verifying its invariant character directly from the transformation law of the components $(\alpha, \beta, \gamma, \delta)$ in (14.5). We wish to show directly that

$$\frac{(\bar{\alpha} - \bar{\beta})^2}{\bar{\alpha}\bar{\beta} - \bar{\gamma}\bar{\delta}} = \frac{(\alpha - \beta)^2}{\alpha\beta - \gamma\delta}. \tag{14.6}$$

The equations relating the quantities $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$ with the components $A^i_{\alpha\beta\gamma}$ are

$$\begin{aligned}
\bar{\alpha} &= A^x_{\alpha\beta\gamma}\, \bar{a}^1_x\, a^\alpha_2\, a^\beta_1\, a^\gamma_1, \\
\bar{\beta} &= A^x_{\alpha\beta\gamma}\, \bar{a}^2_x\, a^\alpha_1\, a^\beta_2\, a^\gamma_2, \\
\bar{\gamma} &= A^x_{\alpha\beta\gamma}\, \bar{a}^1_x\, a^\alpha_1\, a^\beta_2\, a^\gamma_2, \\
\bar{\delta} &= A^x_{\alpha\beta\gamma}\, \bar{a}^2_x\, a^\alpha_2\, a^\beta_1\, a^\gamma_1,
\end{aligned} \tag{14.7}$$

where

$$a^i_j = \partial x^i / \partial \bar{x}^j, \qquad \bar{a}^l_k = \partial \bar{x}^l / \partial x^k.$$

Writing the components $A^x_{\alpha\beta\gamma}$ in the right hand members of the first two equations (14.7) in terms of the quantities $(\alpha, \beta, \gamma, \delta)$ we obtain after some calculation

$$\begin{aligned}
\bar{\alpha} - \bar{\beta} = \Delta[&\alpha(\bar{a}^1_1\, a^1_1 + \bar{a}^2_1\, a^1_2) - \beta(\bar{a}^1_2\, a^2_1 + \bar{a}^2_2\, a^2_2) \\
&+ \delta(\bar{a}^1_2\, a^1_1 + \bar{a}^2_2\, a^1_2) - \gamma(\bar{a}^1_1\, a^2_1 + \bar{a}^2_1\, a^2_2)]
\end{aligned} \tag{14.8}$$

where

$$\Delta = \begin{vmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{vmatrix}.$$

But

$$\begin{aligned}
\bar{a}^1_1\, a^1_1 + \bar{a}^2_1\, a^1_2 &= 1, \qquad \bar{a}^1_2\, a^2_1 + \bar{a}^2_2\, a^2_2 = 1, \\
\bar{a}^1_2\, a^1_1 + \bar{a}^2_2\, a^1_2 &= 0, \qquad \bar{a}^1_1\, a^2_1 + \bar{a}^2_1\, a^2_2 = 0.
\end{aligned} \tag{14.9}$$

Hence

$$\bar{\alpha} - \bar{\beta} = \Delta(\alpha - \beta). \tag{14.10}$$

In a similar manner we find first that

$$\bar{\alpha}\bar{\beta} - \bar{\gamma}\bar{\delta} = \Delta^3 [(\alpha \bar{a}_1^1 + \delta \bar{a}_2^1)(\gamma \bar{a}_1^2 + \beta \bar{a}_2^2) - (\alpha \bar{a}_1^2 + \delta \bar{a}_2^2)(\gamma \bar{a}_1^1 + \beta \bar{a}_2^1)],$$

which upon simplification becomes

$$\bar{\alpha}\bar{\beta} - \bar{\gamma}\bar{\delta} = \Delta^3 (\bar{a}_1^1 \bar{a}_2^2 - \bar{a}_2^1 \bar{a}_1^2)(\alpha\beta - \gamma\delta).$$

Finally, noting that $\bar{a}_1^1 \bar{a}_2^2 - \bar{a}_2^1 \bar{a}_1^2 = 1/\Delta$, we have

$$\bar{\alpha}\bar{\beta} - \bar{\gamma}\bar{\delta} = \Delta^2(\alpha\beta - \gamma\delta). \tag{14.11}$$

The relation (14.6) then follows from (14.10) and (14.11).

Incidentally we have shown that $(\alpha - \beta)$ and $(\alpha\beta - \gamma\delta)$ are relative scalar affine differential invariants of weight 1 and 2 respectively. Hence the absolute invariant $G(\alpha, \beta, \gamma, \delta)$ is formed by means of the ratio of two relative invariants each of which is of weight 2.

In case the geometry of paths reduces to the Riemannian geometry, the absolute scalar invariant $G(\alpha, \beta, \gamma, \delta)$ vanishes identically. This statement can be verified by observing that in the Riemannian geometry the contracted curvature tensor $B^i_{ikl}$ is identically zero. By calculation we have in the case of the two dimensional Riemannian geometry that

$$B^i_{ikl} = A^i_{ikl} - A^i_{ilk} = 2(\beta - \alpha). \tag{14.12}$$

Hence in the Riemannian geometry

$$\alpha - \beta \equiv 0, \qquad n = 2. \tag{14.13}$$

By a further calculation it can be verified that $\alpha\beta - \gamma\delta$ does not vanish identically in the Riemannian geometry of two dimensions. Hence

$$G(\alpha, \beta, \gamma, \delta) \equiv 0, \qquad n = 2$$

in the Riemannian geometry.

In § 13 we saw that our invariant theories were formally equivalent to certain $n^2$ parameter continuous groups of transformations, where $n$ is the dimensionality of the affinely connected manifold. We showed this under the assumption that we did not have the combination $n = 2, p = 1$ where $p$ denotes the order of the differential invariant. We shall now show that even for the singular case $n = 2$, $p = 1$ we are concerned with the invariant theory of a continuous group of transformations.

We have shown that equations (14.4) are dependent. However, the only *constant* quantities $K_1$, $K_2$, $K_3$, $K_4$ which can satisfy the equations

$$
\begin{aligned}
(K_1 + K_4)\,\alpha + K_2\,\delta + K_3\,\gamma &= 0 \\
(K_1 + K_4)\,\beta + K_2\,\delta + K_3\,\gamma &= 0 \\
K_2(\alpha + \beta) + 2\,K_4\,\gamma &= 0 \\
2\,K_1\,\delta + K_3(\alpha + \beta) &= 0
\end{aligned}
\tag{14.14}
$$

identically have the values

$$K_1 = K_2 = K_3 = K_4 = 0.$$

Hence the infinitesimal operators

$$
\begin{aligned}
Z_1^1(1)\,G &= \alpha\frac{\partial G}{\partial \alpha} + \beta\frac{\partial G}{\partial \beta} + 2\,\delta\frac{\partial G}{\partial \delta}, \\
Z_1^2(1)\,G &= \delta\frac{\partial G}{\partial \alpha} + \delta\frac{\partial G}{\partial \beta} + (\alpha + \beta)\frac{\partial G}{\partial \gamma}, \\
Z_2^1(1)\,G &= \gamma\frac{\partial G}{\partial \alpha} + \gamma\frac{\partial G}{\partial \beta} + (\alpha + \beta)\frac{\partial G}{\partial \delta}, \\
Z_2^2(1)\,G &= \alpha\frac{\partial G}{\partial \alpha} + \beta\frac{\partial G}{\partial \beta} + 2\,\gamma\frac{\partial G}{\partial \gamma}
\end{aligned}
\tag{14.12}
$$

are the symbols for the four independent infinitesimal transformations of a four parameter continuous group whose structure is determined by the constants (12.2) for $\varkappa, p, \tau, \mu, q, \sigma = 1, 2$. On following out the methods of § 12 it is clear now that (14.15) gives the infinitesimal transformations of the finite transformations (14.7) when the right hand members of these equations are expressed in terms of the independent variables $(\alpha, \beta, \gamma, \delta)$ and the four essential parameters $(a_1^1,\ a_2^1,\ a_1^2,\ a_2^2)$.

The treatment of the relative scalar invariant $J$ of the first order for $n = 2$ requires only a slight modification of the preceding work. For this case the differential equations are

$$\binom{i\tau}{\alpha\beta\gamma\sigma}\frac{\partial J}{\partial A^i_{\alpha\beta\gamma}} = \delta^\tau_\sigma K J, \qquad n = 2, \tag{14.13}$$

which are obtained from (13.8) for $r = 1$. Introducing the independent components $(\alpha, \beta, \gamma, \delta)$ as before we have from (14.13) the following independent equations

$$
\begin{aligned}
\alpha\frac{\partial \psi}{\partial \alpha} + \beta\frac{\partial \psi}{\partial \beta} + \Big(\qquad\Big) + 2\,\delta\frac{\partial \psi}{\partial \delta} &= 1, \\
\delta\frac{\partial \psi}{\partial \alpha} + \delta\frac{\partial \psi}{\partial \beta} + (\alpha + \beta)\frac{\partial \psi}{\partial \gamma} + \Big(\qquad\Big) &= 0, \\
\alpha\frac{\partial \psi}{\partial \alpha} + \beta\frac{\partial \psi}{\partial \beta} + 2\,\gamma\frac{\partial \psi}{\partial \gamma} \quad + \Big(\qquad\Big) &= 1,
\end{aligned}
\tag{14.14}
$$

where

$$\psi = \frac{\log J}{K}.$$

A particular solution of the system (14.14) is $\log(\alpha-\beta)$. Since the most general solution of the homogeneous system associated with (14.14) is an arbitrary function of the expression $(\alpha-\beta)^2/(\alpha\beta-\gamma\delta)$, the most general solution of the complete system (14.14) is

$$\psi = \log(\alpha-\beta) + F\left[\frac{(\alpha-\beta)^2}{\alpha\beta-\gamma\delta}\right],$$

where $F$ denotes an arbitrary function of its argument. Hence the general relative scalar invariant of order 1 and weight $K$ for $n = 2$ has the form

$$J = (\alpha-\beta)^K F\left[\frac{(\alpha-\beta)^2}{\alpha\beta-\gamma\delta}\right]. \tag{14.15}$$

As particular cases of (14.15) we see that $(\alpha-\beta)$ is an invariant of weight 1, and also that $(\alpha-\beta)^2$ and $(\alpha\beta-\gamma\delta)$ are invariants of weight 2; these results have been shown in the first part of this paragraph by actual calculation.

The skew-symmetric tensor invariant $H_{ij}$ of order 1 for $n = 2$ contains only one non-vanishing component $H_{12}$. Reference to the equations (13.16) shows that the differential equations for the determination of $H_{ij}$ are identical with (14.14) when

$$\psi = \log H_{12}.$$

Hence the most general invariant $H_{ij}$ is given by

$$H_{12} = -H_{21} = (\alpha-\beta)F\left[\frac{(\alpha-\beta)^2}{\alpha\beta-\gamma\delta}\right]; \quad H_{11} = H_{22} = 0.$$

**15. Differential equations of invariants involving arbitrary functions.** The notion of an affine differential invariant can be extended to those cases in which a set of arbitrary scalar functions

$$F^{(k)}(x^1, \cdots, x^n) \qquad (k = 1, 2, \cdots, w) \tag{15.1}$$

and their derivatives occur in the invariants in addition to the functions $\Gamma^i_{\alpha\beta}$ and the derivatives of $\Gamma^i_{\alpha\beta}$. The scalar character of the $F$'s is expressed by

$$F^{(k)}(x) = \bar{F}^{(k)}(\bar{x}). \tag{15.2}$$

Consider, for example, any particular function

$$(15.3)\quad G\left(\Gamma^i_{\alpha\beta};\ \frac{\partial\,\Gamma^i_{\alpha\beta}}{\partial\,x^\gamma};\ \cdots;\ \frac{\partial^r\,\Gamma^i_{\alpha\beta}}{\partial\,x^\gamma\cdots\partial\,x^\sigma};\ F^{(k)};\ \frac{\partial\,F^{(k)}}{\partial\,x^\lambda};\ \cdots;\ \frac{\partial^s\,F^{(k)}}{\partial\,x^\lambda\cdots\partial\,x^\mu}\right)$$

with reference to the system of coördinates $(x^1, \cdots, x^n)$. The function $G$ will be a scalar differential invariant if

$$(15.4)\quad G\left(\Gamma^i_{\alpha\beta};\ \cdots;\ \frac{\partial^s\,F^{(k)}}{\partial\,x^\lambda\cdots\partial\,x^\mu}\right) = G\left(\bar\Gamma^i_{\alpha\beta};\ \cdots;\ \frac{\partial^s\,\bar F^{(k)}}{\partial\,\bar x^\lambda\cdots\partial\,\bar x^\mu}\right)$$

under transformations $\mathfrak{G}$. The order of the invariant $G$ will be defined in the following manner.

*Definition.* An absolute affine scalar differential invariant $G$ of the form (15.3) will be said to be of order $(r, s)$ if the highest derivatives of the $\Gamma^i_{\alpha\beta}$ and the $F^{(k)}$ are of order $r$ and $s$ respectively.

By the replacement Theorem II any invariant of the form (15.3) can be written as

$$(15.5)\quad G(A^i_{\alpha\beta\gamma};\ \cdots;\ A^i_{\alpha\beta\gamma\cdots\delta};\ F^{(k)};\ F^{(k)}_\lambda;\ \cdots;\ F^{(k)}_{\lambda\cdots\mu})$$

where the quantities $F^{(k)}_\lambda, \cdots, F^{(k)}_{\lambda\cdots\mu}$ are the (covariant tensor) extensions of the functions $F^{(k)}$.

It is evident that all differential invariants $G$ of the form (15.3) are algebraic invariants of the group of transformations

$$\begin{aligned}
\bar A^\sigma_{\alpha\beta\gamma_1\cdots\gamma_p}\,u^i_\sigma &= A^i_{\mu\nu\sigma_1\cdots\sigma_p}\,u^\mu_\alpha\,u^\nu_\beta\,u^{\sigma_1}_{\gamma_1}\cdots u^{\sigma_p}_{\gamma_p} && (p = 1, 2, \cdots, r),\\
\bar F^{(k)}_{\lambda_1\cdots\lambda_p} &= F^{(k)}_{\mu_1\cdots\mu_p}\,u^{\mu_1}_{\lambda_1}\cdots u^{\mu_p}_{\lambda_p} && (p = 1, 2, \cdots, s),\\
\bar F^{(k)} &= F^{(k)}
\end{aligned}$$

in the variables $A^i_{\mu\sigma_1\cdots\sigma_p}$, $F^{(k)}_{\mu_1\cdots\mu_p}$, $F^{(k)}$ and the parameters $u^i_\alpha$ where $k$ takes on all values $(1, 2, \cdots, w)$.

By obvious extensions of the methods of § 5 and § 6 it is clear that a necessary and sufficient condition for a function $G$ of the form (15.5) to be an absolute affine scalar differential invariant is that

$$(15.6)\quad \left[\frac{d\,G(\bar A, \bar F)}{d\,\varepsilon}\right]_{\varepsilon=0} = 0$$

under the deformations (6.2) for all functions $\xi$. To write down the system of partial differential equations from (15.6) which are satisfied by the invariants $G$ we note that

$$\left[\frac{d\,\bar F^{(k)}}{d\,\varepsilon}\right]_{\varepsilon=0} = 0 \qquad (k = 1, 2, \cdots, w);$$

also that

$$\left[\frac{d\,\overline{F}^{(k)}_{\alpha\beta\ldots\nu}}{d\,\varepsilon}\right]_{\varepsilon=0} = (\delta^\tau_\alpha F^{(k)}_{\sigma\beta\ldots\nu} + \delta^\tau_\beta F^{(k)}_{\alpha\sigma\ldots\nu} + \cdots + \delta^\tau_\nu F^{(k)}_{\alpha\beta\ldots\sigma})\frac{\partial \xi^\sigma}{\partial x^\tau}$$

which follows from (5.6). Then introducing the symbol

$$\begin{bmatrix} \tau\,(k) \\ \sigma\alpha\beta\ldots\nu \end{bmatrix} = \delta^\tau_\alpha F^{(k)}_{\sigma\beta\ldots\nu} + \delta^\tau_\beta F^{(k)}_{\alpha\sigma\ldots\nu} + \cdots + \delta^\tau_\nu F^{(k)}_{\alpha\beta\ldots\sigma}$$

we have that the equations

$$[(Z^\tau_\sigma(r) + Y^\tau_\sigma(s))\,G]\frac{\partial \xi^\sigma}{\partial x^\tau} = 0 \tag{15.7}$$

are satisfied by the invariant $G$ for arbitrary functions $\xi$. Here

$$Z^\tau_\sigma(r)\,G = \sum_{u=3}^{r+2} \begin{pmatrix} i\tau \\ \alpha_1\ \alpha_2\ \cdots\ \alpha_u \end{pmatrix} \frac{\partial G}{\partial A^i_{\alpha_1\alpha_2\cdots\alpha_u}}$$

and

$$Y^\tau_\sigma(s)\,G = \sum_{u=1}^{s} \begin{bmatrix} \tau\,(k) \\ \sigma\ \alpha_1\ \alpha_2\ \cdots\ \alpha_u \end{bmatrix} \frac{\partial G}{\partial F^{(k)}_{\alpha_1\alpha_2\cdots\alpha_u}}$$

where the index $k$ is summed over the values $(1, 2, \cdots, w)$ and $\alpha_j$ over the values $(1, 2, \cdots, n)$. Hence we have the theorem.

THEOREM XIV. *A necessary and sufficient condition that a function $G$ of the form* (15.5) *be an absolute affine scalar differential invariant of order $(r, s)$ is that it satisfy the system of $n^2$ linear partial differential equations of the first order:*

$$(Z^\tau_\sigma(r) + Y^\tau_\sigma(s))\,G = 0. \tag{15.8}$$

**16. Completeness and independence of the differential equations.** We shall proceed to prove the following theorem in a direct manner.

THEOREM XV. *The system of $n^2$ differential equations* (15.8) *is a complete system.*

By calculation we have

$$(Z^\varkappa_\mu(r) + Y^\varkappa_\mu(s),\ Z^p_q(r) + Y^p_q(s))\,G = (Z^\varkappa_\mu(r),\ Z^p_q(r))G + (Y^\varkappa_\mu(s),\ Y^p_q(s))\,G. \tag{16.1}$$

Furthermore the relations (7.9) give

$$(Z^\varkappa_\mu(r).\ Z^p_q(r))\,G = \delta^\varkappa_q\, Z^p_\mu(r)\,G - \delta^p_\mu\, Z^\varkappa_q(r)\,G. \tag{16.2}$$

Now

$$(Y^x_\mu(s),\ Y^p_q(s))\,G$$

$$= \sum_{u=1}^{s}\sum_{v=1}^{s}\left\{\begin{bmatrix} x\,(k) \\ \mu\ \alpha_1\ \alpha_2\ \cdots\ \alpha_u \end{bmatrix}\frac{\partial\begin{bmatrix} p\,(t) \\ q\ \beta_1\ \beta_2\ \cdots\ \beta_v \end{bmatrix}}{\partial F^{(k)}_{\alpha_1\alpha_2\cdots\alpha_u}} - \begin{bmatrix} p\,(k) \\ q\ \alpha_1\ \alpha_2\ \cdots\ \alpha_u \end{bmatrix}\frac{\partial\begin{bmatrix} x\,(t) \\ \mu\ \beta_1\ \beta_2\ \cdots\ \beta_v \end{bmatrix}}{\partial F^{(k)}_{\alpha_1\alpha_2\cdots\alpha_u}}\right\}\frac{\partial G}{\partial F^{(t)}_{\beta_1\beta_2\cdots\beta_v}}$$

$$= \sum_{u=1}^{s}\left\{\begin{bmatrix} x\,(k) \\ \mu\ \alpha_1\ \alpha_2\ \cdots\ \alpha_u \end{bmatrix}\frac{\partial\begin{bmatrix} p\,(t) \\ q\ \beta_1\ \beta_2\ \cdots\ \beta_u \end{bmatrix}}{\partial F^{(k)}_{\alpha_1\alpha_2\cdots\alpha_u}} - \begin{bmatrix} p\,(k) \\ q\ \alpha_1\ \alpha_2\ \cdots\ \alpha_u \end{bmatrix}\frac{\partial\begin{bmatrix} x\,(t) \\ \mu\ \beta_1\ \beta_2\ \cdots\ \beta_u \end{bmatrix}}{\partial F^{(k)}_{\alpha_1\alpha_2\cdots\alpha_u}}\right\}\frac{\partial G}{\partial F^{(t)}_{\beta_1\beta_2\cdots\beta_u}},$$

where the indices $k$ and $t$ are summed over the values $(1, 2, \cdots, w)$. First consider the special case in which $s = 2$. We have

$$\begin{bmatrix} p\,(t) \\ q\ \beta_1 \end{bmatrix} = \delta^p_{\beta_1} F^{(t)}_q, \qquad \begin{bmatrix} p\,(t) \\ q\ \beta_1\ \beta_2 \end{bmatrix} = \delta^p_{\beta_1} F^{(t)}_{q\beta_2} + \delta^p_{\beta_2} F^{(t)}_{\beta_1 q}. \tag{16.4}$$

Hence

$$\frac{\partial\begin{bmatrix} p\,(t) \\ q\ \beta_1 \end{bmatrix}}{\partial F^{(k)}_{\alpha_1}} = \delta^t_k\,\delta^{\alpha_1}_q\,\delta^p_{\beta_1},$$

$$\frac{\partial\begin{bmatrix} p\,(t) \\ q\ \beta_1\ \beta_2 \end{bmatrix}}{\partial F^{(k)}_{\alpha_1\alpha_2}} = \delta^t_k\,\delta^{\alpha_1}_q\,\delta^{\alpha_2}_{\beta_2}\,\delta^p_{\beta_1} + \delta^t_k\,\delta^{\alpha_1}_{\beta_1}\,\delta^{\alpha_2}_q\,\delta^p_{\beta_2}, \tag{16.5}$$

In virtue of (16.3), (16.4), and (16.5) we obtain

$$\begin{aligned}(Y^x_\mu(2),\ Y^p_q(2))\,G = {} & \{F^{(k)}_\mu\,\delta^x_{\alpha_1}\,\delta^t_k\,\delta^{\alpha_1}_q\,\delta^p_{\beta_1} - F^{(k)}_q\,\delta^p_{\alpha_1}\,\delta^t_k\,\delta^{\alpha_1}_\mu\,\delta^x_{\beta_1}\}\frac{\partial G}{\partial F^{(t)}_{\beta_1}} \\ & + \{(F^{(k)}_{\mu\alpha_2}\,\delta^x_{\alpha_1} + F^{(k)}_{\alpha_1\mu}\,\delta^x_{\alpha_2})(\delta^t_k\,\delta^{\alpha_1}_q\,\delta^{\alpha_2}_{\beta_2}\,\delta^p_{\beta_1} + \delta^t_k\,\delta^{\alpha_1}_{\beta_1}\,\delta^{\alpha_2}_q\,\delta^p_{\beta_2}) \\ & - (F^{(k)}_{q\alpha_2}\,\delta^p_{\alpha_1} + F^{(k)}_{\alpha_1 q}\,\delta^p_{\alpha_2})(\delta^t_k\,\delta^{\alpha_1}_\mu\,\delta^{\alpha_2}_{\beta_2}\,\delta^x_{\beta_1} + \delta^t_k\,\delta^{\alpha_1}_{\beta_1}\,\delta^{\alpha_2}_\mu\,\delta^x_{\beta_2})\}\frac{\partial G}{\partial F^{(t)}_{\beta_1\beta_2}}.\end{aligned} \tag{11.6}$$

Finally on simplifying and collecting terms

$$(Y^x_\mu(2),\ Y^p_q(2))\,G = \delta^x_q\,Y^p_\mu(2)\,G - \delta^p_\mu\,Y^x_q(2)\,G. \tag{16.7}$$

By calculations analogous to the above one can show readily that

$$(16.8)\qquad (Y^{\varkappa}_{\mu}(s),\ Y^{p}_{q}(s))\,G = \delta^{\varkappa}_{q}\,Y^{p}_{\mu}(s)\,G - \delta^{p}_{\mu}\,Y^{\varkappa}_{q}(s)\,G.$$

Hence by a reference to relations (16.1), (16.2), (16.8) we see that

$$(16.9)\qquad \begin{aligned}&(Z^{\varkappa}_{\mu}(r)+Y^{\varkappa}_{\mu}(s),\ Z^{p}_{q}(r)+Y^{p}_{q}(s))\,G\\ &= \delta^{\varkappa}_{q}\,(Z^{p}_{\mu}(r)+Y^{p}_{\mu}(s))\,G - \delta^{p}_{\mu}\,(Z^{\varkappa}_{q}(r)+Y^{\varkappa}_{q}(s))\,G.\end{aligned}$$

This proves our theorem.

As a consequence of Theorems X and XI we have the following two statements:

(1) If $n > 2$ there are $n^2$ independent equations (15.8) for the determination of the scalar invariants $G$ which contain first or higher derivatives of the $\Gamma^{i}_{\alpha\beta}$.

(2) If $n = 2$ there are $n^2$ independent equations (15.8) for the determination of the scalar invariants $G$ which contain second or higher derivatives of the $\Gamma^{i}_{\alpha\beta}$.

If no derivatives of the $\Gamma^{i}_{\alpha\beta}$ occur in $G$, then $G$ is independent of the $\Gamma$'s themselves as well as their derivatives. In this case $G$ has no connection with the affine manifold and need not be considered.

The following theorem is necessary to our discussion.

THEOREM XVI. *The differential equations for the determination of the absolute scalar differential invariants of order* (1,1) *for the general affinely connected manifold of two dimensions are independent.*

The differential equations in question are

$$(16.10)\qquad \begin{pmatrix} i\,\tau \\ \alpha\,\beta\,\gamma\,\sigma \end{pmatrix} \frac{\partial G}{\partial A^{i}_{\alpha\beta\gamma}} + \sum_{k=1}^{w} \begin{bmatrix} \tau(k) \\ \sigma\,\alpha \end{bmatrix} \frac{\partial G}{\partial F^{(k)}_{\alpha}} = 0, \quad n = 2.$$

Using the notation of (14.2) and (14.3) for the independent components of the normal tensor $A^{i}_{\alpha\beta\gamma}$ and letting

$$(16.11)\qquad F^{(1)}_{1} = \varepsilon, \qquad F^{(1)}_{2} = \zeta,$$

equations (16.10) take the following form

$$(16.12a)\qquad \begin{aligned}&\alpha\frac{\partial G}{\partial \alpha}+\beta\frac{\partial G}{\partial \beta}+\Big(\qquad\Big)+2\delta\frac{\partial G}{\partial \delta}\qquad+\varepsilon\frac{\partial G}{\partial \varepsilon}+\Big(\quad\Big)=0,\\ &\delta\frac{\partial G}{\partial \alpha}+\delta\frac{\partial G}{\partial \beta}+(\alpha+\beta)\frac{\partial G}{\partial \gamma}+\Big(\qquad\Big)+\Big(\quad\Big)+\varepsilon\frac{\partial G}{\partial \zeta}=0,\end{aligned}$$

$$(16.12b)\quad \begin{aligned} &\gamma\frac{\partial G}{\partial \alpha}+\gamma\frac{\partial G}{\partial \beta}+\Big(\qquad\Big)+(\alpha+\beta)\frac{\partial G}{\partial \delta}+\zeta\frac{\partial G}{\partial \varepsilon}+\Big(\quad\Big)=0,\\ &\alpha\frac{\partial G}{\partial \alpha}+\beta\frac{\partial G}{\partial \beta}+2\gamma\frac{\partial G}{\partial \gamma}\quad+\Big(\qquad\Big)+\Big(\quad\Big)+\zeta\frac{\partial G}{\partial \zeta}=0, \end{aligned}$$

for $k=1$. The determinant of the matrix of this system

$$\begin{vmatrix} 0 & 2\delta & \varepsilon & 0\\ (\alpha+\beta) & 0 & 0 & \varepsilon\\ 0 & (\alpha+\beta) & \zeta & 0\\ 2\gamma & 0 & 0 & \zeta \end{vmatrix} \neq 0.$$

Hence the equations (16.12) are all independent. As an immediate consequence of this we have that the equations (16.10) are independent for $w>1$. This completes the proof of our theorem.

The only singular case arises in the determination of the absolute affine scalar differential invariants of order (1,0) for $n=2$. The form of the system of equations is the same as that of the equations (14.4). Hence there are three independent equations.

17. **Fundamental sets of invariants involving arbitrary functions.** To determine the number of independent variables in equations (15.8) we observe that the number* of independent components of the normal tensor $A^i_{\alpha\beta\gamma_1\gamma_2\cdots\gamma_\nu}$ is given by the number $N(n,\nu)$. If in equations (15.8) there are $w$ functions then the number of independent components of the $w$ covariant tensors $F^{(k)}_{\alpha_1\alpha_2\cdots\alpha_\mu}$ is given by the number $wK(n,\mu)$ where

$$(17.1)\qquad K(n,\mu)=\frac{n(n+1)\cdots(n+\mu-1)}{\mu!},\qquad K(n,0)=1.$$

Hence the total number of independent variables in the differential equations (15.8) is

$$(17.2)\qquad D(r,s,w,n)=\sum_{\alpha=1}^{r}N(n,\alpha)+w\sum_{\alpha=0}^{s}K(n,\alpha).$$

Consequently the total number of functionally independent solutions of the equations (15.8) is

$$(17.3)\qquad \mathfrak{D}(r,s,w,n)=D(r,s,w,n)-n^2$$

for $r\geqq 1$, $s\geqq 0$, $w\geqq 1$, $n\geqq 2$ with the exception of the combination $r=1$, $s=0$, $n=2$ for which we have

$$(17.4)\qquad \mathfrak{D}(1\ 0,w,2)=D(1,0,w,2)-3=w+1\qquad (w\geqq 1).$$

* Cf. § 4.

Corresponding to the definition of § 10 we have the following definition.

*Definition.* A fundamental set of affine scalar differential invariants of order $(r, s)$ is a set of functionally independent affine scalar differential invariants *each* of which is of order $(r, s)$ such that any other affine scalar differential invariant of order $(r, s)$ is expressible as a function of the members of the set.

By an obvious extension of the reasoning of § 10 it is readily seen that the *number of invariants in a fundamental set* of absolute affine scalar differential invariants of order $(r, s)$ is given by the number $\mathfrak{D}(r, s, w, n)$ defined by (17.3) and (17.4).

The determination of the differential equations for relative tensor invariants involving arbitrary functions $F^{(k)}$ and their derivatives offers no difficulty. More generally it is clear now how one can proceed in order to deduce the differential equations satisfied by a set of invariant quantities $Q_{\alpha\beta\gamma\cdots\sigma}^{abc\cdots k}$ whenever their law of transformation is given. A complete treatment of the differential equations of the general invariant $Q$ leading to the determination of the number of functionally independent quantities $Q$ would require a considerable extension of the methods which we have used; yet at present there appears to us to be no need for such general considerations, for in the scalar invariants $I$ which we have treated in detail we have all that is necessary for the complete characterization of the affinely connected manifold—within the invariants $I$ all the properties of the manifold are contained.

18.* **Finiteness of complete sets of invariants.** Lately we have directed our attention to the question of the finiteness of a complete set of invariants. It is possible to give an algebraic characterization of the affinely connected manifold $\mathfrak{M}$ by means of a finite number of tensor or scalar invariants. The order of the invariants required in this characterization does not exceed a certain value depending on the dimensionality of the manifold.

It can be shown† that a necessary and sufficient condition for the equivalence of the manifolds $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ is the existence of an integer $N \geqq 1$ such that the first $N$ sets of equations (2.2) are compatible in the

* Added December 2, 1926.

† This has been proved in substance by Wright in connection with quadratic differential forms. See, J. E. Wright, *Invariants of Quadratic Differential Forms*, Cambridge (1908) pp. 15—18. A clearer statement and proof of this theorem has recently been given by Veblen and Thomas with a view towards application to a larger class of equivalence problems. More particularly, these authors have proved the corresponding theorem involving the curvature tensor and its successive covariant derivatives, but their theorem implies the above theorem and vice versa. Cf. O. Veblen and J. M. Thomas, Annals of Math., vol. 27 (1926), § 7, p. 288.

variables $x^i$ and $u^i_\alpha$ considered as functions of the independent variables $\bar{x}^i$ and that all their solutions satisfy the $(N+1)$st set of equations (2.2). Now the greatest number of equations in the first $N$ sets of equivations (2.2) which can be independent in the variables $x^i$ and $u^i_\alpha$ is $n(n+1)$. Also the addition of each successive set of equations to those preceding in the sequence (2.2) in the process of obtaining the first $N$ sets of equations (2.2) so as to make $N$ have a minimum value, must increase the total number of independent equations by *one* at least. Hence

$$N \leqq n(n+1).$$

This gives the following theorem.

THEOREM A. *The first* $n^2+n+1$ *normal tensors A constitute a complete set of invariants of the manifold* $\mathfrak{M}$.

For the purpose of extending the finiteness theorems to the case of scalar invariants it is desirable to employ a *second fundamental set of invariants* of order $p \geqq 1$ which contains scalar invariants of all orders from 1 to $p$ inclusive. To distinguish this new fundamental set of invariants from the one previously defined in § 10, which is composed entirely of scalar invariants $I$ of order $p$, we shall refer to the latter as the *first fundamental set of invariants* of order $p$.

*Definition.* A second fundamental set of scalar invariants of order 1 is a fundamental set of solutions of the differential equations (6.11).

As thus defined a second fundamental set of invariants of order 1 is likewise a first fundamental set of invariants of order 1 and vice versa.

*Definition.* A second fundamental set of scalar invariants of order $p>1$ is a fundamental set of solutions of the differential equations (10.1) such that a subset of these solutions considered as a set of scalar invariants constitutes a second fundamental set of scalar invariants of order $p-1$,

Hence a first fundamental set of scalar invariants of order $p>1$ is never a second fundamental set of scalar invariants.

The statements of § 11 remain true if the $\mathfrak{N}(n,p)$ invariants $I$ of a first fundamental set of scalar invariants of order $p$ are replaced by the $\mathfrak{N}(n,p)$ invariants $I$ of a second fundamental set of scalar invariants of order $p$. In the following we shall consider for the sake of definiteness that the invariants $I$ of § 11 denote invariants of a *second* fundamental set of scalar invariants.

Evidently a necessary condition for the equivalence of the manifolds $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ is the existence of an integer $\omega$ such that the equations (11.2) for $p=\omega$ are compatible in the variables $x^i$ as functions of the independent variables $\bar{x}^i$ and that all their solutions (11.7) satisfy the equations (11.2) for $p=\omega+1$. Assuming that this condition is satisfied, it follows by

the argument of § 11 that if (11.7) denotes a solution of the equations (11.2) for $p=\omega$, the first $\omega$ sets of equations (2.2) have the solution (11.7) and (11.8) where $\theta^i_\alpha$ is determined* by (11.7) and the set of equations (11.5). Now if the components $A$ of the first $p \geqq \omega$ normal tensors are given and likewise the components $\bar{A}$ of the set $r(\bar{A})$, then all other components $\bar{A}$ of the first $p$ normal tensors are determined as functions of the components $A$ and the components $\bar{A}$ of the set $r(\bar{A})$, for example, by solving the equations (11.5) for $u^i_\alpha$ and then finding the components $\bar{A}$ in question by means of the first $p$ sets of equations of the sequence (2.2). Hence it must be possible to solve the equations (11.2) for $p=\omega+1$ so as to express all components $\bar{A}$ in terms of the components $A$ and the components $\bar{A}$ of the above set $r(\bar{A})$. Consequently the above solution (11.7) and (11.8) of the first $\omega$ sets of equations (2.2) which represents any solution $x=f(\bar{x})$ and $u=\theta(\bar{x})$ of these equations, must also satisfy the $(\omega+1)$ st set of equations (2.2). This proves the sufficiency of our conditions.

THEOREM B. *A necessary and sufficient condition for two manifolds $\mathfrak{M}$ and $\bar{\mathfrak{M}}$ to be equivalent is that an integer $\omega \geqq 1$ exist such that the equations* (11.2) *for $p=\omega$ are compatible in the variables $x^i$ as functions of the independent variables $\bar{x}^i$ and that all their solutions* (11.7) *satisfy the equations* (11.2) *for $p=\omega+1$.*

The greatest number of independent equations (11.2) for $p=\omega$ *is* $n$. Hence by a consideration similar to that made in the derivation of the inequality satisfied by $N$, we have†

$$\omega \leqq n.$$

THEOREM C. *A second fundamental set of scalar invariants of order $n+1$ constitutes a complete set of scalar invariants of the manifold $\mathfrak{M}$.*

The solutions of the first $N$ sets of equations (2.2) or of the equations (11.2) for $p=\omega$ may be finite in number or may depend on a certain number of arbitrary parameters. In the latter case we have a finite continuous group of transformations analogous to a group of motions of a Riemann space.‡

* In case $n=2$ and $\omega=1$, this determination is not unique. See footnote in § 11.

† In a subsequent paper we shall show that $N$ and $\omega$ are equal and that they are arithmetical invariants of the manifold.

‡ Cf. L. P. Eisenhart, Riemannian Geometry, Princeton (1926), chap. VI.

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# LINEAR CONGRUENCES IN A GENERAL ARITHMETIC.*

By Horace L. Olson.

Let $\mathfrak{A}$ be any rational, associative, semi-simple algebra of finite order $n$ having a modulus. Then there exist integral elements $e_1, \cdots, e_n$ of $\mathfrak{A}$ such that any integral element of $\mathfrak{A}$ is expressible as a linear homogeneous function of $e_1, \cdots, e_n$ with integer coefficients.† Let the multiplication table of the units $e_1 \cdots e_n$ be

$$e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k \qquad (i, j = 1, \cdots, n). \tag{1}$$

Let $a$, $b$, $c$, and $m$ be given integral elements of $\mathfrak{A}$ and consider the congruence

$$axb \underset{l}{\equiv} c \pmod{m}, \tag{2}$$

which is to be understood as equivalent to the equation

$$axb = c + md, \tag{3}$$

where $d$ is an arbitrary integral element of $\mathfrak{A}$. If the norms of $a$ and $b$ are relatively prime to the norm of $m$, multiply both members of equation (3) on the left by $a'$, the product of the conjugates of $a$, and on the right by $b'$, the product of the conjugates of $b$, in any order. Then

$$AxB = a'cb' + a'mdb', \tag{4}$$

where $A$ and $B$ are integers, the norms of $a$ and $b$ respectively. According to Euler's theorem,

$$A^{\varphi(M)} B^{\varphi(M)} \equiv 1 \pmod{M}$$

where $M$ is the norm of $m$. Hence, multiplying both members of equation (4) by $A^{\varphi(M)-1} B^{\varphi(M)-1}$, we obtain

$$x \equiv A^{\varphi(M)-1} B^{\varphi(M)-1} a'cb' + A^{\varphi(M)-1} B^{\varphi(M)-1} a'mdb' \pmod{M}. \tag{5}$$

Similarly, under the same hypotheses, the congruence

$$axb \underset{r}{\equiv} c \pmod{m}, \tag{6}$$

* Presented to the American Mathematical Society September 8, 1926; received October 1, 1926.

† Dickson, *Algebras and their Arithmetics*, p. 163.

which means that there exists an integral element $d$ of $\mathfrak{A}$ such that

$$axb = c + dm,$$

has as its general solution

$$(7) \quad x \equiv A^{\varphi(M)-1} B^{\varphi(M)-1} a' cb' + A^{\varphi(M)-1} B^{\varphi(M)-1} a' dmb' \pmod{M}.$$

It may be of some interest to observe that, contrary to the situation in ordinary scalar congruences, a congruence of the form (2) or (6), with $c = 0$, may have a solution which is not congruent to 0 either on the right or on the left, even when $a$ and $b$ are relatively prime to $m$ in both senses. For example, in the arithmetic of rational quaternions, the congruence

$$(1-i+k)\, x\, (1+j) \underset{l}{\equiv} 0 \pmod{2+i}$$

has as a particular solution $x = i-2j$, which is not divisible by $2+i$ either on the right or on the left.

If the norms of $a$ and $b$ are not both relatively prime to $M$, multiply both members of equation (3) on the left by $m'$, the product of the conjugates of $m$ in any order. Thus

$$(8) \quad m'axb = m'c + Md,$$

which becomes, by the use of equation (1),

$$(9) \quad \sum_{i,j,k,p,q,r=1}^{n} \gamma_{iks}\gamma_{jrk}\gamma_{pqr} m_i a_j x_p b_q \equiv \sum_{i,j=1}^{n} \gamma_{ijs} m_i' c_j \pmod{M}, \quad (s = 1, \cdots, n).$$

In a manner precisely analogous to the well known discussion of systems of linear equations, it can be shown that a necessary and sufficient condition for the solvability in integers of a system of linear congruences with respect to a modulus $M$ is that the modular rank of the matrix of the coefficients with respect to each factor of $M$ be equal to the modular rank of the augmented matrix with respect to the same faktor of $M$; by the modular rank of a matrix with respect to a modulus $\mu$ we mean the highest order of a determinant of that matrix which is not divisible by $\mu$. Thus a necessary and sufficient condition for the existence of integral solutions of the congruence (2) is that the modular rank, with respect to each factor of $M$, of the matrix

$$\left\| \sum_{i,j,k,q,r=1}^{n} \gamma_{iks}\gamma_{jrk}\gamma_{pqr} m_i' a_j b_q \right\| \qquad (p, s = 1, \cdots, n)$$

be equal to the modular rank, with respect to the same factor of $M$, of the augmented matrix

$$\left\| \sum_{i,j,k,q,r=1}^{n} \gamma_{iks}\gamma_{jrk}\gamma_{pqr} m_i' a_j b_q, \sum_{i,j=1}^{n} \gamma_{ijs} m_i' c_j \right\| \quad (p, s = 1, \cdots, n).$$

Similarly, a necessary and sufficient condition for the existence of integral solutions of the congruence (6) is that the modular ranks, with respect to each factor of $M$, of the matrices

$$\left\| \sum_{i,j,k,q,r=1}^{n} \gamma_{ris}\gamma_{kqr}\gamma_{jpk} a_j b_q m_i' \right\| \quad (p, s = 1, \cdots, n)$$

and

$$\left\| \sum_{i,j,k,q,r=1}^{n} \gamma_{ris}\gamma_{kqr}\gamma_{jpk} a_j b_q m_i', \sum_{i,j=1}^{n} \gamma_{ijs} c_i m_j' \right\| \quad (p, s = 1, \cdots, n)$$

be equal.

To solve a congruence in an algebra over any field of complex numbers in which factorization into primes is unique, we need merely to adjoin to the basal units of the algebra certain units of the field, thus representing the algebra as one over the field of all rational numbers.

The solution of a linear congruence of type (2) or (6), when the norms of the coefficients $a$ and $b$ are not both relatively prime to the norm of the modulus, can usually be obtained as follows: consider the scalar congruences (9) or their analogues, and find, if possible, a coefficient of one of the $x$'s which is relatively prime to $M$, the norm of the modulus, and multiply the congruence in which it appeares by the $(M-2)$th power of that coefficient; thus, by means of Euler's theorem, we obtain an expression for one of the $x$'s, with integral coefficients, in terms of the other $x$'s. Substitute this expression in the other congruences, and repeat this process until congruences are obtained which are either identically satisfied or impossible. In the former case we have some of the $x$'s expressed in terms of certain arbitrary integers and the remaining $x$'s, which are arbitrary integers; in the latter case, of course, the given congruence has no solution.

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# NOTE ON A DIFFERENTIAL EQUATION.*

By Joseph Miller Thomas.†

1. The partial differential equation

$$aX' + bX + cY' + dY = 0, \tag{1}$$

in which $a$, $b$, $c$, and $d$ are given functions of the independent variables $x$ and $y$, $X$ and $Y$ are unknown functions of the variables $x$ and $y$ respectively, and $X'$ and $Y'$ their derivatives, is of frequent occurrence in the differential geometry of surfaces. Although in most of the cases that have arisen this equation can be solved by particular methods, it seems worth while to give a general method of testing whether it admits solutions, other than the trivial solution $X = Y = 0$, and of finding the solutions if they exist.‡

2. Suppose both $a$ and $c$ are different from zero. Equation (1) can then be replaced by the system of partial differential equations:

$$\frac{\partial X}{\partial x} = BX + CY + DZ, \qquad \frac{\partial X}{\partial y} = 0,$$

$$\frac{\partial Y}{\partial x} = 0, \qquad \frac{\partial Y}{\partial y} = Z,$$

where

$$B = -b/a, \quad C = -d/a, \quad D = -c/a.$$

By expressing the condition

$$\frac{\partial^2 X}{\partial x\, \partial y} = \frac{\partial^2 X}{\partial y\, \partial x}$$

we find

$$\frac{\partial Z}{\partial y} = \beta X + \gamma Y + \delta Z, \tag{2}$$

where

$$\beta = -B_y/D, \quad \gamma = -C_y/D, \quad \delta = -(C + D_y)/D,$$

the subscript denoting partial differentiation. Hence equation (1) is equivalent to the following system of total differential equations

* Presented to the Amer. Math. Soc., September 9, 1926, received October 4, 1926.

† National Research Fellow in Mathematics.

‡ This problem was suggested to me by Professor W. C. Graustein.

$$
(3)\qquad
\begin{aligned}
&\frac{\partial X}{\partial x} = BX+CY+DZ, && \frac{\partial X}{\partial y} = 0,\\
&\frac{\partial Y}{\partial x} = 0, && \frac{\partial Y}{\partial y} = Z,\\
&\frac{\partial Z}{\partial x} = 0, && \frac{\partial Z}{\partial y} = \beta X+\gamma Y+\delta Z.
\end{aligned}
$$

The well known methods of testing the compatibility of a system of total differential equations and of integrating it can be applied to the system (3). All integrability conditions are found to be identically satisfied except the condition

$$\frac{\partial^2 Z}{\partial x\,\partial y} = \frac{\partial^2 Z}{\partial y\,\partial x}$$

which can be written

$$(4)\qquad \xi X+\eta Y+\zeta Z = 0,$$

where

$$\xi = \beta_x+\beta B, \qquad \eta = \gamma_x+\beta C, \qquad \zeta = \delta_x+\beta D.$$

In the treatment of equation (4) three distinct cases arise.

*Case I.* If (4) is identically satisfied, that is, if

$$(5)\qquad \beta_x+\beta B = \gamma_x+\beta C = \delta_x+\beta D = 0,$$

equations (3) are completely integrable and their general solution contains three arbitrary constants.*

Conditions (5) are satisfied by the equation $X'-Y' = 0$, of which the general solution is

$$X = kx+l, \qquad Y = ky+m,$$

where $k$, $l$, and $m$ are the arbitrary constants.

*Case II.* If (4) is not identically satisfied, we differentiate it with respect to $x$ and $y$, eliminate the derivatives of the unknown functions by means of (3), and obtain

$$(6)\qquad \begin{aligned} \xi' X+\eta' Y+\zeta' Z &= 0,\\ \xi'' X+\eta'' Y+\zeta'' Z &= 0, \end{aligned}$$

in which

$$\xi' = \xi_x+\xi B, \qquad \eta' = \eta_x+\xi C, \qquad \zeta' = \zeta_x+\xi D,$$
$$\xi'' = \xi_y+\zeta\beta, \qquad \eta'' = \eta_y+\zeta\gamma, \qquad \zeta'' = \zeta_y+\eta+\zeta\delta.$$

* For a proof of this statement and a method of integrating such a system cf. Bianchi, *Lezioni sulla teoria dei gruppi continui finiti*, Chapter I.

If relations (6) are algebraic consequences of (4), on eliminating one of the unknowns from the system (3) by means of (4) we get a *completely integrable* system of total differential equations in the two remaining unknowns. The general solution of this system together with relation (4) gives the general solution of (3)* and consequently of (1). It contains two arbitrary constants. An example is furnished by the equation

$$xX' + X + Y' - \frac{2}{y^2}(x+y)\,Y = 0,$$

of which the general solution is

$$X = kx + \frac{l}{x}, \qquad Y = ky^2.$$

*Case III.* If one† of the equations (6) is independent of (4), we treat it in the way we did (4), and obtain thus two equations linear and homogeneous in $X$, $Y$, and $Z$. These equations must be algebraic consequences of (4) and (6), if equation (1) is to have a non-trivial solution. The solution, if it exists, is found by eliminating two of the unknowns from (3) by means of (4) and (6). The remaining unknown is then found by the integration of a perfect differential. The general solution contains in this case one arbitrary constant.‡ An example is furnished by the equation

$$y^2X' + (x+y-1)\,X + x^2Y' + (x+y+1)\,Y = 0,$$

whose solution is

$$X = k(2x+1), \qquad Y = k(-2y+1).$$

3. If $a \neq 0$, $c = 0$, $d \neq 0$, equations (3) and (4) are to be replaced by

$$(3') \qquad \frac{\partial X}{\partial x} = BX + CY, \qquad \frac{\partial X}{\partial y} = 0,$$
$$\frac{\partial Y}{\partial x} = 0, \qquad \frac{\partial Y}{\partial y} = \beta' X + \gamma' Y,$$

$$(4') \qquad (\beta'_x + \beta' B)\,X + (\gamma'_x + \beta' C)\,Y = 0,$$

where

$$\beta' = -B_y/C, \qquad \gamma' = -C_y/C.$$

* Cf. the theorem on partial differential equations in the paper *Projective Invariants for Affine Geometry of Paths* by O. Veblen and J. M. Thomas in these Annals, March 1926.

† Obviously all three relations (4) and (6) cannot be independent if (1) is to have a solution other than $X = Y = 0$.

‡ From the linear and homogeneous form of equation (1) it is clear that its general solution contains at least one arbitrary constant.

If (4′) is identically satisfied, the system (3′) is completely integrable and its general solution contains two arbitrary constants.

If (4′) is not identically satisfied, the linear equations arising from it by differentiation and elimination of the derivatives $X'$ and $Y'$ must be algebraic consequences of (4′) in order that a non-trivial solution exist. The solution is found by eliminating one of the unknowns from (3′) by means of (4′) and integrating the resulting system in the other unknown. The general solution contains one arbitrary constant.

4. There remain for discussion two cases of a somewhat trivial nature.

If $a \neq 0$, $c = d = 0$, the ratio $b/a$ must be a function of $x$ alone in order that a solution exist. If this condition is fulfilled, $X$ is given by an ordinary equation of the first order, and $Y$ is an arbitrary function of $y$.

If $a = c = 0$, the equation ceases to be differential. The condition for the existence of a solution is that the ratio $b/d$ be of the form $\varphi(y)/f(x)$, and the corresponding solution is

$$X = kf(x), \qquad Y = -k\varphi(y),$$

where $k$ is an arbitrary constant.

We can therefore state the following

THEOREM. *The general solution of the differential equation*

$$aX' + bX + cY' + dY = 0$$

*contains at least one and at most three arbitrary constants, except in the trivial case when one of the unknowns and its derivative is missing from the equation.*

5. A method of solution useful in practice is the following. Substitute a particular value for $y$ in equation (1) and solve the resulting equation for $X$. In general there will be three constants in this solution: the values of $Y'$ and $Y$ for the chosen value of $y$, and the constant of integration of the ordinary equation of the first order in $X$. This value of $X$ is then to be substituted in (1). The solution of the resulting equation in $Y$ is independent of $x$ or can be made so by assuming relations along the constants in the expression for $X$. The values of $X$ and $Y$ so obtained form the general solution of (1). In solving we impose the condition that the differential equation in $Y$ be an identity in $x$.

Applying the method of the present section to the second example of section 2, we find

$$X = kx + \frac{l}{x} + 2k - m,$$

where $Y(1) = k$, $Y'(1) = m$. The equation for $Y$ is accordingly

$$Y' - \left(\frac{2}{y} + \frac{2x}{y^2}\right) Y + 2kx + 2k - m = 0.$$

The vanishing of the coefficient of $x$ and of the term independent of $x$ gives $Y = ky^2$ and $m = 2k$.

6. The methods outlined above can be extended to the general equation

$$\sum_{i=1}^{n} (a_i X_i' + b_i X_i) = 0,$$

where $a_i$ and $b_i$ are given functions of the $n$ independent variables $x_i$, and each unknown $X_i$ is a function only of the independent variable with the same subscript.

PARIS.

August, 1926.

# ON THE CLASS OF A COVARIANT TENSOR.*

By J. W. Alexander.

1. The components of a covariant tensor $A_{i_1 i_2 \ldots i_m}$ in $n$-space may always be thought of as the coefficients of a differential form,

$$A_{i_1 i_2 \ldots i_m}\, d_1 x^{i_1}\, d_2 x^{i_2} \cdots d_m x^{i_m} \tag{1}$$

of degree $m$. Let $k\ (\leqq n)$ be the minimum number of variables such that the form (1) is expressible in terms of these $k$ variables and their differentials. We shall call the number $k$ the *class* of the tensor $A_{i_1 i_2 \ldots i_m}$.

To determine the class of the tensor $A_{i_1 i_2 \ldots i_m}$, let us first inquire under what conditions the form (1) may be expressed in terms of less than $n$ variables and their differentials. This amounts to determining under what conditions there exists a coördinate system $K_x$ with reference to which the form (1) is independent of the variable $x^n$ and the differentials of $x^n$. With reference to such a coördinate system, the components of the $A_{i_1 i_2 \ldots i_m}$ must satisfy the conditions

$$\begin{aligned} A_{n i_2 i_3 \ldots i_m} &= 0 \\ A_{i_1 n i_3 \ldots i_m} &= 0 \\ &\cdots\cdots \\ A_{i_1 i_2 i_3 \ldots i_m} &= 0, \end{aligned} \tag{2}$$

which express the absence of terms in the differentials of $x^n$, and the further conditions

$$\frac{\partial A_{i_1 i_2 i_3 \ldots i_m}}{\partial x^n} = 0, \tag{3}$$

which express that the coefficients $A_{i_1 i_2 \ldots i_m}$ are independent of $x^n$ itself. From equations (3), let us subtract the derivatives of equations (2) with respect to $x^{i_1}, x^{i_2}, \cdots, x^{i_m}$ respectively. We thus obtain, after a change of sign, the relations

$$\frac{\partial A_{n i_2 \ldots i_m}}{\partial x^{i_1}} + \cdots + \frac{\partial A_{i_1 i_2 \ldots n}}{\partial x^{i_m}} - \frac{\partial A_{i_1 i_2 \ldots i_m}}{\partial x^n} = 0$$

* Received October 12, 1926.

which we shall express in the abridged notation

$$\Gamma_{i_1 i_2 \ldots i_m n} = 0. \tag{4}$$

Evidently, conditions (2) and (4) are together equivalent to conditions (2) and (3). The expressions $\Gamma_{i_1 i_2 \ldots i_m n}$ are generalizations of the Christoffel bracket symbols, to which they reduce (except for a numerical factor) when the tensor $A_{i_1 i_2 \ldots i_m}$ is a symmetrical tensor of the second order.

Now, let

$$\delta^i_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \tag{5}$$

be the Kronecker tensor, and let $dx^i = X^i$ be that contravariant vector which reduces to the normal form

$$X^i = \delta^i_n, \qquad (i = 1, 2, \cdots, n), \tag{6}$$

in the coördinate system $K_x$. With the help of the vector $dx^i = X^i$, conditions (2) and (4) may be written as

$$\begin{aligned} A_{s i_2 i_3 \ldots i_m}\, dx^s &= 0 \\ A_{i_1 s i_3 \ldots i_m}\, dx^s &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_{i_1 i_2 i_3 \ldots s}\, dx^s &= 0 \end{aligned} \tag{7a}$$

and

$$\Gamma_{i_1 i_2 i_3 \ldots s}\, dx^s = 0 \tag{7b}$$

respectively. Equations (7a) are tensor equations and, therefore, valid in all coördinate systems. Treated as equations in an unknown vector $dx^i$, their rank is an invariant, known as the *rank** of the tensor $A_{i_1 i_2 \ldots i_m}$. In general, equations (7b) are not tensor equations; nevertheless, the form of the combined equations (7a) and (7b)—which equations we shall hereafter refer to collectively as the equations (7)—is invariantive under all transformations of coördinates. For let $B_{i_1 i_2 \ldots i_m}$ be the components of the tensor $A_{i_1 i_2 \ldots i_m}$ in any new coördinate system $K_y$, and let $\mathcal{A}_{i_1 i_2 \ldots i_m n}$ be the generalized Christoffel symbols formed from the derivatives of the functions $B_{i_1 i_2 \ldots i_m}$. Then, from the relations

$$B_{i_1 i_2 \ldots i_m} = A_{s_1 s_2 \ldots s_m} \frac{\partial x^{s_1}}{\partial y^{i_1}} \frac{\partial x^{s_2}}{\partial y^{i_2}} \cdots \frac{\partial x^{s_m}}{\partial y^{i_m}},$$

* Cf. my note On the Decomposition of Tensors, Annals of Math., (2), vol. 27 (1926), p. 421.

we obtain, by straightforward calculation,

$$A_{i_1 i_2 \cdots i_m s} = \Gamma_{s_1 s_2 \cdots s_m \sigma} \frac{\partial x^\sigma}{\partial y^s} \frac{\partial x^{s_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{s_m}}{\partial y^{i_m}}$$
$$+ A_{s_1 s_2 \cdots s_m} \frac{\partial x^{s_1}}{\partial y^s} \frac{\partial}{\partial y^{i_1}} \left( \frac{\partial x^{s_2}}{\partial y^{i_2}} \cdots \frac{\partial x^{s_m}}{\partial y^{i_m}} \right)$$
$$+ \cdots \cdots \cdots \cdots \cdots,$$

where the omitted terms are similar to the last one written out in full. We thus have

$$A_{i_1 i_2 \cdots i_m s}\, dy^s = \Gamma_{s_1 s_2 \cdots s_m \sigma}\, dx^\sigma \left( \frac{\partial x^{s_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{s_m}}{\partial y^{i_m}} \right)$$
$$+ A_{s_1 s_2 \cdots s_m}\, dx^{s_1} \frac{\partial}{\partial y^{i_1}} \left( \frac{\partial x^{s_2}}{\partial y^{i_2}} \cdots \frac{\partial x^{s_m}}{\partial y^{i_m}} \right) \tag{8}$$
$$+ \cdots \cdots \cdots \cdots \cdots,$$

which vanishes because of (7). This establishes the invariantive form of equations (7) as a whole. By analogy with certain definitions in the theory of Pfaffian forms, we shall call the system of equations (7) the *characteristic system* of the tensor $A_{i_1 i_2 \cdots i_m}$, and the vector solutions $dx^i = X^i$ of (7) the *characteristic vectors* of $A_{i_1 i_2 \cdots i_m}$.

2. *A necessary and sufficient condition that there exist a coördinate system $K_x$ with reference to which the form* (1) *is free from the variable $x_n$ and its differentials is that the characteristic equations of* (1) *admit a solution $dx^i = X^i$.*

Proof: By what we have already shown, the condition is necessary. For if a coördinate system such as $K_x$ exists, the characteristic equations (7) admit as a solution that vector $dx^i = X^i$ which reduces to the normal form

$$X^i = \delta^i_n \tag{9}$$

with reference to $K_x$. The condition is also sufficient. For if the characteristic equations admit any solution $dx^i = X^i$, there is always some coördinate system $K_x$ with reference to which the vector $X^i$ reduces to the normal form (9). In fact, such a coördinate system may immediately be found by selecting, as the $n-1$ first variables any $n-1$ independent first integrals of the differential equations

$$\frac{dx^i}{ds} = X^i$$

and as the remaining variable the parametric variable $s$. But with reference to the system $K_x$, conditions (2) and (4) are fulfilled, whence the form (1) is free from the variables $x^n$ and the differentials of $x^n$.

*The class of a tensor $A_{i_1 i_2 \cdots i_m}$ is equal to the rank of its characteristic system.*

Proof. If the tensor $A_{i_1 i_2 \cdots i_m}$ is of class $k$, coördinates may be so chosen that the form (1) is independent of the variables $x^{k+1}, x^{k+2}, \cdots, x^n$ and their differentials. With reference to these coördinates, the characteristic equations (7) obviously admit the $n-k$ independent solutions

$$X^i_{(s)} = \delta^i_s, \qquad (k < s \leqq n), \tag{10}$$

(cf. (6)); therefore, the rank of the characteristic system cannot exceed $n-(n-k) = k$, which is the class of the tensor $A_{i_1 i_2 \cdots i_m}$. Nor can the rank of the characteristic system be less than the class of the tensor. For if $l$ is the rank of the characteristic system, this last admit a set of $n-l$ linearly independent solutions

$$X^i_{(s)}, \qquad (l < s \leqq n). \tag{11}$$

Let coördinates by chosen in such a manner that the solution $X^i_{(n)}$ is in the normal form

$$X^i_{(n)} = \delta^i_n. \tag{12}$$

Then by adding a suitable scalar multiple of this solution to each of the others we may obviously transform the others into new solutions of such a form that their $n$-th components all vanish. If we now hold the variable $x^n$ fixed, $x^n = c^n$, and regard the first $n-1$ components of the transformed solution $X^i_{(n-1)}$ as determining the components of a vector in $(n-1)$-space, it is clear that by a transformation of the variables $x^1, x^2, \cdots, x^{n-1}$ alone (which will not alter the form of (12)) we may reduce the solution $X^i_{(n-1)}$ to the normal form

$$X^i_{(n-1)} = \delta^i_{n-1}, \text{ (valid for } x^n = c^n).$$

In a similar manner we may ultimately reduce the solutions (11) until they fulfil the conditions

$$\begin{aligned}
&X^i_{(n)} = \delta^i_n, \\
&X^i_{(n-1)} = \delta^i_{n-1}, \quad (x^n = c^n), \\
&X^i_{(n-2)} = \delta^i_{n-2}, \quad (x^n = c^n,\ x^{n-1} = c^{n-1}), \\
&\cdots\cdots\cdots \\
&X^i_{(n-l)} = \delta^i_{n-l}, \quad (x^n = c^n, \cdots, x^{n-l+1} = c^{n-l+1}).
\end{aligned} \tag{13}$$

But, as we have already shown, if the first of relations (13) determines a solution of the characteristic system (7), the form (1) must be independent of the variable $x^n$ and the differentials of $x^n$. Therefore, if we substitute the constant $c^n$ for $x^n$ in relations (13), the new vectors $X^i_{(s)}$ thus obtained are solutions of the characteristic system for all values of $x^n$, and not only for $x^n = c^n$. By repeated use of a similar argument, we show that the solutions

$$X^i_{(s)} = \delta^i_s \tag{14}$$

are valid for all values of the independent variables $x^i$ and that the form (1) must be independent of the last $n-l$ variables $x^s$,

$$(l < s \leqq n),$$

and their differentials. In other words, the class of the tensor $A_{i_1 i_2 \cdots i_m}$ cannot exceed the rank of the characteristic system. This completes the argument.

*The characteristic system of a tensor $A_{i_1 i_2 \cdots i_m}$ is a complete set of Pfaffian equations.*

For if coördinates are so chosen that relations (13) determine a set of linearly independent solutions of the characteristic system, this last is equivalent to the system

$$dx^1 = dx^2 = \cdots = dx^k = 0.$$

3. The special case where the tensor $A_{i_1 i_2 \cdots i_m}$ is skew symmetrical has already been treated from a somewhat different angle.* If $A_{i_1 i_2 \cdots i_m}$ is skew symmetrical, the expressions $\Gamma_{i_1 i_2 \cdots i_m s}$ are the components of a tensor differing by a numerical factor only from the *curl*

$$\left[\frac{\partial A_{i_1 i_2 \cdots i_m}}{\partial x^s}\right] \tag{12}$$

of the tensor $A_{i_1 i_2 \cdots i_m}$. The brackets in (12) are here used to symbolize the sum of all terms derivable from the one which they enclose by permuting the indices $s i_1 i_2 \cdots i_m$ in all possible ways and changing the signs of all terms obtained as the result of odd permutations. The characteristic system of a skew symmetrical tensor simplifies, after discarding certain redundant equations, to

$$\begin{aligned} A_{s i_2 \cdots i_m}\, dx^s &= 0, \\ \left[\frac{\partial A_{i_1 i_2 \cdots i_m}}{\partial x^s}\right] dx^s &= 0. \end{aligned} \tag{13}$$

* Cf. Goursat: *Leçons sur le Problème de Pfaff*, Chap. IV.

4. *A necessary and sufficient condition that a contraviant vector $X^i$ be a characteristic vector of a covariant tensor $A_{i_1 i_2 \ldots i_m}$ is that the form* (1) *be a differential invariant (in the sense of Cartan) of the system of equations*

$$\frac{dx^1}{X^1} = \frac{dx^2}{X^2} = \ldots = \frac{dx^n}{X^n}. \tag{14}$$

This may be seen at once by choosing coördinates with reference to which the vector $X^i$ is in the normal form

$$X^i = \delta^i_n.$$

Another way of proving the theorem is to write equations (14) in the form

$$\frac{dx^i}{dt} = \lambda X^i, \tag{15}$$

where $\lambda$ is an arbitrary scalar. Conditions (7) express the fact that we have

$$\frac{dA_{i_1 i_2 \ldots i_m}}{dt} = 0$$

for all choices of the scalar $\lambda$ in (15), as may readily be verified by direct calculation.

PRINCETON, NEW JERSEY.
September 1926.

# SEXTIC SURFACES WITH A DOUBLE SEPTIMIC CURVE.*

By B. C. Wong.

1. Non-ruled sextic surfaces with a double curve of order 9 and those with a double curve of order 8 have been studied by Professor Sisam.† It is the purpose of this paper to study non-ruled sextic surfaces with a nodal curve of order 7. For this purpose we make use of a certain two-dimensional variety, $\Sigma_2^6$, of order 6 belonging to four-space. In two previous papers‡ the writer has studied this variety and has, by means of the involutorial quartic transformation in four-space studied by Miss Alderton,|| established a one-to-one correspondence between its points and the points of a plane $\sigma$. If we project $\Sigma_2^6$ from any general point $Z$ not on it upon a given 3-space $S_3$ not through $Z$, we have in $S_3$ a sextic surface which contains a double curve of order 7,¶ and whose points are clearly in one-to-one correspondence with the points of the plane $\sigma$. The geometry of this surface, which we shall designate by $\Sigma$, can be easily inferred from that of $\Sigma_2^6$ and hence from that of $\sigma$ with reference to the transformation employed.

2. It is to be noted that, as is shown in the papers referred to above, corresponding to each plane $\sigma$ in 4-space there is a definite $\Sigma_2^6$. If $\sigma$ takes a special position relative to the singular manifold of transformation whose points have lines for images, then the corresponding $\Sigma_2^6$ has special properties. The locus of points whose transforms are lines is a $J_2^{10}$, a two-dimensional manifold of order 10 in $S_4$. A general 3-space $S_3$ meets $J_2^{10}$ in a curve $j_1^{10}$. Any plane $\sigma$ in $S_3$ meets $j_1^{10}$ in 10 points, $P_i[i = 0, 1, \cdots, 9]$, which are the base points in $\sigma$ and they go into lines on $\Sigma_2^6$. But $\sigma$ may contain one or two trisecants, a quadrisecant, or a quadrisecant and a trisecant of $j_1^{10}$, or may take such a position as to have 6 or 7 of the intersections with $j_1^{10}$ lying on a conic, etc. These different positions of $\sigma$ give rise to different species of $\Sigma_2^6$, and the projections of the latter upon any $S_3$, and, in particular, upon the $S_3$ containing $\sigma$, are also different species of sextic surfaces with a nodal septic curve. Furthermore, many other types of the

* Received October 11, 1927.

† Amer. Journ. Math., vol. 37, pp. 445–456 and vol. 38, pp. 373–386.

‡ Bull. Amer. Math. Soc., vol. 32, pp. 156–158, and Annals of Math. (2), vol. 27, p. 330.

|| Univ. of Cal. Publ. in Math., vol. 1, pp. 345–358.

¶ For $\Sigma_2^6$ is such that any 3-space section of it is a twisted sextic curve with 7 apparent double points.

surface arise if we project each species of $\Sigma_2^6$ from various special positions in $S_4$ of the center of projection. We shall in the following discuss in detail the most general case, that is, the case where the plane $\sigma$ and also the center $Z$ of projection are in general positions, and then consider briefly some of the other cases.

3. On $\Sigma_2^6$ there are 10 non-intersecting lines $\bar{\pi}_i[i = 0, 1, \cdots, 9]$ corresponding to the 10 base points $P_i$ in $\sigma$*; 45 conics $\bar{\lambda}_{ik}[i, k = 0, 1, \cdots, 9]$ corresponding to the 45 lines $l_{ik}$ in $\sigma$ joining the 10 base points two at at time; 10 non-singular plane cubics $\bar{\mu}_i$ corresponding to the 10 cubics $m_i$ in $\sigma$ each through 9 of the 10 base points and not through the $i$th. Each line $\bar{\pi}_i$ meets 9 plane cubics $\bar{\mu}_i[k \neq i]$ and 9 conics $\bar{\lambda}_{ik}$ all each once; each plane cubic $\bar{\mu}_i$ meets the other 9 plane cubics $\bar{\mu}_k$ each once, also 9 lines $\bar{\pi}_k$ each once, 9 conics $\bar{\lambda}_{ik}$ each twice and the remaining 36 conics $\bar{\lambda}_{jk}[i \neq j \neq k]$ each once; and each conic $\bar{\lambda}_{ik}$ meets 2 lines $\bar{\pi}_i$ and $\bar{\pi}_k$ each once, 8 plane cubics $\bar{\mu}_j$ each once and the remaining 2 plane cubics $\bar{\mu}_i$ and $\bar{\mu}_k$ each twice, and also 28 conics $\bar{\lambda}_{jl}[l \neq i \neq j \neq k]$ each once.

4. Let $\pi_i$, $\lambda_{ik}$, $\mu_i$ denote the projections of $\bar{\pi}_i$, $\bar{\lambda}_{ik}$, $\bar{\mu}_i$ in $S_3$ respectively. The configuration they form is the projection of the one formed by $\bar{\pi}_i$, $\bar{\lambda}_{ik}$, $\bar{\mu}_i$ and it lies on the projection $\Sigma$ of $\Sigma_2^6$. Hence there are 10 points, the base points $P_i$, in $\sigma$ which have for images on $\Sigma$ the 10 line $\pi_i$. Corresponding to the 45 lines $l_{ik}$ and the 10 cubics $m_i$ in $\sigma$ are the 45 conics $\lambda_{ik}$ and the 10 plane cubics $\mu_i$ on $\Sigma$ respectively. Their incidences are the same as those on $\Sigma_2^6$. There are no other lines and conics on $\Sigma$.

5. Since every 3-space section of $\Sigma_2^6$ is a space sextic curve with 7 apparent nodes, there are on the variety $\infty^4$ such curves. Of these $\infty^3$ lie in the $\infty^3$ $\Sigma$-spaces through the center $Z$ of projection and their projections upon $S_3$ are plane sextic curves with 7 nodes, i. e., plane sections of $\Sigma$. Hence, on $\Sigma$ there is an $\infty^4$-family, and it may be noted that this is the only family, of sextic curves of genus 3, which includes the $\infty^3$-family of plane sections. Every member, whether twisted or plane, of this $\infty^4$-family of sextic curves corresponds to a quartic curve through the 10 base points $P_i$ in $\sigma$; and, because of this fact, it meets each of the 10 lines $\pi_i$ once, each of the 45 conics $\bar{\lambda}_{ik}$ twice, and each of the 10 plane cubics $\mu_i$ three times. Any two meet six times.

6. The surface $\Sigma$ has upon it a double curve of order 7, since it is met by any plane in a sextic curve with 7 nodes. Every point on this double curve, $I^7$, is the projection of two distinct points on $\Sigma_2^6$, which correspond to two distinct points $C$ and $C'$ in $\sigma$. The curve on $\Sigma_2^6$ of which $I^7$ is

* Any proposition made in this paper concerning $\Sigma_2^6$ without demonstration is quoted from one or the other of the papers cited in footnote ‡ (p. 251).

the projection is evidently a curve of order 14. The locus of $C$ and $C'$ in $\sigma$ is evidently of order $M$ satisfying the relation

$$14 = 4M - \sum \alpha_i$$

where $\alpha_i$ is the number of times the locus passes through the point $P_i$. Since all the 10 base points play equal role in the transformation, all the $\alpha_i$'s are equal and hence the above relation may be written

$$14 = 4M - 10\alpha$$

where $\alpha$ is the number of times the locus passes through each of the 10 points $P_i$. The values 6,1; 11,3; 16,5; etc., may be assigned to $M$ and $\alpha$ respectively. As $M = 11$, $\alpha = 3$* is the only set of values that does not lead to contradiction, we conclude that the curve in $\sigma$ to which corresponds the double curve $\Gamma^7$ on $\Sigma$ is of order 11 and has 10 triple points coincident with the 10 points $P_i$. We denote this curve by $\gamma^{11}$.

7. The double curve $\Gamma^7$ has evidently one triple point $T$ which is also triple on $\Sigma$, and has 9 apparent nodes and is consequently of rank 18.† Corresponding to $T$ there are three distinct points on $\Sigma_2^6$ and hence there are three distinct points $T_1$, $T_2$, $T_3$ in $\sigma$. As $\Gamma^7$ passes through $T$ three times, the corresponding curve $\gamma^{11}$ in $\sigma$ passes through each of the three points $T_1$, $T_2$, $T_3$ twice. Hence, $T_1$, $T_2$, $T_3$ are nodes on $\gamma^{11}$ and there are no other nodes besides the 10 triple points at $P_i$. The genus of $\gamma^{11}$ is 12.

8. Return for a moment to the $\infty^4$-family of sextic curves of genus 3 on $\Sigma$ to which correspond the $\infty^4$-family of quartics through the point $P_i$ in $\sigma$. The question presents itself as to how to distinguish those quartics which give rise to plane sextics from those which give rise to space sextics. Consider a plane section $\zeta$ of $\Sigma$. Its 7 nodes are on the double curve $\Gamma^7$. Therefore, the corresponding quartic $z$ in $\sigma$ meets the curve $\gamma^{11}$ in 14 points grouped in 7 pairs. As a point $Q$ describes $\Gamma^7$ on $\Sigma$, there are two points $Q_1$ and $Q_2$ describing the curve $\gamma^{11}$ in $\sigma$. Hence the points of $\gamma^{11}$ all go in pairs. Then we conclude that if any quartic $z$ of the $\infty^4$-family of quartics through the 10 base points $P_i$ in $\sigma$ passes through 6 points forming 3 pairs

* The formulas for $M$ and $\alpha$ given by Clebsch and Cayley [Cayley, Collected Math. Papers, vol. VIII, pp. 388-393] are $(n-4)n'+3$ and $(n-4)r+1$ respectively, where $n$ is the order of the surface, $n'$ the order of transformation, $r$ the number of times a general curve of order $n'$ passes through each of the base points in the plane. For our problem $n = 6$, $n' = 4$, $r = 1$.

† These results can be obtained from formulas given by Cayley in the paper referred to above. The fact that there is only one triple point is a consequence of the fact that no two trisecants of a space sextic curve of genus 3 can meet except on the curve.

on $\gamma^{11}$, it will meet $\gamma^{11}$ in 4 other pairs, and hence the corresponding sextic curve on $\Sigma$ is plane. Since any quartic of the family is determined by 4 more points in addition to the 10 base points $P_i$, any 4 independent points on $\gamma^{11}$ determine a quartic whose image is a sextic curve on $\Sigma$; but if the 4 given points are such that they belong to two pairs, the sextic is plane. If a quartic passes through the three nodes $T_1$, $T_2$, $T_3$ on $\gamma^{11}$, the sextic lies in a plane through $T$ and therefore has a triple point.

9. Since the variety $\Sigma_2^6$ has 27 tangent 3-spaces passing through a given plane corresponding to the number of quartics of a pencil of plane quartics that have a node, the surface $\Sigma$ is of class 27. The tangent cone with vertex at any point of $S_3$ is of order 16, and corresponding to the curve $\tau$ of contact is a curve $t$ of order 9 passing through each of the 10 base points $P_i$ twice in $\sigma$. Since every $\tau$ passes through the pinch-points on $\Gamma^7$, every $t$ in $\sigma$ passes through the points which have the pinch-points for images on $\Sigma$. Take any two curves $t$ and $t'$ in $\sigma$ and let their images be $\tau$ and $\tau'$ on $\Sigma$ respectively. Of the 81 intersections of $t$ and $t'$ 40 are at the nodes $P_i$, 27 correspond to the 27 points of contact between $\Sigma$ and its tangent planes through the line joining the vertices of the two tangent cones whose curves of contact are $\tau$ and $\tau'$, and the remaining 14 correspond to the pinch-points on $\Gamma^7$. Hence the surface has 14 pinch-points, and they lie on the double curve.

10. Now consider the triple tangent planes to the surface. A plane through a cubic $\mu_\alpha$ which is the image of $m_\alpha$ in $\sigma$ meets the surface in another cubic $\mu'_\alpha$ which, being the image of a certain line $g_\alpha$ through the point $P_\alpha$ in $\sigma$, is unicursal. The degenerate plane sextic has 10 nodes, of which one is the node on $\mu'_\alpha$ and the remaining 9 are the intersections of $\mu_\alpha$ and $\mu'_\alpha$. Of the 9 intersections 3 correspond to the 3 points common to $g_\alpha$ and $m_\alpha$ in $\sigma$ and are therefore points of contact between the plane and the surface. Since there are 10 plane cubics $\mu_i$ and consequently 10 others, $\mu'_i$, which are unicursal, on $\Sigma$ there are 10 triple tangent planes to the surface meeting it in pairs of cubics.

11. A plane meeting $\Sigma$ in a conic $\lambda_{\alpha\beta}$ meets it in a quartic $\varrho_{\alpha\beta}$ besides. This quartic $\varrho_{\alpha\beta}$ is the image of a certain cubic $\gamma_{\alpha\beta}$ through 8 of the base points and not through the remaining 2, $P_\alpha$ and $P_\beta$, and is therefore binodal. The degenerate plane sextic composed of $\lambda_{\alpha\beta}$ and $\varrho_{\alpha\beta}$ has 10 nodes, of which two are on $\varrho_{\alpha\beta}$ and 8 are the intersections of the two components. Of the latter, 3 are the images of the points of $\sigma$ in which the line $l_{\alpha\beta}$ meets the cubic $\gamma_{\alpha\beta}$, and are therefore points of contact between the plane and the surface. The existence of 45 conics on $\Sigma$ gives rise to 45 triple tangent planes to the surface each meeting it in a conic and a binodal quartic curve.

12. Now a plane through a line $\pi_\alpha$ of $\Sigma$ meets the surface in a quadrinodal quintic $\omega_\alpha$ besides. This quintic, $\omega_\alpha$, is the image of a certain quartic $w_\alpha$ in $\sigma$ passing through the point $P_\alpha$ twice and each of the other 9 base points once. The line $\pi_\alpha$ meets the quintic $\omega_\alpha$ in 5 points 3 of which correspond to the triple point of $\gamma^{11}$ in $\sigma$ coincident with $P_\alpha$ and the other 2 are points of contact between the plane and the surface. Hence all the planes through $\pi_\alpha$ are double tangent planes. Of these a finite number are triply tangent. Such a plane contains a quintic $\omega'_\alpha$ with an extra node and that requires the corresponding quartic to have two nodes, one already at $P_\alpha$ and the other elsewhere. Since the number of binodal quartics of a pencil of uninodal quartics with their nodes coincident is 20, we infer that there are through the line $\pi_\alpha$ 20 triple tangent planes to the surface; and since there are 10 lines $\pi_i$, the number of triple tangent planes of this type is 200. The total number of triple tangent planes to the surface meeting it in degenerate sextic curves is 255, including those of the preceding paragraphs.

13. Now consider some of the curves on the surface. To obtain the curves of a given order $N$ on the surface, assign values to $M$ and $\sum\alpha_i$ within certain limits in the formula [Paragraph 6]

$$N = 4M - \sum\alpha_i.$$

Thus, the 45 conics are given by $M = 1$, $\sum\alpha_i = 2$, that is, they are given by the lines determined by the 10 base points in $\sigma$. If $M = 1$, $\sum\alpha_i = 1$ or $M = 2$, $\sum\alpha_i = 5$ or $M = 3$, $\sum\alpha_i = 9$, we obtain all the cubics on $\Sigma$. The first case gives rise to 10 $\infty$-families of twisted cubics corresponding to the 10 pencils of lines with vertices at the 10 points $P_i$ in $\sigma$. The surface $\Sigma$ may be considered as being generated by any one of these 10 families of twisted cubics. Each family has one of the 10 lines $\pi_i$ for directrix and contains one member that is a plane nodal cubic [Paragraph 10] and 8 members that touch a given plane.

14. For the case $M = 2$, $\sum\alpha_i = 5$, we have the 252 conics in $\sigma$ through the 10 base points $P_i$ 5 at a time, forming 126 degenerate quartics each having one component through 5 and the other through the remaining 5 of the points $P_i$. These degenerate quartics give rise to 126 degenerate space sextics on $\Sigma$ each composed of two twisted cubics with 4 points in common. There are in all 256 twisted cubics of this system, and each meets 5 of the 10 lines $\pi_i$ once. The case $M = 3$, $\sum\alpha_i = 9$ gives the 10 plane cubics $\mu_i$.

15. To obtain the quartic curves on $\Sigma$, put $N = 4$. Then if $M = 1$, $\sum\alpha_i = 0$, we have $\infty^2$ unicursal quartics on the surface. Any given plane

touches 28 of them twice, osculates 24 and touches once 12 that pass through a given point on the surface. For $M = 2$, $\sum \alpha_i = 4$, there are 210 $\infty$-families of unicursal quartics corresponding to the 210 pencils of conics each through 4 of the 10 points $P_i$ in $\sigma$. Each family has 4 of the lines $\pi_i$ for directrices. There are 2520 curves that touch a given plane, 12 belonging to each family. For $M = 3$, $\sum \alpha_i = 8$ there are 45 $\infty$-families of quartics of the first species on $\Sigma$ corresponding to the 45 pencils of cubics each through 8 of the base points $P_i$ in $\sigma$. Since each pencil of cubics contains 12 nodal members, there are 540 quartics of the second species in this case. There are also 540 that are touched by any given plane, 12 belonging to each family. Each family has 8 of the 10 lines for directrices. There is another case for $M = 3$, $\sum \alpha_i = 8$. A cubic in $\sigma$ passing through 6 base points each once and a 7th twice yields a unicursal quartic on $\Sigma$ meeting 6 lines each once and a 7th twice. There are in all 84 such quartics of this system. We have also the case $M = 4$, $\sum \alpha_i = 12$, for which there are 45 quartics in $\sigma$ through 8 base points each once and the remaining 2 each twice. The corresponding quartics on $\Sigma$ are of the first species and are 45 in number. Each has 8 of the 10 lines for unisecants and the remaining 2 for bisecants.

16. For quintics and sextics we have the following:

$N = 5$.

| | $M$ | $\sum \alpha_i$ | | |
|---|---|---|---|---|
| I | 2 | 3 | 120 | $\infty^2$-families of unicursal quintics. Each curve cuts across 3 lines. |
| II | 3 | 7 | 120 | $\infty^2$-families of quintics of genus 1. Each curve cuts across 7 lines. |
| III | 3 | $5+1(2)$* | 1260 | $\infty$-families of unicursal quintics. Each curve has one line for bisecant and each line is bisecant to 126 families. |
| IV | 4 | $9+1(2)$ | 10 | $\infty^2$-families of quintics of genus 2, each containing an $\infty$-family of plane quintics with 4 nodes and 20 quintics with 5 nodes [Paragraph 12]. |
| V | 4 | $7+2(2)$ | 210 | $\infty$-families of quintics of genus 1. Each curve has 2 lines for bisecants. |
| VI | 4 | $8+1(3)$ | 90 | unicursal quintics each with a line for trisecant. |

* $\sum \alpha_i = \alpha + \alpha'(\nu)$ means that the curve in $\sigma$ passes through $\alpha$ of the base points each once and $\alpha'$ others each $\nu$ times.

| | $M$ | $\sum \alpha_i$ | | |
|---|---|---|---|---|
| VII | 4 | 5+3(2) | 2520 | unicursal quintics each with 3 lines for bisecants. |
| VIII | 5 | 5+5(2) | 252 | quintics of genus 1, each with 5 lines for bisecants. |

$N = 6.$

| | | | | |
|---|---|---|---|---|
| I | 2 | 2 | 45 | $\infty^3$-families of unicursal sextics. |
| II | 3 | 6 | 210 | $\infty^3$-families of sextics of genus 1. |
| III | 3 | 4+1(2) | 1260 | $\infty^2$ families of sextics of genus 0. |
| IV | 4 | 10 | 1 | $\infty^4$-family of sextics of genus 3. $\infty^3$ are plane. |
| V | 4 | 8+1(2) | 10 | $\infty^3$-families of sextics of genus 2. |
| VI | 4 | 6+2(2) | 1260 | $\infty^2$-families of sextics of genus 1. |
| VII | 4 | 7+1(3) | 360 | $\infty$-families of unicursal sextics. Each curve has 1 line for trisecant. |
| VIII | 4 | 4+3(2) | 4200 | $\infty$-families of unicursal sextics. |
| IX | 5 | 6+4(2) | 210 | $\infty^2$-families of sextics with 8 apparent nodes. |
| X | 5 | 7+2(2)+1(3) | 1 | $\infty$-family of sextics of genus 1. |
| XI | 5 | 4+5(2) | 1260 | $\infty$-families with 5 apparent nodes. |
| XII | 5 | 2+6(2) | 1260 | unicursal sextics. |
| XIII | 6 | 2+8(2) | 45 | $\infty$-families of sextics of genus 2. |

17. It is of interest to find the maximum genus that any non-multiple curve $C^N$ of a given order $N$ on the surface can have. If $M$ is the order of the plane curve $K^M$ in $\sigma$ whose image is $C^N$, then we have from Paragraph 13

$$M = \frac{N+\sum \alpha_i}{4}.$$

The genus of $C^N$ is equal to the genus of $K^M$ and the latter is given by

$$\frac{(M-1)(M-2)}{2} - \sum \frac{\alpha_i(\alpha_i-1)}{2}$$

or

$$\frac{(N+\sum \alpha_i-4)(N+\sum \alpha_i-8)}{32} - \sum \frac{\alpha_i(\alpha_i-1)}{2}.$$

The value of this expression is greatest when

$$\alpha_0 = \alpha_1 = \cdots = \alpha_9 = \frac{N+2}{6}$$

and it is then equal to

$$\frac{2N^2-4N+29}{24}.$$

The greatest integer in this last expression is the limit which the genus of any non-multiple curve of order $N$ on the surface cannot exceed and this limit holds for all values of $N$ and suitable values of the $\alpha$'s.

18. We shall now consider briefly the different types of the surface arising from letting the center $Z$ of projection take special positions in 4-space. If the surface described above be thought of as belonging to Type I, we have Type II if we let $Z$ be a point of an $S_3$ determined by $\overline{\pi}_\alpha$ and $\overline{\pi}_\beta$, any two of the lines of $\Sigma_2^6$. The two corresponding lines $\pi_\alpha$ and $\pi_\beta$ on $\Sigma$ meet and their point of meeting is on the nodal curve $\Gamma^7$. Their plane is quadruply tangent to the surface and it is the only one. This plane meets $\Sigma$ in the two lines and a binodal quartic corresponding to a quartic in $\sigma$ through 8 of the base points each once and the remaining 2 each twice. Two of the 4 points of tangency are on $\pi_\alpha$ and the other two on $\pi_\beta$.

19. Suppose $Z$ be in the plane of two $\Sigma$-spaces $S_3$ and $S_3'$ each containing two lines of $\Sigma_2^6$. The surface $\Sigma$ now has two quadruple tangent planes and is said to belong to Type III. If the 4 lines on $\Sigma_2^6$ determining $S_3$ and $S_3'$ are all distinct, there are on $\Sigma$ 2 pairs of intersecting lines and their planes are the quadruple tangent planes; but if $S_3$ and $S_3'$ have one line in common on $\Sigma_2^6$, then on $\Sigma$ there is one line cutting across two others and the two quadruple tangent planes pass through the common transversal.

20. Type IV or Type V arises according as $Z$ is on the line of three 3-spaces or is coincident with the point common to four 3-spaces, each determined by two lines of $\Sigma_2^6$. In the former case $\Sigma$ has 3 quadruple tangent planes and if the three 3-spaces have in pairs a line of $\Sigma_2^6$ in common, then three of the lines of $\Sigma$ are concurrent at the triple point $T$ on $\Gamma^7$. In the latter case $\Sigma$ has four quadruple tangent planes, each containing two of the lines $\pi_i$. There are variations which we need not discuss as they can be inferred readily.

21. According as $Z$ is in a plane met by 5 or 6 of the lines of $\Sigma_2^6$, $\Sigma$ is of Type VI or of Type VII. In the one case, 5 of the lines of $\Sigma$ have a common transversal through which pass 5 double tangent planes; in the other, 6 of the lines of $\Sigma$ have a common transversal through which pass 6 double tangent planes.

22. We have Type VIII, IX, X or XI according as $Z$ is in one, two, three, or four of the 126 3-spaces each meeting $\Sigma_2^6$ in a pair of twisted cubics. The surface $\Sigma$ has respectively one, two, three, or four quadruple tangent planes each meeting it in a pair of nodal cubics.

23. Let $Z$ be in the 3-space determined by the line $\bar{\pi}_\alpha$ and the conic $\bar{\lambda}_{\alpha\beta}$ of $\Sigma_2^6$. This 3-space necessarily contains the cubic $\bar{\mu}_\beta$, for $\bar{\mu}_\beta$ has 2 points in common with $\bar{\lambda}_{\alpha\beta}$ and one point in common with $\bar{\pi}_\alpha$. The projection of this configuration is a degenerate plane sextic on $\Sigma$ with 11 nodes, the components being the line $\pi_\alpha$, the conic $\lambda_{\alpha\beta}$, and the cubic $\mu_\beta$, and its plane is a quadruple tangent plane. This is Type XII.

24. The point $Z$ may be common to two, three or four 3-spaces each of which is determined by a conic and a line of $\Sigma_2^6$ having a point in common. Then we have Type XIII, XIV or XV as the case may be, and the corresponding $\Sigma$ has 2, 3, or 4 quadruple tangent planes each meeting it in a line, a conic, and a cubic (non-singular).

25. Now let $Z$ be in the plane $\varphi_{\alpha\beta}$ of a conic $\bar{\lambda}_{\alpha\beta}$. The corresponding conic $\lambda_{\alpha\beta}$ on $\Sigma$ has now become a double line $\delta$ forming a part of the nodal curve $\Gamma^7$. Since $\bar{\mu}_\alpha$ and $\bar{\mu}_\beta$ on $\Sigma_2^6$ have one point, $Q_3$, in common which is in $\varphi_{\alpha\beta}$, the projecting line $ZQ_3$ meets $\bar{\lambda}_{\alpha\beta}$ in two points $Q_1$ and $Q_2$. Hence $ZQ_3$ meets the $S_3$ containing $\Sigma$ in the point $T$ which is triple on $\Sigma$. Corresponding to $T$ on $\Sigma$ or to $Q_1$, $Q_2$, $Q_3$ on $\Sigma_2^6$ are three points $Q_1'$, $Q_2'$, $Q_3'$ in $\sigma$. $Q_1'$ and $Q_2'$ are on the line $l_{\alpha\beta}$ through $P_\alpha$ and $P_\beta$, and $Q_3'$ is the 9th intersection of the cubics $m_\alpha$ and $m_\beta$. The double line $\delta$ is the image of $l_{\alpha\beta}$. The nodal curve $\Gamma^7$ on $\Sigma$ is now composed of $\delta$ and a $\Gamma^6$ to which corresponds a 10th degree curve $\gamma^{10}$ passing through the points $Q_1'$, $Q_2'$ each once, the points $P_\alpha$, $P_\beta$, $Q_3'$ each twice and the 8 remaining base points each three times. The line $l_{\alpha\beta}$ meets $\gamma^{10}$ in 2 more pairs of points corresponding to the 2 points in which $\delta$ meets $\Gamma^6$. $\Gamma^6$ has a real node on $\delta$ and that is the triple point $T$ on $\Sigma$.

26. Any plane $\psi$ through the line $\delta$ meets $\Sigma$ in a binodal quartic besides. This quartic is the image of a cubic through 8 base points and not through $P_\alpha$ and $P_\beta$ in $\sigma$. There are two positions of $\psi$ in which the binodal quartic decomposes into a line and a non-singular cubic. The planes in these positions are quintuple tangent planes. There are 12 other quintuple tangent planes through $\delta$ as there are 12 nodal members of a pencil of plane cubics. All the other planes through $\delta$ are quadruply tangent to the surface.

27. The type above is XVI and the next type, XVII, is obtained by letting $Z$ coincide with the point of intersection of the two planes $\varphi_{\alpha\beta}$, $\varphi_{\alpha\gamma}$ of two non-intersecting conics $\bar{\lambda}_{\alpha\beta}$, $\bar{\lambda}_{\alpha\gamma}$. The double curve $\Gamma^7$ on $\Sigma$ is now composed of two lines $\delta$ and $\delta'$ and a $\Gamma^5$. The component $\Gamma^5$ meets each of the other two components in 4 points 2 of which are coincident. Corresponding to $\Gamma^5$ is a curve $\gamma^9$ in $\sigma$ which passes through $P_\alpha$ once, $P_\beta$ and $P_\gamma$ twice and the remaining 7 base points each three times and has two other double points. The surface $\Sigma$ has two triple points, and two of its 45 conics have disappeared.

28. If $Z$ be in the plane $\varphi_\alpha$ of a cubic $\bar{\mu}_\alpha$ of $\Sigma_2^6$, there is on $\Sigma$ a triple line $\epsilon$. The double curve $\Gamma^7$ is now composed of $\epsilon$ regarded as a double cubic and a unicursal quartic $\Gamma^4$. There are 6 pinch-points on $\epsilon$. The curve $\gamma^{11}$ in $\sigma$ consists of $m_\alpha$ counted twice and a $\gamma^5$ which passes through $P_\alpha$ three times and each of the remaining 9 base points once. Any plane $\psi$ through $\epsilon$ meets $\Sigma$ in a nodal cubic besides and may be considered as a sextuple tangent plane. Since $\Sigma$ has 9 lines [not $\pi_\alpha$] meeting $\epsilon$, there are 9 planes through $\epsilon$ that are tangent to it again. The plane cubic $\mu_\alpha$ has disappeared. In this case the surface is of type XVIII.

29. Suppose $Z$ be on the line of intersection of the plane of the conic $\bar{\lambda}_{\alpha\beta}$ and the plane of the cubic $\bar{\mu}_\alpha$. The surface $\Sigma$ is now of type XIX. It contains a double line $\delta$ and a triple line $\epsilon$ lying in a plane. The double curve $\Gamma^7$ degenerates into the line $\delta$, the line $\epsilon$ and a cubic $\Gamma^3$. Corresponding to $\Gamma^3$ there is in $\sigma$ a quartic $\gamma^4$ through $P_\alpha$ twice and each of 8 base points once and not through the point $P_\beta$. The plane of $\delta$ and $\epsilon$ meets $\Sigma$ in a degenerate sextic composed of $\delta$, $\epsilon$, $\pi_\beta$ counted twice, three times, and once respectively. The surface contains only 44 conics.

30. Now it remains to consider the special cases of the surface arising from taking special positions of the plane $\sigma$ or, what amounts to the same thing, from imposing certain conditions on the 10 base points in $\sigma$. If $P_\alpha$, $P_\beta$, $P_\gamma$ are three collinear base points, we have Case II (if we regard the general case as Case I) and the surface $\Sigma$ has an extra line $\xi_{\alpha\beta\gamma}$. This line cuts across the three lines $\pi_\alpha$, $\pi_\beta$, $\pi_\gamma$ and is the image of the line $x_{\alpha\beta\gamma}$ through $P_\alpha$, $P_\beta$, $P_\gamma$. Three of the conics, $\lambda_{\alpha\beta}$, $\lambda_{\beta\gamma}$, $\lambda_{\gamma\alpha}$ are degenerate and have the line $\xi_{\alpha\beta\gamma}$ for common component. There remain only 42 conics on the surface.

31. If, in addition, three other base points, $P_{\alpha'}$, $P_{\beta'}$, $P_{\gamma'}$ in $\sigma$ are on another line $x_{\alpha'\beta'\gamma'}$, then $\Sigma$ (Case III) has another line $\xi_{\alpha'\beta'\gamma'}$, which meets the lines $\pi_{\alpha'}$, $\pi_{\beta'}$, $\pi_{\gamma'}$ and also the line $\xi_{\alpha\beta\gamma}$. Three more conics have degenerated and there remain only 39 conics on the surface. If $P_\alpha \equiv P_{\alpha'}$, i. e., if one base point $P_\alpha$ is on the intersection of the two lines $x_{\alpha\beta\gamma}$ and $x_{\alpha\beta'\gamma'}$, the two extra lines $\xi_{\alpha\beta\gamma}$ and $\xi_{\alpha\beta'\gamma'}$ on $\Sigma$ are skew and have the line $\pi_\alpha$ for transversal.

32. Suppose that 4 base points, $P_\alpha$, $P_\beta$, $P_\gamma$, $P_\delta$, in $\sigma$ are on a line $x_{\alpha\beta\gamma\delta}$. Then the corresponding surface $\Sigma$ (Case V) has besides $T$ another triple point $X$. The curve $\gamma^{11}$ in $\sigma$ whose image is $\Gamma^7$ on $\Sigma$ degenerates into the line $x_{\alpha\beta\gamma\delta}$ and a curve $\gamma^{10}$ of order 10 passing through the 4 collinear points each twice and the remaining 6 each three times. The two components meet again twice, and hence the triple point $X$ on $\Sigma$ is on $\Gamma^7$. The 4 lines $\pi_\alpha$, $\pi_\beta$, $\pi_\gamma$, $\pi_\delta$ are concurrent at this triple point, and they form 6 degenerate conics of the surface. Of the 39 proper conics 15 meet at $X$

and the remaining 24 are in 4 groups of 6 each, the members of each group being incident with one of the 4 concurrent lines. Six of the cubics $\mu_i$ have become nodal with their nodes coincident with $X$.

33. It may happen that a base point $P_\alpha$, in addition to being collinear with $P_\beta$, $P_\gamma$, $P_\delta$, is collinear with three others, $P_{\beta'}$, $P_{\gamma'}$, $P_{\delta'}$, then there are on the surface $\Sigma$ (Case VI) three triple points $T$, $X$, $X'$. The curve $\gamma^{11}$ in $\sigma$ is now composed of the two lines $x_{\alpha\beta\gamma\delta}$, $x_{\alpha\beta'\gamma'\delta'}$ and a curve $\gamma^9$ of order 9 passing through $P_\alpha$ once, $P_\beta$, $P_\gamma$, $P_\delta$, $P_{\beta'}$, $P_{\gamma'}$, $P_{\delta'}$ each twice and the three remaining base points each three times and meeting each of the other two components again twice. Hence the double curve $\Gamma^7$ passes through the triple points $X$, $X'$ each once and meets the line $\pi_\alpha$ once. The three free base points in $\sigma$ and the three nodes $T_1$, $T_2$, $T_3$ [Paragraph 7] on $\gamma^9$ lie on a conic $\varkappa$. There are only 24 proper conics on the surface, and 3 of them meet twice, once at $X$ and once at $X'$. The six cubics $\mu_\beta$, $\mu_\gamma$, $\mu_\delta$, $\mu_{\beta'}$, $\mu_{\gamma'}$, $\mu_{\delta'}$ are nodal, the former 3 having their nodes at $X$, the latter 3 having theirs at $X'$. Any plane $\theta$ through $\pi_\alpha$ meets $\Sigma$ again in a unicursal quintic with one node at $X$, another at $X'$. Corresponding to this quintic is a conic $\eta$ through the three free base points. If $\eta$ passes through $P_\alpha$, the plane $\theta$ contains $\pi_\alpha$ counted double and a unicursal quartic; if $\eta$ passes through $P_\beta$, $\theta$ contains $\pi_\alpha$ and also the line $\pi_\beta$ and a unicursal quartic; if $\eta$ is the conic $\varkappa$ mentioned above, $\theta$ passes through the other triple point $T$ also.

34. We shall take up two more cases, VII and VIII. In the former, six, and in the latter, seven, of the base points of $\sigma$ lie on a conic. The surface $\Sigma$ has in the one case an extra conic meeting 6 lines and in the other case an extra line meeting 7 others. The plane of the conic meets the surface in a unicursal quartic besides and is therefore a quadruple tangent plane. Any plane through the extra line meets the surface in a unicursal quintic besides and is therefore a quadruple tangent plane. There are 10 positions in which this plane is quintuply tangent.

35. There are various other cases that might be considered, which arise if $\sigma$ touches or osculates the curve $j_1^{10}$ [Paragraph 2] or is in some other relation with it. Each of these cases and of those already considered can have different types if we take different positions for the center $Z$ of projection. We shall not discuss these types which are indeed numerous but shall proceed to show that in the general case the surface $\Sigma$ can have at most 9 isolated nodes, i. e., nodes apart from the double curve $\Gamma^7$. We recall that the tangent cone is of order 16 and class 27. Since the 9th degree curve $t$ in $\sigma$ corresponding to the curve $\tau$ of contact has only 10 nodes, the cone is of genus 18 and consequently has 48 nodal and 39 cuspidal elements, the former touching the surface doubly and the latter

inflexionally. Any plane section of this cone not through its vertex, being a curve of order 16 with 48 nodes and 39 cusps, can have as many as 9 additional nodes.* Hence the surface can have as many as 9 isolated nodes and consequently can be of class as low as 9. The existence of nodes, any number up to 9, or of points of higher singularity such as binodes and unodes resulting from the union of two or more nodes, necessarily causes changes in the relative positions of the lines, conics, etc., on the surface. The number of species of the surface of the general type according to the number and kinds of singular points is over 350, and since each of the types discussed in Paragraphs 17–29 can be so classified, the total number of species is enormous. As we do not attempt to give a complete classification of these surfaces we shall omit further details which can be obtained or inferred readily in any particular case.

* Hollcroft, Singularities of curves of given order, Bull. Amer. Math. Soc., vol. 29, pp. 407–414.

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# DETERMINATION OF A BASIS FOR THE INTEGRAL ELEMENTS OF CERTAIN GENERALIZED QUATERNION ALGEBRAS.*

By M. D. Darkow.

1. The algebras of generalized quaternions form a two-parameter family, each of which consists of the totality of elements $X = \sum_{i=0}^{3} x_i e_i$ with rational coördinates $x_i$, subject to the multiplication table

$$e_0 e_i = e_i = e_i e_0 \ (i = 0, 1, 2, 3), \quad e_1^2 = \alpha, \quad e_2^2 = -\beta, \quad e_3^2 = \alpha\beta,$$
$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_1 e_3 = -e_3 e_1 = \alpha e_2 \quad e_2 e_3 = -e_3 e_2 = \beta e_1,$$

where $\alpha$ and $\beta$ are fixed integers, which by a linear transformation may be made to contain no square factor $>1$.

Latimer† has developed a method for determining a basis for the integral elements of those algebras for which $\alpha \equiv \beta \equiv 1, \mod 2$, and $\alpha \equiv 2$, $\beta \equiv 1, \mod 8$. The purpose of the following paper is to obtain corresponding results for those algebras $\mathfrak{A}$ for which $\alpha \equiv \beta \equiv 2, \mod 4$, and in particular to throw some light on the train of thought employed where the method is tentative.

2. Conjugate, norm, rank equation, integral element are defined as in Latimer's paper. The preliminary theorems for which the proofs parallel those in Latimer's paper, will be briefly enumerated.

If $X$ is an integral element, the coefficients of the rank equation for $e_i X$ must be integers. Hence

Theorem 1. *A necessary condition that $X$ be an integral element of $\mathfrak{A}$ is that it may be written in the form* (1), *with the $u_i$ integers:*

$$X = \frac{1}{2}\left(u_0 + \frac{u_1 e_1}{\alpha} + \frac{u_2 e_2}{\beta} + \frac{u_3 e_3}{\alpha\beta}\right). \tag{1}$$

If $X$ is an integral element, $N(X)$ is an integer. Hence

Theorem 2. *An element of form* (1) *has the property $R$ if and only if the $u_i$ are integers satisfying*

$$\alpha\beta u_0^2 - \beta u_1^2 + \alpha u_2^2 - u_3^2 \equiv 0 \mod 4\alpha\beta. \tag{2}$$

* Received October 11, 1926.

† Amer. Journ. of Math., vol. 48 (1926), pp. 57.

Write $\alpha = 2\mu\delta$, $\beta = 2\nu\delta$, where $\mu, \nu, \delta$ are odd integers, and $2\delta$ is the positive g. c. d. of $\alpha$ and $\beta$. Then $\mu, \nu, \delta$ are relatively prime in pairs. These, in theorem 2, give

THEOREM 3. *An element of form* (1) *has property R if and only if* $u_0, u_1, u_2$, *and* $v_3 = \frac{u_3}{2\delta}$ *be integers satisfying*

$$2\mu\nu\delta u_0^2 - \nu u_1^2 + \mu u_2^2 - 2\delta v_3^2 \equiv 0 \bmod 8, \tag{3}$$

$$\nu u_1^2 + 2\delta v_3^2 \equiv 0 \bmod \mu, \tag{4}$$

$$\mu u_2^2 - 2\delta v_3^2 \equiv 0 \bmod \nu, \tag{5}$$

$$-\nu u_1^2 + \mu u_2^2 \equiv 0 \bmod \delta. \tag{6}$$

3. Let $M_i(N_j, D_k)$ be the positive prime factors of $\mu(\nu, \delta)$ of which $-\beta(\alpha, \mu\nu)$ are quadratic non-residues. Write $M = \prod_i M_i (N = \prod_j N_j$, $D = \prod_k D_k)$. If there are no positive prime factors having this property, write $M = 1$, $(N = 1, D = 1)$. Write $\mu = \mu' M$, $\nu = \nu' N$, $\delta = \delta' D$. $M, \mu', N, \nu', D, \delta'$ are odd and relatively prime in pairs. By the use of quadratic residues, we show that $u_1, u_2, v_3$ are multiples of $MD$, $ND$, $MN$, respectively. Hence

THEOREM 4. *X is an integral element of* $\mathfrak{A}$, *only if it may be written in the form* (7)

$$X = \frac{1}{4}\left(2w_0 + \frac{w_1 e_1}{\mu'\delta'} + \frac{w_2 e_2}{\nu'\delta'} + \frac{w_3 e_3}{\mu'\nu'\delta}\right) \tag{7}$$

*with* $w_i (i = 0, 1, 2, 3)$ *integers.*

THEOREM 5. *An element of form* (7) *has property R if and only if* $w_i (i = 0, 1, 2, 3)$ *are integers satisfying*

$$2\mu\nu w_0^2 - \nu\delta w_1^2 + \mu\delta w_2^2 - 2w_3^2 \equiv 0 \bmod 8, \tag{8}$$

$$\nu' D w_1^2 + 2N\delta' w_3^2 \equiv 0 \bmod \mu', \tag{9}$$

$$\mu' D w_2^2 - 2M\delta' w_3^2 \equiv 0 \bmod \nu', \tag{10}$$

$$-M\nu' w_1^2 + \mu' N w_2^2 \equiv 0 \bmod \delta'. \tag{11}$$

In order to reduce quadratic congruences (9), (10), (11) to linear congruences, we show that there exist odd integers $\xi, \eta, \zeta$, prime to $\mu', \nu', \delta'$ respectively, such that $\beta\xi, \alpha\eta, \mu\nu\zeta$ are solutions of

$$x^2 + \beta \equiv 0 \bmod \mu', \tag{12}$$

$$y^2 - \alpha \equiv 0 \bmod \nu', \tag{13}$$

$$z^2 - \mu\nu \equiv 0 \bmod \delta', \tag{14}$$

respectively. Factor $|\mu'|, |\nu'|, |\delta'|$ into their positive prime factors:

$$|\mu'| = \mu_1' \cdots \mu_m', \quad |\nu'| = \nu_1' \cdots \nu_n', \quad |\delta'| = \delta_1' \cdots \delta_p'.$$

Theorem 5 then leads to

THEOREM 6: *A necessary and sufficient condition that an element of form* (7) *have property R is that the $w_i$ be integers satisfying* (8) *and one congruence from each of the following $m+n+p = q$ pairs of congruences*

$$(15) \quad w_1 \equiv 2N\delta'\xi w_3, \quad w_1 \equiv -2N\delta'\xi w_3, \bmod \mu_i' \qquad (i = 1, \cdots, m),$$

$$(16) \quad w_2 \equiv 2M\delta'\eta w_3, \quad w_2 \equiv -2M\delta'\eta w_3, \bmod \nu_j' \qquad (j = 1, \cdots, n),$$

$$(17) \quad w_2 \equiv M\nu'\zeta w_1, \quad w_2 \equiv -M\nu'\zeta w_1, \bmod \delta_k' \qquad (k = 1, \cdots, p).$$

4. Choose an arbitrary set $K_i\,(i = 1, \cdots, 2^q)$ of congruences, one from each pair (15), (16), (17). Supplementary quantities $E_i, F_i, H_i$ are introduced, to enable us, by the elimination of $\xi, \eta, \zeta$, to unite the $m$ congruences of $K_i$, chosen from (15), into one congruence, and similarly the $n$ congruences chosen from (16) and the $p$ chosen from (17). The only properties of $E_i, F_i, H_i$, that enter, are the following: $E_i, F_i, H_i$ are odd, prime to $\mu', \nu', \delta'$ respectively, and satisfy

$$(18) \quad \begin{aligned} 1 + \beta E_i^2 &\equiv 0 \bmod \mu', \\ 1 - \alpha F_i^2 &\equiv 0 \bmod \nu', \\ 1 - \mu\nu H_i^2 &\equiv 0 \bmod \delta'. \end{aligned}$$

Thus we obtain

THEOREM 7. *A sufficient condition that an element X of form* (7) *have property R is that the coördinates $w_i$ of X relative to form* (7) *be integers which satisfy* (8) *and one congruence of each pair of equivalent congruences*

$$(19) \quad w_1 \equiv 2N\delta' E_i w_3 \quad \text{or} \quad (19') \quad w_3 \equiv -\nu' D E_i w_1, \bmod \mu',$$

$$(20) \quad w_2 \equiv 2M\delta' F_i w_3 \quad \text{or} \quad (20') \quad w_3 \equiv \mu' D F_i w_2, \bmod \nu',$$

$$(21) \quad w_2 \equiv M\nu' H_i w_1 \quad \text{or} \quad (21') \quad w_1 \equiv \mu' N H_i w_2, \bmod \delta'.$$

5. Define the class $ijk$ of algebras $\mathfrak{A}$ as the class of algebras for which

$$\begin{aligned} \mu\delta &\equiv i \bmod 8, \\ \nu\delta &\equiv j \bmod 8, \\ \mu\nu &\equiv k \bmod 8. \end{aligned}$$

Since $ij \equiv k \bmod 8$, we can define algebras of four types:

| $\mathfrak{A}$ of type | $\mathfrak{A}$ of class | | | |
|---|---|---|---|---|
| A | 155 | 375 | 515 | 735 |
| B | 133 | 357 | 573 | 717 |
| C | 111 | 331 | 551 | 771 |
| D | 177 | 313 | 537 | 753 |

The following table shows the sets of congruences to which (8) is equivalent, for $\mathfrak{A}$ of type A, B, C, D respectively.

| | (8) equivalent to | $\mathfrak{A}$ of type |
|---|---|---|
| (22a) | $w_0 \equiv w_3 \bmod 2,\quad w_1 \equiv w_2 \equiv 0 \bmod 4,$ | A |
| or | | |
| (22b) | $w_0 \equiv w_3 \bmod 2,\quad w_1 \equiv w_2 \equiv 2 \bmod 4.$ | |
| (23a) | $w_0 \equiv w_3 \equiv 1 \bmod 2,\quad w_1 \equiv 2 \bmod 4,\quad w_2 \equiv 0 \bmod 4,$ | B |
| or | | |
| (23b) | $w_0 \equiv w_3 \equiv 1 \bmod 2,\quad w_1 \equiv 0 \bmod 4,\quad w_2 \equiv 2 \bmod 4,$ | |
| or | | |
| (23c) | $w_0 \equiv w_3 \equiv 0 \bmod 2,\quad w_1 \equiv w_2 \equiv 2 \bmod 4,$ | |
| or | | |
| (23d) | $w_0 \equiv w_3 \equiv 0 \bmod 2,\quad w_1 \equiv w_2 \equiv 0 \bmod 4.$ | |
| (24a) | $w_0 \equiv w_3 \bmod 2,\quad w_1 \equiv w_2 \equiv 0 \bmod 4,$ | C |
| or | | |
| (24b) | $w_0 \equiv w_3 \bmod 2,\quad w_1 \equiv w_2 \equiv 2 \bmod 4,$ | |
| or | | |
| (24c) | $w_0 \equiv w_3 \bmod 2,\quad w_1 \equiv w_2 \equiv 1 \bmod 2.$ | |
| (25a) | $w_0 \equiv w_3 \equiv 0 \bmod 2,\quad w_1 \equiv w_2 \equiv 2 \bmod 4,$ | D |
| or | | |
| (25b) | $w_0 \equiv w_3 \equiv 0 \bmod 2,\quad w_1 \equiv w_2 \equiv 0 \bmod 4,$ | |
| or | | |
| (25c) | $w_0 \equiv w_3 \equiv 1 \bmod 2,\quad w_1 \equiv 2 \bmod 4,\quad w_2 \equiv 0 \bmod 4,$ | |
| or | | |
| (25d) | $w_0 \equiv w_3 \equiv 1 \bmod 2,\quad w_1 \equiv 0 \bmod 4,\quad w_2 \equiv 2 \bmod 4,$ | |
| or | | |
| (25e) | $w_0 \equiv w_3 \equiv 1 \bmod 2,\quad w_1 \equiv w_2 \equiv 1 \bmod 2.$ | |

The problem of sections 6–9 is to use the information obtained in sections 3 and 5 in order to divine the desired basal units. The method is tentative.

**6. $\mathfrak{A}$ of type A.** The bases of sets $S_i$ of integral elements are given by

$$A_0 = 1,$$
$$A_1 = \frac{1}{2}\left(1+\frac{e_3}{2\delta}\right),$$
$$A_2 = \frac{1}{\mu'}(e_1 - E_i e_3),$$
$$A_3 = \frac{1}{2}\left(\frac{NH_i e_1}{\delta'} + \frac{e_2}{\nu'\delta'} + \frac{F_i e_3}{\nu'\delta'}\right).$$

These results are obtained from the following reflections: Every integral element $X$ is expressible in form (7) and also as $\sum a_\lambda U_\lambda$, where the $U_\lambda$ are the desired basal units and the $a_\lambda$ are integers. Since $w_2$ is even by (22), form (7) suggests that we try $(2\nu'\delta')^{-1} e_2$ as a term of one basal unit, $U_3$, the other basal units not to contain $e_2$; for then, by equating coefficients of $e_2$, we obtain $a_3 = \frac{1}{2}w_2 =$ an integer.

$U_3$ must be of form (7). If its coördinates relative to form (7) are $u_\lambda$, $u_2 = 2$. Congruences (20′), (21′), (22) suggest respectively that we take $u_3 = 2\mu' DF_i$, $u_1 = 2\mu' NH_i$, $u_0 = 0$, and hence $U_3 = A_3$. In

$$X - a_3 A_3 = \frac{w_0}{2} + \frac{w_1 - \mu' NH_i w_2}{4\mu'\delta'} e_1 + \frac{w_3 - \mu' DF_i w_2}{4\mu'\nu'\delta} e_3, \tag{26}$$

the coefficient of $(\mu')^{-1} e_1$ is an integer, by (21′), (22). This suggests that we try $(\mu')^{-1} e_1$ as a term of a unit, $U_2$, $U_0$ and $U_1$ not to contain $e_1$, since then we obtain $a_2 =$ the coefficient of $(\mu')^{-1} e_1 =$ an integer. If the coördinates of $U_2$ relative to form (7) be now taken as $u_\lambda$, we have $u_1 = 4\delta'$. Congruences (19′), (20), (21), (22) suggest that we try $u_3 = -4\nu'\delta E_i$, $u_2 = 0$, $u_0 = 0$, and hence $U_2 = A_2$. In

$$X - a_3 A_3 - a_2 A_2 = \frac{w_0}{2} + \frac{w_3 - \mu' DF_i w_2 + 4\nu'\delta E_i a_2}{4\mu'\nu'\delta} e_3,$$

by (19′), (20′), the coefficient of $(4\delta)^{-1} e_3$ is an integer $= a_1 \equiv w_3 \bmod 2$. As above, we try $(4\delta)^{-1} e_3$ as a term of the unit $U_1$, the coördinates relative to form (7) of which are therefore $u_3 = \mu'\nu'$, and, by (19), (20), (22), $u_1 = u_2 = 0$, $u_0 = 1$. Hence $U_1 = A_1$, and

$$X - a_1 A_1 - a_2 A_2 - a_3 A_3 = \frac{1}{2}(w_0 - a_1).$$

By (22), $\frac{1}{2}(w_0 - a_1)$ is an integer $= a_0$. Therefore try $U_0 = A_0 = 1$.

We have now expressed every element of form (7) that satisfies (19) or (19′), (20) or (20′), (21) or (21′), (22), and hence every integral element in an algebra of type A, as a linear combination of the $A_j$ with integral coefficients. The totality of such linear combinations of the $A_j$ may now be easily proved to have the properties $R$, $C$, $U$, $M$, and the $A_j$ therefore to be a basis of $\mathfrak{A}$.

7. $\mathfrak{A}$ **of type** B. The bases of sets $S_i$ of integral elements are given by

$$B_0 = 1, \quad B_1 = e_1, \quad B_2 = (4\mu'\delta)^{-1}(2\mu'\delta + \beta E_i e_1 + e_2), \quad B_3 = A_3.$$

These units were obtained as follows. Since $w_2$ is again even when $\mathfrak{A}$ is of type B, we can argue as in the case of type A, with (23) in place of (22), that we may try $A_3$ as a unit. Then, in (26), the coefficient of $(4\mu'\delta)^{-1}e_3$ is an integer $b_2$ by (20′). This suggests that we try $(4\mu'\delta)^{-1}e_3$ as a term of a unit $U_2$ having coördinates $u_\lambda$ relative to form (7), the units $U_1$ and $U_0$ not to contain $e_3$. Thus we try $u_3 = \nu'$, and by (19), (20), (23), $u_1 = 2\nu\delta' E_i$, $u_2 = 0$, $u_0 = 1$, and $U_2 = B_2$. In

$$X - a_3 A_3 - b_2 B_2 = \frac{w_0}{2} - \frac{w_3 - \mu' D F_i w_2}{2\nu'} + \left(\frac{w_1 - \mu' N H_i w_2}{4\mu'\delta'} - \frac{w_3 - \mu' D F_i w_2}{2\mu'} \cdot N E_i\right) e_1,$$

by (20′), (23), and by (21′), (19), (23), respectively, the coefficients of 1 and $e_1$ are integers $b_0$, $b_1$.

Hence every integral element of an algebra of type B is a linear combination of the $B_j$ with integral coefficients. The set of all linear combinations of the $B_j$ with integral coefficients may then be proved to have properties $R$, $C$, $U$, $M$, and hence the $B_j$ to form a basis of $\mathfrak{A}$.

8. $\mathfrak{A}$ **of type** C. In order to obtain bases of sets $S_i[T_i]$, of integral elements, it was found convenient to subdivide (24c) into

$$(24c_1) \quad w_0 \equiv w_3 \bmod 2, \quad w_1 \equiv w_2 \equiv 1 \bmod 2, \quad w_1 \equiv w_2 \bmod 4,$$
$$[(24c_2)] \quad w_0 \equiv w_3 \bmod 2, \quad w_1 \equiv w_2 \equiv 1 \bmod 2, \quad w_1 \not\equiv w_2 \bmod 4,$$

and to associate with (24c$_1$) and [(24c$_2$)] the normalizations

$$(27) \qquad N\delta' E_i \equiv M\delta' F_i \equiv M\nu' H_i \equiv 1\,[3] \bmod 8,$$

respectively. The bases of sets $S_i[T_i]$ of integral elements are

$$C_0 = 1, \quad C_1 = e_2, \quad C_2 = (4\delta')^{-1}(e_1 + \mu H_i e_2),$$
$$C_3 = \frac{1}{4}\left(2 + \frac{2NE_i e_1}{\mu'} + \frac{2MF_i e_2}{\nu'} + \frac{e_3}{\mu'\nu'\delta}\right).$$

These are obtained from the following considerations. Form (7) suggests that we may try a unit $U_3$ containing $(4\mu'\nu'\delta)^{-1}e_3$, no other unit to contain $e_3$. If $u_\lambda$ are the coördinates of $U_3$ relative to form (7), $u_3 = 1$. By (19), (20), (24), we try $u_1 = 2N\delta' E_i$, $u_2 = 2M\delta' F_i$, $u_0 = 1$ and hence $U_3 = C_3$. In

$$X - w_3 C_3 = \frac{1}{2}(w_0 - w_3)$$
$$+ (4\mu'\nu'\delta)^{-1}[(\nu' D w_1 - \beta E_i w_3)e_1 + (\mu' D w_2 - \alpha F_i w_3)e_2],$$

by (19), the coefficient of $(4\delta')^{-1}e_1$ is an integer $c_2$. If $u_\lambda$ are the coördinates relative to form (7) of a unit $U_2$ containing the term $(4\delta')^{-1}e_1$, $u_1 = \mu'$, and by (19'), (21), (24), we may try $u_3 = 0$, $u_2 = \mu\nu' H_i$, $u_0 = 0$, and hence $U_2 = C_2$. In

$$X - w_3 C_3 - c_2 C_2 = \frac{1}{2}(w_0 - w_3)$$
$$+\left[\frac{w_2 - 2M\delta' F_i w_3}{4\nu'\delta'} - \frac{MH_i(w_1 - 2N\delta' E_i w_3)}{4\delta'}\right]e_2,$$

the coefficients of 1 and $e_2$ are integers as a result of (20), (21), (24), (27).

Hence every integral element of an algebra of type C is a linear combination with integral coefficients of the $C_j$. The set of all such linear combinations may then be proved to have properties $R$, $C$, $U$, $M$, and the $C_j$ to form a basis of $\mathfrak{A}$.

9. $\mathfrak{A}$ **of type** D. In order to obtain bases of sets $S_i[T_i]$ of integral elements, (25e) was subdivided into the pairs

(25e$_1$) $w_0 \equiv 0 \bmod 2$, $w_1 \equiv w_2 \equiv w_3 \equiv 1 \bmod 2$, $w_1 \equiv w_2 \bmod 4$,
(25e$_2$) $w_3 \equiv 0 \bmod 2$, $w_1 \equiv w_2 \equiv w_0 \equiv 1 \bmod 2$, $w_1 \not\equiv w_2 \bmod 4$,
and
[(25e$_3$)] $w_0 \equiv 0 \bmod 2$, $w_1 \equiv w_2 \equiv w_3 \equiv 1 \bmod 2$, $w_1 \not\equiv w_2 \bmod 4$,
[(25e$_4$)] $w_3 \equiv 0 \bmod 2$, $w_1 \equiv w_2 \equiv w_0 \equiv 1 \bmod 2$, $w_1 \equiv w_2 \bmod 4$.

With these pairs, as indicated by the brackets, the following normalizations of $E_i$, $F_i$, $H_i$ were associated:

(28) $$\nu' D E_i \equiv \mu' D F_i \equiv \mu' N H_i \equiv 1 \ [3] \bmod 8.$$

The bases of sets $S_i[T_i]$ of integral elements are given by

$$D_0 = 1, \quad D_1 = e_1, \quad D_2 = \frac{1}{4}\left(2 + \frac{2\nu E_i e_1}{\mu'} + \frac{e_3}{\mu'\delta}\right),$$
$$D_3 = \frac{1}{4\delta'}\left(NH_i e_1 + \frac{e_2}{\nu'} + \frac{F_i e_3}{\nu'}\right).$$

These are obtained from the following considerations: Form (7) suggests that we may try a unit $U_3$, having coördinates $u_\lambda$ relative to form (7), with a term $(4\nu'\delta')^{-1}e_2$. Then $u_2 = 1$, and (20′), (21′), (25) suggest that $u_3 = \mu' DF_i$, $u_1 = \mu' NH_i$, $u_0 = 0$, and hence that $U_3 = D_3$. In

$$X - w_2 D_3 = \frac{1}{2} w_0 + \frac{w_1 - \mu' NH_i w_2}{4\mu'\delta'} e_1 + \frac{w_3 - \mu' DF_i w_2}{4\mu'\nu'\delta} e_3,$$

the coefficient of $(4\mu'\delta)^{-1}e_3$ is by (20′) an integer $d_2$. If a unit $U_2$ having coördinates $u_\lambda$ relative to form (7), contain $(4\mu'\delta)^{-1}e_3$ as a term, $u_3 = \nu'$, and by (19), (20), $u_1 = 2\nu\delta' E_i$, $u_2 = 0$, $u_0 = 1$, and hence $U_2 = D_2$. In

$$X - w_2 D_3 - d_2 D_2 = \left(\frac{w_0}{2} - \frac{w_3 - \mu' DF_i w_2}{2\nu'}\right) + \frac{w_1 - \mu' NH_i w_2 - (w_3 - \mu' DF_i w_2) 2N\delta' E_i}{4\mu'\delta'} e_1,$$

the coefficients of 1 and $e_1$ are integers by (20′), (25), (19), (21′).

Hence every integral element of an algebra of type D is a linear combination with integral coefficients of the $D_j$. The set of all such linear combinations may then be proved to have properties $R$, $C$, $U$, $M$, and the $D_j$ to form a basis of $\mathfrak{A}$.

# EXPANSIONS IN BESSEL FUNCTIONS.*

By M. H. Stone.

The object of the present paper is to carry through a discussion of expansions in Bessel functions by methods closely allied to those which have proved successful in the study of Sturm-Liouville and Birkhoff series. The first step in the treatment is the formulation of the expansion problem in terms of a differential equation with boundary conditions. The procedure is heuristic in the sense that it leans heavily on analogies. The question of convergence is then considered in a manner which has many points of contact with the discussion given by Watson in his recent treatise.† The method is that used by the author in a memoir on Birkhoff series.‡ Our treatment is valid for functions of complex order and seems to represent an improvement in certain other respects, since it gives fuller results concerning convergence and summability and throws more light on the behavior of the expansions at $x = 1$. It appears also to be rather more direct.

Most of the facts deduced concerning expansions in Bessel functions are matters of record in the case of functions of real order. W. H. Young has already obtained them in an entirely different manner.§ In favor of the method set forth here, it may be said that we require weaker asymptotic formulae and arrive at the results more directly.

## I. The form of the expansion problem.

The differential equation

$$u'' + \left(\varrho^2 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right) u = 0, \qquad 0 \leqq x \leqq 1, \tag{1}$$

where $\varrho$ and $\nu$ are complex parameters, is satisfied by the function

$$u = c_1 \sqrt{x}\, H_\nu^{(1)}(\varrho x) + c_2 \sqrt{x}\, H_\nu^{(2)}(\varrho x)$$

where $H_\nu^{(1)}$, $H_\nu^{(2)}$ are the Bessel functions of the third kind.|| We shall require the real part of $\nu$ to be greater than or equal to $-\frac{1}{2}$. We write

* Presented, in part, to the American Mathematical Society, October 31, 1925. Received November 4, 1926.

† Watson, The Theory of Bessel Functions, Cambridge, 1922, Chapter XVII. We refer to this work as W throughout the paper.

‡ Stone, Trans. Amer. Math. Soc., vol. 28 (1926), pp. 695–761; this paper will be referred to henceforth as S.

§ W. H. Young, Proc. Lond. Math. Soc., (2), vol. 18 (1920), pp. 163–200, referred to henceforth as Y.

|| W 3.6, 3.61, 3.62, 4.3.

$W(u) \equiv u(1)$ or $W(u) \equiv u'(1) + Au(1)$ where $A$ is a real or complex constant. To the equation (1) we adjoin the conditions

$$c_1 - c_2 = 0, \tag{2}$$

$$W(u) = 0. \tag{3}$$

The problem of representing an arbitrary function in terms of the solutions of this differential system leads to Fourier-Bessel and Dini expansions and generalisations thereof. For instance, if we take $\nu$ real and $W(u) \equiv u(1)$, the solutions of the differential system are $\sqrt{x}\, J_\nu(\varrho_k x)$, $J_\nu(\varrho_k) = 0$; and the problem of expanding the function $\sqrt{x} f(x)$ in terms of these solutions is that of expanding $f(x)$ in Fourier-Bessel series:

$$f(x) \sim a_1 J_\nu(\varrho_1 x) + a_2 J_\nu(\varrho_2 x) + \cdots, \qquad a_k = \frac{\int_0^1 y f(y) J_\nu(\varrho_k y)\, dy}{\int_0^1 y J_\nu^2(\varrho_k y)\, dy}.$$

A special case of Fourier-Bessel series defined in this manner is the sine series on $(0, 1)$; and a special case of Dini series is the cosine series on that interval. If in (1) we take $\nu = \pm \frac{1}{2}$, the condition (2) restricts us to solutions linearly dependent upon

$$\sqrt{\varrho x}\, J_{-\frac{1}{2}}(\varrho x) = \sqrt{2/\pi} \cos \varrho x, \qquad \sqrt{\varrho x}\, J_{+\frac{1}{2}}(\varrho x) = \sqrt{2/\pi} \sin \varrho x.$$

Consequently, if in the system (1), (2), (3) we take $\nu = \frac{1}{2}$ and $W(u) \equiv u(1)$, we obtain the sine series; and if we take $\nu = -\frac{1}{2}$, $W(u) \equiv u'(1)$, we obtain the cosine series. We shall show that on any interval $(a, 1)$, $a > 0$, these trigonometric series are models for the corresponding series in Bessel functions.

By the method of variation of constants we find that the general solution of the non-homogeneous equation

$$u'' + \left(\varrho^2 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right) u = \varphi(x)$$

is given by

$$u = c_1 \sqrt{x}\, H_\nu^{(1)}(\varrho x) + c_2 \sqrt{x}\, H_\nu^{(2)}(\varrho x) + \int_0^1 g_\nu(x, y; \varrho)\, \varphi(y)\, dy,$$

where

$$g_\nu(x, y; \varrho) = \pm \frac{1}{8} \pi i \sqrt{xy}\, [H_\nu^{(1)}(\varrho x)\, H_\nu^{(2)}(\varrho y) - H_\nu^{(1)}(\varrho y)\, H_\nu^{(2)}(\varrho x)],$$

$$+x < y, \quad -x \geqq y,$$

and where $\varphi(x)$ is restricted so that all the integrals involved exist. If we apply conditions (2) and (3) to the general solution in this form we find

$$u = \int_0^1 G_\nu(x, y; \varrho)\, \varphi(y)\, dy$$

where

$$G_\nu(x, y; \varrho) = -\frac{\sqrt{x}\, J_\nu(\varrho x) W(g_\nu)}{W(\sqrt{x}\, J_\nu(\varrho x))} + g_\nu(x, y; \varrho),$$

a function which reminds one of the Green's function for a linear homogeneous differential system.*

Since this function $G_\nu$ is not expressible in the asymptotic forms used by Birkhoff and others in the discussion of certain regular differential systems†, the study of the expansion problem cannot be effected in terms of it. It is possible, however, to define a function $\Gamma_\nu(x, y; \varrho^2)$ whose poles and residues are the same as those of $G_\nu(x, y; \varrho)$ and which in addition has the requisite behavior for values of $\varrho$ of large modulus. If we set

$$\gamma_\nu(x, y; \varrho) = -\frac{\sqrt{x}\, J_\nu(\varrho x) W(g_\nu(x, y; \varrho))}{W(\sqrt{x}\, J_\nu(\varrho x))},$$

then

$$\Gamma_\nu(x, y; \varrho^2) = \{\gamma_\nu(y, x; \varrho);\ \gamma_\nu(x, y; \varrho)\} \tag{4}$$

where the notation $\{A; B\}$ indicates that $A$ is to be taken if $x \geqq y$, $B$ if $x < y$. The expansion problem can be set up in terms of this function.

First we show that $\Gamma_\nu$ depends analytically upon $\varrho^2$, by proving that $\gamma_\nu$ is an even function of $\varrho$:

$$\gamma_\nu(x, y; e^{\pi i}\varrho) = \gamma_\nu(x, y; \varrho).$$

This fact is a matter of computation, whose details we shall merely indicate. We have by simple manipulations, involving the relation between Bessel functions of the first and third kinds,

$$\begin{aligned}(5)\qquad \gamma_\nu(x, y; \varrho) &= +\frac{\pi i\sqrt{xy}\, J_\nu(\varrho x) J_\nu(\varrho y) W(\sqrt{x}\, H_\nu^{(1)}(\varrho x))}{4 W(\sqrt{x}\, J_\nu(\varrho x))} \\ &\qquad -\frac{1}{4}\pi i\sqrt{xy}\, J_\nu(\varrho x)\, H_\nu^{(1)}(\varrho y) \\ &= -\frac{\pi i\sqrt{xy}\, J_\nu(\varrho x) J_\nu(\varrho y) W(\sqrt{x}\, H_\nu^{(2)}(\varrho x))}{4 W(\sqrt{x}\, J_\nu(\varrho x))} \\ &\qquad +\frac{1}{4}\pi i\sqrt{xy}\, J_\nu(\varrho x)\, H_\nu^{(2)}(\varrho y).\end{aligned}$$

* For the developments of this paragraph see Bocher, *Leçons sur les méthodes de Sturm*, Paris, 1917.

† Birkhoff, Trans. Amer. Math. Soc., vol. 9 (1908), pp. 373–395; Tamarkin, Rendiconti di Palermo, vol. 34 (1912), pp. 345–395.

In case $W(u) \equiv u(1)$, we have

$$W(\sqrt{x}\, H_\nu^{(1)}(\varrho x)) = H_\nu^{(1)}(\varrho), \tag{6}$$

$$W(\sqrt{x}\, H_\nu^{(2)}(\varrho x)) = H_\nu^{(2)}(\varrho), \tag{7}$$

$$W(\sqrt{x}\, J_\nu(\varrho x)) = J_\nu(\varrho); \tag{8}$$

and in case $W(u) \equiv u'(1) + A\,u(1)$,

$$W(\sqrt{x}\, H_\nu^{(1)}(\varrho x)) = -\varrho H_{\nu+1}^{(1)}(\varrho) + \left(A + \nu + \frac{1}{2}\right) H_\nu^{(1)}(\varrho), \tag{9}$$

$$W(\sqrt{x}\, H_\nu^{(2)}(\varrho x)) = -\varrho H_{\nu+1}^{(2)}(\varrho) + \left(A + \nu + \frac{1}{2}\right) H_\nu^{(2)}(\varrho), \tag{10}$$

$$W(\sqrt{x}\, J_\nu(\varrho x)) = -\varrho J_{\nu+1}(\varrho) + \left(A + \nu + \frac{1}{2}\right) J_\nu(\varrho), \tag{11}$$

by the use of the formulae of W 3.2 and W 3.6. It is now an easy matter to replace $\varrho$ by $e^{\pi i}\varrho$ in (5). From W 3.62 we find that

$$J_\nu(e^{\pi i}\varrho x) = e^{\nu\pi i} J_\nu(\varrho x), \quad J_\nu(e^{\pi i}\varrho y) = e^{\nu\pi i} J_\nu(\varrho y), \quad H_\nu^{(1)}(e^{\pi i}\varrho y) = -e^{-\nu\pi i} H_\nu^{(2)}(\varrho y),$$

$$W(\sqrt{x}\, H_\nu^{(1)}(e^{\pi i}\varrho x)) = -e^{-\nu\pi i} W(\sqrt{x}\, H_\nu^{(2)}(\varrho x)),$$

$$W(\sqrt{x}\, J_\nu(e^{\pi i}\varrho x)) = e^{\nu\pi i} W(\sqrt{x}\, J_\nu(\varrho x)).$$

On substituting these expressions, we see at once that $\gamma_\nu$ is an even function of $\varrho$. Consequently $\Gamma_\nu$ is a meromorphic function of $\varrho^2$.

Concerning the distribution of the poles of $\Gamma_\nu$ we can state the following theorem.

THEOREM I. *The function $W(\sqrt{x}\, J_\nu(\varrho x))$ has on the right half-plane, $\Re(\varrho) \geqq 0$, the asymptotic form*

$$\left(\varrho^{\mu - \frac{1}{2}} / \sqrt{2\pi}\right)\left(e^{i\left(\varrho - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}\,[i^\mu] + e^{-i\left(\varrho - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}\,[(-i)^\mu]\right)$$

*where $\mu = 0$ or $\mu = 1$ according as $W(u) \equiv u(1)$ or $W(u) \equiv u'(1) + A\,u(1)$. The equation $W(\sqrt{x}\, J_\nu(\varrho x)) = 0$ has infinitely many roots on the right half-plane; they are distributed asymtotically near the roots of*

$$e^{2i\left(\varrho - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)} + (-1)^\mu = 0$$

*so that for large $|\varrho|$ there is one simple root near each root of this equation. The roots of the equation on the left half-plane, $\Re(\varrho) < 0$, are the negatives of those on the right half-plane.**

* The notation $[c]$ will be used to indicate a function of the asymptotic form $[c] + \frac{E(\varrho)}{\varrho}$, where $E$ is bounded for the region considered, here $\Re(\varrho) \geqq 0$, for all $\varrho$ of sufficiently great modulus.

The asymptotic form is readily obtained by substituting in (11) the asymptotic forms for $H_\nu^{(1)}$ and $H_\nu^{(2)}$ as given by W 7.2, formulae (3) and (4). It is then possible to use the methods of Birkhoff to prove the existence and distribution of roots from the asymptotic form.* The equation $W(\sqrt{x}\,J_\nu(e^{\pi i}\varrho x)) = e^{\nu\pi i}\,W(\sqrt{x}\,J_\nu(\varrho x))$ shows us that if $\varrho$ is a root $-\varrho$ is also a root.

For more detailed study of the roots when $\nu$ and $A$ are real we refer to W, Chapter XV, and W 18.3.

We shall now consider the residues of $\Gamma_\nu$. If $\varrho = \bar{\varrho}$ is a pole of this function distinct from the origin, (5) shows us that the corresponding residue is $c(\bar{\varrho})\sqrt{xy}\,J_\nu(\bar{\varrho}\,x)J_\nu(\bar{\varrho}\,y)$. If $\bar{\varrho}$ is a simple pole, then

$$c(\bar{\varrho}) = \left.\frac{\pi i\,W(\sqrt{x}\,H_\nu^{(1)}(\varrho x))}{4\dfrac{d}{d\varrho}\,W(\sqrt{x}\,J_\nu(\varrho x)}\right]_{\varrho=\bar{\varrho}}.$$

When $W(u) \equiv u(1)$ this coefficient becomes $\dfrac{\pi i H_\nu^{(1)}(\bar{\varrho})}{4J_\nu'(\bar{\varrho})}$; and from the equation given in W 3.63 (4), namely,

$$H_\nu^{(1)}(\varrho)\frac{dH_\nu^{(2)}(\varrho)}{d\varrho} - H_\nu^{(2)}(\varrho)\frac{dH_\nu^{(1)}(\varrho)}{d\varrho} = -\frac{4i}{\pi\varrho},$$

and the boundary condition $H_\nu^{(1)}(\bar{\varrho}) + H_\nu^{(2)}(\bar{\varrho}) = 0$, we can evaluate this expression as $1/(2J_\nu'^2(\bar{\varrho}))$. Similarly, when $W(u) \equiv u'(1) + Au(1)$, we find

$$\begin{aligned} c(\bar{\varrho}) &= \left.\frac{\pi i\left(\varrho\dfrac{d}{d\varrho}H_\nu^{(1)}(\varrho) + (A+\tfrac{1}{2})H_\nu^{(1)}(\varrho)\right)}{4(\varrho J_\nu''(\varrho) + (A+\tfrac{3}{2})J_\nu'(\varrho))}\right]_{\varrho=\bar{\varrho}} \\ &= \frac{-1}{2\bar{\varrho}\,J_\nu(\bar{\varrho})(\bar{\varrho}\,J_\nu''(\bar{\varrho}) + (A+\tfrac{3}{2})J_\nu'(\bar{\varrho}))}. \end{aligned}$$

By reference to certain of the formulae given in W 18.3, we see that the residue can be put in the form

$$\frac{1}{2}\,\frac{\sqrt{xy}\,J_\nu(\bar{\varrho}\,x)\,J_\nu(\bar{\varrho}\,y)}{2\bar{\varrho}\int_0^1 y\,J_\nu^2(\bar{\varrho}\,y)\,dy}.$$

While we do not need to obtain more exact information than this, it is apparent that the statement that, when $\nu$ and $A$ are real, $\Gamma_\nu$ has only

* Birkhoff, loc. cit., pp. 383–386; Tamarkin, loc. cit., pp. 353–358; Birkhoff, Rendiconti di Palermo, vol. 36 (1913), p. 117.

simple poles except possibly at the origin is implicit in some of the formulae written down; and the behavior of $\Gamma_\nu$ at the origin might be investigated fully at this point. Since such knowledge can be obtained most readily by an application of the main theorems of this paper, it is unnecessary to pursue the investigation farther.

It is important to notice the special differential systems giving rise to the sine and cosine series. If we set $\nu = \frac{1}{2}$, $W(u) \equiv u(1)$, $\Gamma_\nu$ is very easily computed and is seen to have only simple poles, the origin being a regular point; when $\nu = -\frac{1}{2}$, $W(u) \equiv u'(1) + Au(1)$, $A = 0$, $\Gamma_\nu$ is found to have only simple poles, among them the origin, where the residue is $+1$.

In the $\varrho$-plane we define a system of concentric circles $C_1, C_2, \cdots$, with their centers at the origin and their radii $R_1 < R_2 < \cdots$ forming a monotone sequence with limit $+\infty$. We may suppose that the annular region bounded by successive circles contains at least two poles of $\Gamma_\nu$ and as few others as possible. No circle $C$ shall pass through a pole. For any function such that $\sqrt{x}f(x)$ is summable on $(0, 1)$ we consider the behavior of

$$\frac{1}{2\pi i}\int_0^1 \sqrt{y}f(y)\int_{C_n} 2\varrho\,\Gamma_\nu(x, y; \varrho^2)\,d\varrho\,dy \tag{12}$$

when $n \to \infty$; in this way we are able to apply the theory of residues to formulate the convergence problem in connection with expansions in Bessel functions. Similarly, we are able to study the term-by-term derived series of Fourier-Bessel and of Dini by considering the expressions

$$\frac{1}{2\pi i}\frac{\partial^k}{\partial x^k}\int_0^1 \sqrt{y}f(y)\int_{C_n} 2\varrho\,(1-\varrho^2/R_n^2)^{k+l}\,\Gamma_\nu(x, y; \varrho^2)\,d\varrho\,dy \tag{13}$$

for $k = 0, 1, 2, \cdots, l \geqq 0$, where the branch of $(1-\varrho^2/R_n^2)^{k+l}$ reducing to $+1$ at the origin is employed. This factor is used to introduce the Riesz typical means. For more detailed information concerning integrals of this character we refer the reader to other discussions.*

## II. Discussion of the expansion problem.

The treatment of the integrals obtained in the preceding section rests upon the following theorem:

THEOREM II. *Let $\varphi(x, y, R)$ be defined for $x$ and $y$ on $(0, 1)$ and for real positive values of $R$ belonging to a set $\mathfrak{R}$, one of whose limit points is $+\infty$; and let this function be summable, $0 \leqq y \leqq 1$, for each pair of values $(x, R)$. Then a sufficient condition that as $R$ becomes infinite in any manner in $\mathfrak{R}$*

* S, §§ II and III; W 18.5; Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge Tracts, Nr. 18, 1915, Chapters IV and V, pp. 19–38.

$$\lim_{R\to\infty}\int_0^1 F(y)\,\varphi(x,y,R)\,dy = 0$$

*uniformly for all* $x$ *on* $(a,1)$, $a>0$, *where* $F(x)$ *is any summable function, is that*

(1) $|\varphi(x,y,R)| < M(a,\varepsilon)$, $0 \leqq y \leqq a' \leqq a-\varepsilon$, $a \leqq x \leqq 1$, *where* $M$ *is finite for each pair of values* $(a,\varepsilon)$, $0<\varepsilon<a$;

(2) $|\varphi(x,y,R)| < M(a')$, $a' \leqq y \leqq 1$, $a' \leqq x \leqq 1$, *where* $M$ *is finite for each positive value of* $a'$;

(3) $\lim_{R\to\infty}\int_\alpha^\beta \varphi(x,y,R)\,dy = 0$ uniformly $a \leqq x \leqq 1$, $a' \leqq \alpha \leqq \beta \leqq 1$.

If $a$ is given we can choose $\varepsilon$ and $a'$ so that

$$\left|\int_0^{a'} F(y)\,\varphi(x,y,R)\,dy\right| \leqq M(a,\varepsilon)\int_0^{a'}|F(y)|\,dy < \eta/2.$$

Then we can choose $R'$ so that for $R \geqq R'$

$$\left|\int_{a'}^1 F(y)\,\varphi(x,y,R)\,dy\right| < \eta/2$$

by a theorem of Lebesgue which makes this inequality follow from conditions (2) and (3).* On adding these two inequalities we find

$$\left|\int_0^1 F(y)\,\varphi(x,y,R)\,dy\right| < \eta, \qquad R \geqq R',$$

as we wished to show.

Before proceeding to the application of this theorem to the expansion problem, we shall introduce certain notations and a few fundamental lemmas.

About each pole of $\Gamma_\nu$ as center we describe a circle $\sigma$ of small radius $r$. We denote by $\Sigma'$ the region obtained by deleting the interiors of these circles from the $\varrho$-plane. By $S_1$ and $S_4$ we shall mean the first and fourth quadrants of the $\varrho$-plane, respectively; and by $S_1'$ and $S_4'$ the parts of $\Sigma'$ on these quadrants. We represent by $C$ a circle with center at the origin and radius $R$ lying entirely on $\Sigma'$. The arcs of $C$ on $S_1'$ and $S_4'$ we denote by $\gamma_1$ and $\gamma_4$ respectively.

The lemmas which we shall now state are all more or less apparent. Since we have given explicit proofs of entirely similar lemmas elsewhere, we shall not give demonstrations here.†

LEMMA I. *If* $m(x,y,\varrho)$ *is bounded* $0 \leqq y \leqq 1$, $a \leqq x \leqq 1$, *for each* $a>0$, *and for all* $\varrho$ *on* $S_1'$, *then*

* Lebesgue, Annales de Toulouse (3), vol. 1 (1909), p. 52–55.

† 8, Lemmas III, V, VI, VI′, IX, IX′.

$$\int_{\gamma_1} e^{\varrho i(x-y)} m(x, y, \varrho)\, d\varrho$$

*is uniformly bounded* $0 \leqq y \leqq a-\varepsilon$, $a \leqq x \leqq 1$, *for each pair of values* $(a, \varepsilon)$, $a > \varepsilon > 0$, *and for all* $R$.

LEMMA II. *If* $m(x, y, \varrho)$ *is bounded for all* $x$ *and* $y$ *on* $(a', 1)$, *for each* $a' > 0$, *and for all* $\varrho$ *on* $S_1'$, *then*

$$\int_{\gamma_1} m(x, y, \varrho)/\varrho\, d\varrho$$

*is bounded for* $x$ *and* $y$ *on* $(a', 1)$ *and for all* $R$.

LEMMA III. *If* $m(x, y, \varrho)$ *is bounded for all* $x$ *and* $y$ *on* $(a', 1)$, $a' > 0$, *and for all* $\varrho$ *on* $S_1'$, *then*

$$\int_{\gamma_1} e^{\varrho i x} m(x, y, \varrho)\, d\varrho$$

*is uniformly bounded for all* $x$ *and* $y$ *on* $(a', 1)$ *and for all* $R$.

LEMMA IV. *If* $m(x, y, \varrho)$ *is a function bounded for all* $x$ *and* $y$ *on* $(a', 1)$, $a' > 0$, *and for all* $\varrho$ *on* $S_1'$, *then*

$$\lim_{R\to\infty} \int_\alpha^x \int_{\gamma_1} e^{\varrho i(x-y)} m(x, y, \varrho)/\varrho\, d\varrho\, dy = 0, \quad a' \leqq \alpha \leqq x \leqq 1,$$

$$\lim_{R\to\infty} \int_\alpha^x \int_{\gamma_1} e^{-\varrho i(x-y)} m(x, y, \varrho)/\varrho\, d\varrho\, dy = 0, \quad a' \leqq x \leqq \alpha \leqq 1,$$

*uniformly.*

LEMMA V. *If* $m(x, y, \varrho)$ *is bounded for all* $x$ *and* $y$ *on* $(a', 1)$, $a' > 0$, *and for all* $\varrho$ *on* $S_1'$, *then*

$$\lim_{R\to\infty} \int_\alpha^x \int_{\gamma_1} e^{\varrho i(1-y)} m(x, y, \varrho)/\varrho\, d\varrho\, dy = 0$$

*uniformly* $a' \leqq x \leqq 1$, $a' \leqq \alpha \leqq 1$.

We shall now prove

LEMMA VI. *On* $S_1'$

$$\frac{1}{e^{2i\left(\varrho - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}[i^\mu] + [(-i)^\mu]} = \left[\frac{1}{(-i)^\mu}\right] + e^{2i\varrho} M(\varrho)$$

*where* $M$ *is bounded.*

The fraction on the left is easily shown to be bounded on $S_1'$ by following a proof given by Birkhoff.* Thus we find

* Birkhoff, Rendiconti di Palermo, vol. 36 (1913), pp. 120–121.

$$\frac{1}{e^{2i\left(\varrho-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)}[i^\mu]+[(-i)^\mu]}-\frac{1}{(-i)^\mu}$$

$$=\frac{(-i)^\mu-[(-i)^\mu]-e^{2i\left(\varrho-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)}[i^\mu]}{(-i)^\mu\left(e^{2i\left(\varrho-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)}[i^\mu]+[(-i)^\mu]\right)}$$

$$=[0]+e^{2\varrho i}M(\varrho)$$

on $S_1'$, as we were to show.

In the six lemmas just stated we may replace $S_1'$ by $S_4'$, $\gamma_1$ by $\gamma_4$, and $+i$ by $-i$, obtaining forms useful on the fourth quadrant.

Lastly we shall write

$$(14)\quad \mathfrak{G}(x,y;\varrho)=\frac{1}{2}i(-1)^\mu e^{\varrho i(2-x-y)}+\left\{-\frac{1}{2}e^{\varrho i(x-y)};-\frac{1}{2}e^{-\varrho i(x-y)}\right\}$$

for all $\varrho$ on the upper half-plane, and

$$\mathfrak{G}(x,y;\varrho)=-\mathfrak{G}(x,y;-\varrho)$$

for all $\varrho$ on the lower half-plane.

We shall next prove the fundamental theorem of this section.

THEOREM III. *If $f(x)$ is any function such that $\sqrt{x}f(x)$ is summable on $(0,1)$, then*

$$lim_{R\to\infty}\int_0^1\sqrt{y}f(y)\int_C(2\varrho\Gamma_\nu(x,y;\varrho^2)-\mathfrak{G}(x,y;\varrho))\,d\varrho\,dy=0$$

*uniformly on any interval $(a,1)$, $a>0$.*

Since $\varrho$ may be replaced by $-\varrho$ without changing the integral, it is at once apparent that we can establish the theorem by proving it true when the circle $C$ is replaced by the semicircle $\gamma_1+\gamma_4$. We shall confine our attention to the case where $C$ is replaced by $\gamma_1$; the proof when $C$ is replaced by $\gamma_4$ is entirely similar. Thus we shall show that

$$\varphi(x,y,R)=\int_{\gamma_1}(2\varrho\Gamma_\nu-\mathfrak{G})\,d\varrho$$

satisfies conditions (1)–(3) of Theorem II, so that

$$\lim_{R\to\infty}\int_0^1\sqrt{y}f(y)\,\varphi(x,y,R)\,dy=0$$

uniformly, $0<a\leqq x\leqq 1$.

*Condition* (1) *is satisfied.* From W 3.13, 7.21 (1) we obtain the fact that

$$(15)\quad J_\nu(\varrho z)=\frac{1}{\sqrt{2\pi\varrho z}}e^{-i\left(\varrho z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)}\omega_\nu(\varrho z),\qquad |\omega_\nu|\leqq\Omega_\nu,$$

for $\Re(\nu) \geqq -\frac{1}{2}$, $\varrho$ on $S_1$, $z \geqq 0$; and from W 7.2, formulae (3) and (4), the fact that

$$H_\nu^{(1)}(\varrho z) = \sqrt{\frac{2}{\pi \varrho z}}\, e^{i\left(\varrho z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}[1], \tag{16}$$

$$H_\nu^{(2)}(\varrho z) = \sqrt{\frac{2}{\pi \varrho z}}\, e^{-i\left(\varrho z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}[1], \tag{17}$$

for $\varrho$ on $S_1$, $z \geqq a > 0$. By using the last two formulae, with $z = 1$, in conjunction with equations (9) and (10) we see that

$$\frac{W(\sqrt{x}\, H_\nu^{(1)}(\varrho x))}{W(\sqrt{x}\, J_\nu(\varrho x))} = \frac{2 e^{2i\left(\varrho - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}[(i)^\mu]}{e^{2i\left(\varrho - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}[(i)^\mu] + [(-i)^\mu]}. \tag{18}$$

Now for $0 \leqq y \leqq a - \varepsilon$, $a \leqq x \leqq 1$, where $a > \varepsilon > 0$ are fixed, we find by substituting from equations (15)–(18) in the explicit formulae for $\Gamma_\nu$,

$$2\varrho\Gamma_\nu - \mathfrak{G} = \frac{i e^{\varrho i(2-x-y)}\, \omega_\nu(\varrho x)\, \omega_\nu(\varrho y)\, [(i)^\mu]}{2\left(e^{2i\left(\varrho - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}[(i)^\mu] + [(-i)^\mu]\right)} - \frac{i}{2} e^{\varrho i(x-y)}\, \omega_\nu(\varrho y)\,[1]$$
$$- \frac{1}{2} i(-1)^\mu e^{\varrho i(2-x-y)} + \frac{i}{2} e^{\varrho i(x-y)}.$$

Writing $e^{\varrho i(2-x-y)} = e^{\varrho i(x-y)}\, e^{2\varrho i(1-x)}$ and recalling Lemma VI, we see that

$$2\varrho\Gamma_\nu - \mathfrak{G} = e^{\varrho i(x-y)}\, m(x, y, \varrho)$$

where $m$ is defined and bounded, $0 \leqq y \leqq 1$, $a \leqq x \leqq 1$, for all $\varrho$ on $S_1'$. By Lemma I it follows that $|\varphi(x, y, R)| < M(a, \varepsilon)$, $0 \leqq y \leqq a - \varepsilon$, $a \leqq x \leqq 1$, for all $R$, as we wished to show.

*Condition* (2) *is satisfied.* For $x$ and $y$ on $(a', 1)$, $a' > 0$, and for $\varrho$ on $S_1'$, we can write, by (16), (17), (18)

$$2\varrho\Gamma_\nu - \mathfrak{G}$$
$$= \frac{(i)^{\mu+1}\left(e^{\varrho i(1-x)}[1] + e^{i\left(\varrho + \varrho x - \nu\pi - \frac{1}{2}\pi\right)}[1]\right)\left(e^{\varrho i(1-y)}[1] + e^{i\left(\varrho + \varrho y - \nu\pi - \frac{1}{2}\pi\right)}[1]\right)}{2\left(e^{2i\left(\varrho - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}[i^\mu] + [(-i)^\mu]\right)}$$

$$+ \left\{-\frac{i}{2}\left(e^{\varrho i(x-y)}[1] + e^{i\left(\varrho x + \varrho y - \nu\pi - \frac{1}{2}\pi\right)}[1]\right);\right. \tag{19}$$
$$\left. -\frac{i}{2}\left(e^{-\varrho i(x-y)}[1] + e^{i\left(\varrho x + \varrho y - \nu\pi - \frac{1}{2}\pi\right)}[1]\right)\right\}$$
$$- \frac{1}{2} i(-1)^\mu e^{\varrho i(2-x-y)} - \left\{-\frac{i}{2} e^{\varrho i(x-y)};\ -\frac{i}{2} e^{\varrho i(x-y)}\right\}.$$

By using Lemma VI and combining terms we see that

$$2\varrho\Gamma_\nu - \mathfrak{G} = e^{\varrho i(2-x-y)}[0] + e^{\varrho i} m_1(x, y, \varrho) + \{e^{\varrho i(x-y)}[0] + e^{\varrho ix} m_2(x, y, \varrho);\ e^{-\varrho i(x-y)}[0] + e^{\varrho ix} m_3(x, y, \varrho)\},$$

and, since $e^{\varrho i} = e^{\varrho ix} e^{\varrho i(1-x)}$, that

$$2\varrho\Gamma_\nu - \mathfrak{G} = M_1(x, y, \varrho)/\varrho + e^{\varrho ix} M_2(x, y, \varrho),$$

where the functions $m$ and $M$ are bounded for $x$ and $y$ on $(a', 1)$ and for $\varrho$ on $S_1'$. By Lemmas II and III, $|\varphi(x, y, R)| < M(a')$, $a' \leqq x \leqq 1$, $a' \leqq y \leqq 1$, for all $R$.

*Condition* (3) *is satisfied.* We consider the integral $\int_\alpha^\beta \varphi(x, y, R)\, dy$ as the sum of the integrals over $(\alpha, x)$ and $(x, \beta)$. Since these two integrals are of precisely the same form, it is sufficient to show that $\lim_{R\to\infty} \int_\alpha^x \varphi(x, y, R)\, dy = 0$, uniformly $a' \leqq x \leqq 1$, $a' \leqq \alpha \leqq 1$. Since we have

$$\int_\alpha^x e^{\varrho i(1-y)}\, dy = [0], \qquad \int_\alpha^x e^{\varrho iy}\, dy = [0],$$

for this range of variables, it is clear that

$$\int_\alpha^x \left(e^{\varrho i(1-y)}[1] + e^{i\left(\varrho+\varrho y-\nu\pi-\frac{1}{2}\pi\right)}[1]\right) dy = [0] + e^{\varrho i}[0]$$

and

$$\int_\alpha^x e^{i\left(\varrho x+\varrho y-\nu\pi-\frac{1}{2}\pi\right)}[1]\, dy = e^{\varrho ix}[0]$$

for $a' \leqq x \leqq 1$, $a' \leqq \alpha \leqq 1$, and for all $\varrho$ on $S_1'$. On using these results together with Lemma VI and equation (19) we find

$$\int_\alpha^x (2\varrho\Gamma_\nu - \mathfrak{G})\, dy = \int_\alpha^x e^{\varrho i(2-x-y)}[0]\, dy + e^{\varrho i}[0] + \int_\alpha^x \{e^{\varrho i(x-y)}[0];\ e^{-\varrho i(x-y)}[0]\}\, dy + e^{\varrho ix}\{[0];\ [0]\}.$$

On writing $e^{\varrho i} = e^{\varrho ix} e^{\varrho i(1-x)}$, we obtain

$$\int_\alpha^x (2\varrho\Gamma_\nu - \mathfrak{G})\, dy = \int_\alpha^x e^{\varrho i(1-y)} M_1(x, y, \varrho)\, dy + e^{\varrho ix} M_2(x, y, \varrho)/R + \int_\alpha^x \{e^{\varrho i(x-y)} M_3(x, y, \varrho)/\varrho;\ e^{-\varrho i(x-y)} M_4(x, y, \varrho)/\varrho\}\, dy,$$

where the functions $M$ are bounded on the ranges considered. By Lemmas III, IV, V, we see that

$$\lim_{R\to\infty}\int_a^x \varphi(x, y, R)\,dy = \lim_{R\to\infty}\int_{\gamma_1}\int_a^x (2\varrho\Gamma_\nu - \mathfrak{G})\,d\varrho\,dy = 0$$

uniformly, $a' \leqq x \leqq 1$, $a' \leqq \alpha \leqq 1$. Consequently,

$$\lim_{R\to\infty}\int_\alpha^\beta \varphi(x, y, R)\,dy = 0$$

uniformly, $a' \leqq x \leqq 1$, $a' \leqq \alpha \leqq \beta < 1$, as we wished to show.

At this point we pause to call attention to the fact that formula (15), based on the hypothesis that $\Re(\nu) \geqq -\frac{1}{2}$, is of fundamental importance in our proof. By its use we are able to eliminate the inconveniences introduced by the singular point of the differential equation (1). It is the lack of a suitable formula analogous to (15) when $\Re(\nu) < -\frac{1}{2}$ that forces us to restrict the parameter $\nu$ to the range considered.*

The conclusions which can be drawn from Theorem III are far-reaching. We denote by $B_{\nu,N}(f)$ the sum of the first $N$ terms of the expansion of $f(x)$ in terms of the functions $J_\nu(\varrho x)$ associated with the differential system (1), (2), (3); and by $S_N(F)$ and $C_N(F)$ the sums of the first $N$ terms of the sine and cosine series for $F(x)$ on (0, 1), respectively. We can then prove the following theorem.

THEOREM IV. *If $\sqrt{x}\,f(x)$ is summable on (0, 1) and if $\Re(\nu) \geqq -\frac{1}{2}$, then*

$$lim_{N\to\infty}\left(\sqrt{x}\,B_{\nu,N}(f) - S_N(\sqrt{x}\,f)\right) = 0$$

*uniformly, $0 < a \leqq x \leqq 1$, when $W(u) \equiv u(1)$ in (3); and*

$$lim_{N\to\infty}\left(\sqrt{x}\,B_{\nu,N}(f) - C_N(\sqrt{x}\,f)\right) = 0$$

*uniformly, $0 < a \leqq x \leqq 1$, when $W(u) \equiv u'(1) + Au(1)$ in (3).*

Let there be given any two functions $\Gamma_\nu$, $\Gamma_{\nu'}$. It is then possible to select a sequence of circles $C_1, C_2, \cdots$, which satisfies the conditions prescribed at the beginning of this section for circles $C$ and which has the further property that for $N$ sufficiently large $C_N$ contains exactly $2N$ poles of $\Gamma_\nu$ and exactly $2(N+n)$ poles of $\Gamma_{\nu'}$, where $n$ is a positive or negative integer or zero, independent of $N$. The possibility of such a selection is apparent from the asymptotic distribution of the poles of $\Gamma_\nu$ and $\Gamma_{\nu'}$ described in Theorem I.

If $W(u) \equiv u(1)$, we take $\nu' = +\frac{1}{2}$. Then by Theorem III,

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_0^1 \sqrt{y}\,f(y)\int_{C_N} 2\varrho\left(\Gamma_\nu - \Gamma_{\frac{1}{2}}\right)d\varrho\,dy = 0$$

* W 18.1.

uniformly, $0<a\leqq x\leqq 1$; in other words,

$$\lim_{N\to\infty}(\sqrt{x}\,B_{\nu,N}(f)-S_{N+n}(\sqrt{x}f))=0$$

uniformly, $0<a\leqq x\leqq 1$. By the theorem of Riemann-Lebesgue

$$\lim_{N\to\infty}(S_{N+n}(\sqrt{x}f)-S_N(\sqrt{x}f))=0$$

uniformly, $0\leqq x\leqq 1$. The first part of the theorem follows at once.

When $W(u)\equiv u'(1)+Au(1)$ we let $\Gamma_{\nu'}$ be formed for the differential system (1), (2), (3) with $\nu'=-\frac{1}{2}$ and $A=0$. The further reasoning parallels that of the preceding case.

COROLLARY I. *If $F(x)$ is summable on $(0,1)$ and is identical to $f(x)$ on an interval $(a',1)$, $0<a'<a$, then*

$$\lim_{N\to\infty}B_{\nu,N}(f)-S_N(F)=0,\qquad W(u)\equiv u(1),$$
$$\lim_{N\to\infty}B_{\nu,N}(f)-C_N(F)=0,\qquad W(u)\equiv u'(1)+Au(1),$$

*uniformly, $0<a\leqq x\leqq 1$.*

We discuss the first limit only, the second being amenable to similar treatment. By a familiar property of the sine series

$$\lim_{N\to\infty}(S_N(\sqrt{x}f)-S_N(\sqrt{x}F))=0$$

uniformly on $(a,1)$, since $F\equiv f$ on $(a',1)$. We extend the definition of $F$ to the interval $(-1,0)$ by the identity $F(-x)\equiv -F(x)$. If $\mathfrak{F}_N$ represents the sum of the first $2N+1$ terms of the Fourier series on $(-1,1)$ we have

$$\mathfrak{F}_N(\sqrt{|x|}\,F)\equiv S_N(\sqrt{x}\,F),\qquad \mathfrak{F}_N(F)\equiv S_N(F),$$

on $(0,1)$. Now

$$\mathfrak{F}(\sqrt{|x|}\,F)-\sqrt{|x|}\,\mathfrak{F}_N(F)=\int_{-1}^{+1}F(y)\,\varphi(x,y,N)\,dy$$

where

$$\varphi(x,y,N)=(\sqrt{|y|}-\sqrt{|x|}\,)\frac{\sin(N+\frac{1}{2})\pi(x-y)}{2\sin\frac{1}{2}\pi(x-y)}.$$

Since $|\varphi(x,y,N)|<M$, for all $x$ and $y$ on $(-1,1)$ and for all integral $N$, and since it can be shown that $\lim_{N\to\infty}\int_\alpha^\beta\varphi(x,y,N)\,dy=0$ uniformly on $(-1,1)$, it is true that

$$\lim_{N\to\infty}\int_{-1}^{+1}F(y)\,\varphi(x,y,N)\,dy=0$$

uniformly on $(-1, 1)$, by the theorem of Lebesgue quoted above.* Thus

$$\lim_{N\to\infty} S_N(\sqrt{x}F) - \sqrt{x}S_N(F) = 0$$

uniformly on (0,1). The corollary results at once.

The importance of this theorem and its corollary is sufficiently obvious that we do not need to dilate upon it. The theorem means that we can translate well-known properties of convergence and summability in the case of the sine or the cosine series to the expansions considered here.†

We may mention an application of the theorem to which reference was made earlier in the paper, namely, that to the evaluation of the residues of $\Gamma_\nu$. The residue at $\rho = \bar{\rho} \neq 0$ has the form $c(\bar{\rho})\sqrt{xy}\,J_\nu(\bar{\rho}x)J_\nu(\bar{\rho}y)$, in accordance with (5). If we take $f(x) \equiv J_\nu(\bar{\rho}x)$ in Theorem IV, we see that the equations given there reduce to

$$c(\bar{\rho})\sqrt{x}\,J_\nu(\bar{\rho}x)\int_0^1 y J_\nu^2(\bar{\rho}y)\,dy = \sqrt{x}\,J_\nu(\bar{\rho}x).$$

We conclude that $c(\bar{\rho}) = 1\Big/\int_0^1 y J_\nu^2(\bar{\rho}y)\,dy$, and, by reference to some of the formulae in the preceding discussion, that the pole is a simple pole. The residue at $\rho = 0$ may be discussed in a similar manner, although the situation is somewhat more complicated.

## III. The derived series.

We shall now indicate with the greatest possible brevity the outlines of a discussion of the expressions

$$\frac{1}{2\pi i}\frac{\partial^k}{\partial x^k}\int_0^1 \sqrt{y}f(y)\int_C 2\rho(1-\rho^2/R^2)^{k+l}\Gamma_\nu(x, y; \rho^2)\,d\rho\,dy$$

where $k = 0, 1, 2, \cdots, l \geqq 0$, for any interval $(a, b)$ completely interior to $(0, 1)$. The reader who was followed the preceding work will be able to fill in the details with the aid of the author's discussion of a similar problem for Birkhoff series.‡

We first generalise two of the lemmas used in the study of the case $k = 0$, $l = 0$.

LEMMA I′. *If $m(x, y, \rho)$ is bounded for all $x$ on $(a, 1)$ and all $y$ on $(0, 1)$ and for all $\rho$ on $S_1'$, then*

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* Lebesgue, Annales de Toulouse, (3), vol. 1 (1909), pp. 52–55.

† See Y, W 18.33.

‡ S, VI.

$$\int_{\gamma_1} \varrho^k (1-\varrho^2/R^2)^{k+l} e^{\varrho i(x-y)} m(x, y, \varrho) d\varrho, \qquad k \geqq 0, \qquad l \geqq 0,$$

*is uniformly bounded* $0 \leqq y \leqq a-\varepsilon$, $a \leqq x \leqq 1$, *for each pair of values* $(a, \varepsilon)$, $0<\varepsilon<a$, *and for all* $R$.

LEMMA III′. *If* $m(x, y, \varrho)$ *is a function bounded for all* $x$ *and* $y$ *on* $(a', 1)$, $a'>0$, *and for all* $\varrho$ *on* $S_1'$, *then the integrals*

$$\int_{\gamma_1} \varrho^k (1-\varrho^2/R^2)^{k+l} e^{\varrho ix} m(x, y, \varrho) d\varrho,$$

$$\int_{\gamma_1} \varrho^k (1-\varrho^2/R^2)^{k+l} e^{\varrho i(1-x)} m(x, y, \varrho) d\varrho,$$

(1) *are uniformly bounded for all* $x, y$ *on* $(a', b)$, $b<1$, *if* $l=0$, *and* (2) *approach zero uniformly as* $R\to\infty$, $0<a' \leqq x \leqq b<1$, *if* $l>0$.

In these statements we can replace $i, \gamma_1, S_1'$ by $-i, \gamma_4, S_4'$ respectively.

We shall also prove a new lemma.

LEMMA VII. *The asymptotic representations for* $\sqrt{z}\, H_\nu^{(1)}(z)$, $\sqrt{z}\, H_\nu^{(2)}(z)$ *are differentiable term-by-term on the regions for which they are valid.*

Starting from the operational equation

$$\frac{d}{dz}(\sqrt{z}\,*) = z^{\nu+1/2}\left(\frac{1}{z}\frac{d}{dz}\right)(z^{-\nu}*) + \frac{2\nu+1}{2\sqrt{z}}*,$$

we can show by induction that for $k = 1, 2, \cdots$,

$$\frac{d^k}{dz^k}(\sqrt{z}\,*) = \sum_{\alpha=0}^{\alpha=k} A_{\alpha k}(z)\left(\frac{1}{z}\frac{d}{dz}\right)^\alpha (z^{-\nu}*)$$

where $z^{-\nu-\frac{1}{2}} A_{\alpha k}(z)$ is representable as a sum of descending powers of $z$ of which the highest is $z^{2\alpha-k}$. When these equivalent operators are applied to the functions of the third kind there result two equations, of which the following is typical:*

$$\frac{d^k}{dz^k}\sqrt{z}\, H_\nu^{(1)}(z) = \sum_{\alpha=0}^{\alpha=k} (-1)^\alpha A_{\alpha k}(z)\, z^{-\nu-\alpha} H_{\nu+\alpha}^{(1)}(z).$$

In these equations we can substitute on the right the asymptotic expansions for the functions of the third kind as given in W 7.2, and thus obtain representations for the derivatives on the left. It is then a matter of standard operations with asymptotic forms to show that these expansions are obtained from those for $\sqrt{z}\, H_\nu^{(1)}(z)$, $\sqrt{z}\, H_\nu^{(2)}(z)$, respectively, by formal differentiation. Thus, for $k = 0, 1, 2, \cdots$,

* W 3.6 (10).

$$(20)\quad \frac{d^k}{dz^k}\sqrt{z}\,H_\nu^{(1)}(z)\sim\sqrt{\frac{2}{\pi}}(i)^k e^{i\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)}\left(1+\frac{c_{1k}}{(zi)}+\frac{c_{2k}}{(zi)^2}+\cdots\right),$$

$$(21)\quad \frac{d^k}{dz^k}\sqrt{z}\,H_\nu^{(2)}(z)\sim\sqrt{\frac{2}{\pi}}(-i)^k e^{-i\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)}\left(1+\frac{c_{1k}}{(-zi)}+\frac{c_{2k}}{(-zi)^2}+\cdots\right),$$

where $c_{1k}, c_{2k}, \cdots,$ are constants, dependent upon $\nu$.

We now require a function $\mathfrak{G}^{(k)}(x, y; \varrho)$ defined as follows: for $0 \leqq y \leqq 1/|\varrho|$, $\mathfrak{G}^{(k)}(x, y; \varrho) \equiv 0$; for $1/|\varrho| \leqq y \leqq 1$, $0 \leqq x \leqq 1$,

$$\mathfrak{G}^{(k)}(x, y; \varrho) = \frac{1}{2}\left\{-e^{\varrho i(x-y)} i \sum_{s=0}^{s=k} (\varrho i)^s \sum_{\alpha+\beta=k-s} (-1)^\beta c_{\alpha k} c_{\beta 0} x^{-\alpha} y^{-\beta};\right.$$
$$\left. -e^{-\varrho i(x-y)} i \sum_{s=0}^{s=k} (-\varrho i)^s \sum_{\alpha+\beta=k-s} (-1)^\beta c_{\alpha k} c_{\beta 0} x^{-\alpha} y^{-\beta}\right\}$$
$$c_{0,k} = 1, \qquad k = 0, 1, 2, \cdots,$$

for all $\varrho$ on the upper half-plane; and $\mathfrak{G}^{(k)}(x, y; \varrho) = -\mathfrak{G}^{(k)}(x, y; -\varrho)$ for all $\varrho$ on the lower half-plane.

We shall now prove

THEOREM V. *If $\sqrt{x}f(x)$ is summable on $(0, 1)$*

$$lim_{R\to\infty}\int_0^1 \sqrt{y}f(y)\int_C (1-\varrho^2/R^2)^{k+l}\left(2\varrho\left\{\frac{\partial^k \Gamma_\nu}{\partial x^k}; \frac{\partial^k \Gamma_\nu}{\partial x^k}\right\} - \mathfrak{G}^{(k)}(x, y; \varrho)\right)$$
$$\times d\varrho\, dy = 0$$

*uniformly, $0 < a \leqq x \leqq b < 1$, for $l \geqq 0$. The expression*

$$\frac{1}{2\pi i}\frac{\partial^k}{\partial x^k}\int_0^1 \sqrt{y}f(y)\int_C 2\varrho\,(1-\varrho^2/R^2)^{k+l}\,\Gamma_\nu(x, y; \varrho^2)\,d\varrho\,dy$$

*is equivalent on any interval $(a, b)$ completely interior to $(0, 1)$ to a linear combination with coefficients $c_{\alpha k} x^{-\alpha}$ of means of order $k+l$, $l > 0$, formed for the sine series and their derived series up to order $k$ for the functions equal to zero on $(0, \eta)$ and equal to $x^{-\beta+\frac{1}{2}}(-1)^\beta c_{\beta 0} f(x)$ on $(\eta, 1)$, $\beta = 0, \cdots, k$, $0 < \eta < a$. On any interval $(a, b)$ the problem of the term-by-term derived series associated with the differential system (1), (2), (3), is reduced to a problem in the derived series of Fourier series (sine series).*

As in Theorem III we shall consider only integrals over $\gamma_1$. We show that the function

$$\varphi(x, y, R) = \int_{\gamma_1} (1-\varrho^2/R^2)^{k+l}\left(2\varrho\left\{\frac{\partial^k \Gamma_\nu}{\partial x^k}; \frac{\partial^k \Gamma_\nu}{\partial x^k}\right\} - \mathfrak{G}^{(k)}\right) d\varrho$$

satisfies conditions (1), (2), (3) of Theorem I on any interval $(0, b)$, $b < 1$.

*Condition* (1) *is satisfied.* We differentiate in formula (5) and proceed to substitute from (15)–(18), (20), (21), thus obtaining, when $0 \leqq y \leqq a-\varepsilon$, $a \leqq x \leqq 1$,

$$(1-\varrho^2/R^2)^{k+l}\left(2\varrho\left\{\frac{\partial^k \Gamma_\nu}{\partial x^k}; \frac{\partial^k \Gamma_\nu}{\partial x^k}\right\} - \mathfrak{G}^{(k)}\right)$$
$$= \varrho^k (1-\varrho^2/R^2)^{k+l} (e^{\varrho i x} m_1 + e^{\varrho i(1-x)} m_2 + e^{\varrho i(x-y)} m_3).$$

It should be noted that the discontinuity introduced in the definition of $\mathfrak{G}^{(k)}$ as a function of $y$ makes possible the inequality

$$\mathfrak{G}^{(k)} = \varrho^k e^{\varrho i(x-y)} \omega(x, y, \varrho), \qquad |\omega| < \Omega, \tag{22}$$

$0 \leqq y \leqq a-\varepsilon$, $a \leqq x \leqq 1$, for all $\varrho$ on $S_1$. By Lemmas I′ and III′ we find

$$|\varphi(x, y, R)| < M(a, \varepsilon),$$

$0 \leqq y \leqq a-\varepsilon$, $a \leqq x \leqq b < 1$.

*Condition* (2) *is satisfied.* Here we show that

$$(1-\varrho^2/R^2)^{k+l}\left(2\varrho\left\{\frac{\partial^k \Gamma_\nu}{\partial x^k}; \frac{\partial^k \Gamma_\nu}{\partial x^k}\right\} - \mathfrak{G}^{(k)}\right) = \varrho^k (1-\varrho^2/R^2)^{k+l} (e^{\varrho i x} m_1 + e^{\varrho i(1-x)} m_2)$$
$$+ \varrho^k (1-\varrho^2/R^2)^{k+l} \{e^{\varrho i x} m_3; e^{\varrho i x} m_4\} + \{m_5/\varrho; m_6/\varrho\},$$

for $x$ and $y$ on $(a', 1)$, in a manner similar to that described in Theorem III under the corresponding head. By Lemmas II and III′, $|\varphi(x, y, R)| < M(a')$, $a' \leqq y \leqq 1$, $a' \leqq x \leqq b$.

*Condition* (3) *is satisfied.* It is found that

$$\int_\alpha^x (1-\varrho^2/R^2)^{k+l}\left(2\varrho\left\{\frac{\partial^k \Gamma_\nu}{\partial x^k}; \frac{\partial^k \Gamma_\nu}{\partial x^k}\right\} - \mathfrak{G}^{(k)}\right) dy$$
$$= \varrho^{k-1} (1-\varrho^2/R^2)^{k+l} (e^{\varrho i x} m_1 + e^{\varrho i(1-x)} m_2)$$
$$+ \varrho^{k-1} (1-\varrho^2/R^2)^{k+l} \{e^{\varrho i x} m_3; e^{\varrho i x} m_4\} + \int_\alpha^x \{e^{\varrho i(x-y)} m_5/\varrho; e^{-\varrho i(x-y)} m_6/\varrho\} dy.$$

By Lemmas III′ and IV

$$\lim_{R\to\infty} \int_\alpha^x \varphi(x, y, R)\, dy = 0$$

and

$$\lim_{R\to\infty} \int_\alpha^\beta \varphi(x, y, R)\, dy = 0$$

uniformly, $0 < a \leqq x \leqq b < 1$, for $a' \leqq \alpha \leqq \beta \leqq 1$.

Thus we are able to state the first part of the theorem. The remainder follows from it by a series of easy manipulations.

First we notice that, by (22),

$$\mathfrak{G}^{(k)} = \varrho^{k+l} e^{\varrho i(x-y)} m(x, y, \varrho)/R^l, \qquad l \geqq 0,$$

$0 \leqq y \leqq \eta < a \leqq x \leqq 1$, for all $\varrho$ on $S_1$. Thus, from Lemma I',

$$\left| \int_{\gamma_1} (1 - \varrho^2/R^2)^{k+l} \mathfrak{G}^{(k)} \, d\varrho \right| \leqq M/R^l$$

$0 \leqq y \leqq \eta$, $a \leqq x \leqq 1$, for all $R$. Consequently

$$\lim_{R \to \infty} \int_0^\eta \sqrt{y} f(y) \int_{\gamma_1} (1 - \varrho^2/R^2)^{k+l} \mathfrak{G}^{(k)} \, d\varrho \, dy = 0$$

uniformly on $(a, b)$, if $l > 0$. In this integral we can replace $\gamma_1$ by $C$.

Next we observe that

$$\frac{\partial^k}{\partial x^k} \int_0^1 \sqrt{y} f(y) \int_C 2\varrho (1 - \varrho^2/R^2)^{k+l} \Gamma_\nu \, d\varrho \, dy$$

$$= \int_0^1 \sqrt{y} f(y) \int_C 2\varrho (1 - \varrho^2/R^2)^{k+l} \left\{ \frac{\partial^k \Gamma_\nu}{\partial x^k}; \frac{\partial^k \Gamma_\nu}{\partial x^k} \right\} d\varrho \, dy.$$

When $\nu = +\frac{1}{2}$, $\mathfrak{G}^{(k)}$ takes on a very simple form, namely,

$$\mathfrak{G}^{(k)}_{\frac{1}{2}}(x, y; \varrho) \equiv 0, \qquad 0 \leqq y \leqq 1/|\varrho|,$$

$$\mathfrak{G}^{(k)}_{\frac{1}{2}}(x, y; \varrho) = \tfrac{1}{2} \{ -i(\varrho i)^k e^{\varrho i(x-y)}; \; -i(-\varrho i)^k e^{-\varrho i(x-y)} \}$$

for $1/|\varrho| \leqq y \leqq 1$ for $\varrho$ on the upper half-plane, and

$$\mathfrak{G}^{(k)}_{\frac{1}{2}}(x, y; \varrho) = -\mathfrak{G}^{(k)}_{\frac{1}{2}}(x, y; -\varrho)$$

for $\varrho$ on the lower half-plane. Consequently,

$$\mathfrak{G}^{(k)}(x, y; \varrho) = \sum_{s=0}^{s=k} \mathfrak{G}^{(s)}_{\frac{1}{2}} \sum_{\alpha+\beta=k-s} (-1)^\beta c_{\alpha k} c_{\beta 0} x^{-\alpha} y^{-\beta}.$$

In the differential system (1), (2), (3), the choice $\nu = +\frac{1}{2}$, $W(u) \equiv u(1)$, yields the sine series; and, by combining the various results obtained in the discussion of the present theorem, we prove the following relation between the general differential system and this special one:

$$\lim_{R\to\infty}\Big[\frac{\partial^k}{\partial x^k}\int_0^1\sqrt{y}\,f(y)\int_C 2\varrho\,(1-\varrho^2/R^2)^{k+l}\Gamma_\nu\,d\varrho\,dy$$
$$-\sum_{s=0}^{s=k}\sum_{\alpha+\beta=k-s}c_{\alpha k}\,x^{-\alpha}\frac{\partial^s}{\partial x^s}\int_\eta^1\sqrt{y}\,f(y)\,(-1)^\beta c_{\beta 0}\,y^{-\beta}$$
$$\times\int_C 2\varrho\,(1-\varrho^2/R^2)^{k+l}\Gamma_{+\frac{1}{2}}\,d\varrho\,dy\Big]=0$$

uniformly on $(a, b)$, for $l>0$. This proves the assertion with regard to the relation between the derived series of Bessel functions and derived Fourier series.

From the preceding theorem it is possible to deduce* the following theorem.

THEOREM VI. *If $\sqrt{x}\,\lambda(x)$ is a $k$-fold integral in the sense of Lebesgue on $(0, 1)$ then*

$$lim_{R\to\infty}\ \frac{1}{2\pi i}\,\frac{\partial^k}{\partial x^k}\int_0^1\sqrt{y}\,\lambda(y)\int_C 2\varrho\,(1-\varrho^2/R^2)^{k+l}\Gamma_\nu(x, y, \varrho^2)\,d\varrho\,dy$$
$$=\frac{d^k}{dx^k}\sqrt{x}\,\lambda(x)$$

*almost everywhere, $0<x<1$; and if the derivative on the right is continuous, the limit is uniform, $0<a\leqq x\leqq b<1$. The Riesz typical sums of order $k+l$, $l>0$, for the sequence $\dfrac{d^k}{dx^k}B_{\nu,N}(\lambda)$ converge in a similar manner to $\lambda^{(k)}$.*

As a result of Theorem V, it is possible to apply the theory of the derived series of Fourier series (sine series) to the problem of derived Bessel series. It is to be noted that the Riesz typical means when $\nu=\frac{1}{2}$, $W(u)\equiv u(1)$, are equivalent to Cesàro means of the same order, so that theorems concerning Cesàro summability of derived Fourier series can be employed here.† Under the hypotheses of the present theorem, if $l>0$,

$$\lim_{R\to\infty}\ \frac{1}{2\pi i}\,\frac{\partial^k}{\partial x^k}\int_0^1\sqrt{y}\,\lambda(y)\int_C 2\varrho\,(1-\varrho^2/R^2)^{k+l}\Gamma_\nu\,d\varrho\,dy\ =\ \Phi_k(x),$$

$0<x<1$, the limit being uniform on $(a, b)$ if $\dfrac{d^k}{dx^k}\sqrt{x}\,\lambda(x)$ is continuous. Then

$$\int_{x_1}^x\Phi_k(x)\,dx\ =\ \Phi_{k-1}(x)-\Phi_{k-1}(x_1),\qquad 0<x<1,\ \ 0<x_1<1,$$
$$\Phi_k(x)\ =\ \Phi'_{k-1}(x)\ =\ \cdots\ =\ \Phi_0^{(k)}(x).$$

* S, Theorem XXXIV.

† Hardy and Riesz, loc. cit. (footnote, p. 276); S, Theorem VI, Corollary I.

Now $\Phi_0(x) = \sqrt{x}\,\lambda(x)$, as we see by taking $k = 0$ in Theorem V. This proves the first part of the theorem. To show the second we use induction in a similar manner. For $k = 0$, the Riesz typical means of order $l > 0$ of $B_{\nu,N}(\lambda)$ have the limit $\lambda$, as a special case under Theorem V. We suppose that the theorem concerning the typical means for $\frac{d^k}{dx^k} B_{\nu,N}(\lambda)$ has been proved for $k = 0, 1, 2, \cdots, K-1$. Then

$$\frac{1}{2\pi i} \frac{\partial^k}{\partial x^k} \int_0^1 \sqrt{y}\,\lambda(y) \int_C 2\varrho\,(1-\varrho^2/R^2)^{K+l} \Gamma_\nu\, d\varrho\, dy$$

is the Riesz mean of order $K+l$, $l > 0$, for

$$\frac{d^k}{dx^k} \sqrt{x}\, B_{\nu,N}(\lambda) = \sum_{\alpha=0}^{\alpha=k} C_\alpha^k \frac{d^{k-\alpha}}{dx^{k-\alpha}} \sqrt{x}\, \frac{d^\alpha}{dx^\alpha} B_{\nu,N}(\lambda);$$

the left hand side has the limit

$$\frac{d^k}{dx^k} \sqrt{x}\,\lambda(x) = \sum_{\alpha=0}^{\alpha=k} C_\alpha^k \frac{d^{k-\alpha}}{dx^{k-\alpha}} \sqrt{x}\, \frac{d^\alpha}{dx^\alpha} \lambda(x),$$

while each term on the right, save the last, is summed to the value

$$C_\alpha^k \frac{d^{k-\alpha}}{dx^{k-\alpha}} \sqrt{x}\, \frac{d^\alpha}{dx^\alpha} \lambda(x)$$

by hypothesis. As a consequence, the last term is summable to the value $\lambda^{(k)}(x)$. Thus we have established the result concerning the summability of $\frac{d^k}{dx^k} B_{\nu,N}(\lambda)$ for all positive integral values of $k$.

# ON THE GREEN'S FUNCTION FOR SYSTEMS OF DIFFERENTIAL EQUATIONS.*

By W. M. Whyburn.

1. **Introduction.** In this paper we shall be concerned with systems of $n$ first order differential equations whose coefficients are summable (in the sense of Lebesgue) functions of $x$ on a closed interval $X$: $a \leqq x \leqq b$ and continuous functions of a parameter $\lambda$ on $L$: $L_1 < \lambda < L_2$. These coefficients also have the property that for every $\lambda$ on $L$ their absolute values are less than a function of $x$ that is itself summable on $X$. We will deal with boundary conditions whose coefficients are continuous functions of the parameter.

Bounitzky† has shown how to set up a matrix of Green's functions for a system of $n$ first order differential equations together with $n$ linear boundary conditions where the coefficients of the differential equations are continuous functions of $x$ and the coefficients of the boundary conditions are constants. Bôcher‡ has shown that for the $n$th order differential system whose coefficients are of the type treated by Bounitzky, it is possible to vary the coefficient of the zero derivative of the dependent variable in the equation in such a way as to render the system incompatible. In a recent paper§ the author defined the Green's function for second order systems whose coefficients are general analytic functions of a parameter $\lambda$ and continuous in $(x, \lambda)$.

It is the purpose of the present paper to generalize the results of the three above-mentioned papers to more general systems. For convenience I have chosen to interpret my functions as $n$-dimensional vectors and tensors|| wherever this interpretation is possible. This simplifies the notation and conforms to the present tendency in various developments of mathematical physics.

The Greek letters together with the letters $U, u, V, v, y, Z, z, M, N,$ are used to denote $n$-dimensional vectors and their components are designated

* Presented to the American Mathematical Society October 8, 1926; received November 19, 1926.

† Bounitzky, Journ. de Math., (6), vol. 5 (1909), pp. 65–125.

‡ Bôcher, Bull. Amer. Math. Soc., vol. 20 (Oct. 1914), p. 1.

§ Whyburn, W. M., Annals of Math., (2), vol. 26 (1924), pp. 125–130.

|| Peano was the first to use this notation for differential systems. It has been used rather extensively by Bliss, Hart, and Courant, cf. Bliss, Annals of Math., (2), vol. 6 (1904–5), p. 49; Hart, Amer. Journ. of Math., vol. 39 (1917), p. 407; Courant und Hilbert, Methoden der Mathematischen Physik, v. 1, Springer, Berlin, (1924).

by the same letters with suitable subscripts. A capital letter, with the exception of those mentioned above, is used to denote a $n$-dimensional tensor (or square matrix) whose element in the $i$th row and $j$th column is represented by the same letter with the subscript $ij$ and whose vector component that makes up the $j$th column is given by the same letter with the single subscript $j$. Ordinary scalar quantities are given by small letters. The symbol $|G|$ represents the $n$th order determinant whose elements are the corresponding elements of the tensor $G$. The notation $i = (1, n)$ means that $i$ takes on all of the integral values from 1 to $n$ inclusive.

A vector or tensor is said to have a property if each of its components has that property, By the *dot* product, $G \cdot H$, of the tensor $G$ by the tensor $H$ we mean the product which yields a tensor $K$ such that $K_{ij} = \sum_{r=1}^{r=n} G_{ir} H_{rj}$. By the *dot* product, $G \cdot u$, of the tensor $G$ by the vector $u$ we mean a vector $z$ such that $z_i = \sum_{r=1}^{r=n} G_{ir} u_r$.

**2. Theorem on compatibility.** Consider the differential equation

$$N = \frac{dy}{dx} - A \cdot y = 0 \tag{2.1}$$

together with the boundary conditions

$$U(y) = \varphi[y(a, \lambda)] - \varphi[y(b, \lambda)] = 0 \tag{2.2}$$

where

$$\varphi[y(x, \lambda)] = R \cdot y.$$

$A$ and $R$ are continuous functions of $\lambda$ on $L$ for each $x$ on $X$ and $A$ is summable on $X$ for each fixed $\lambda$ on $L$. Finally $|A_{ij}| \leqq m(x)$ for all $\lambda$ and all $x$ on $XL$ where $m(x)$ is a summable function of $x$ on $X$. The system (2.1), (2.2) is said to have $r$-fold compatibility for a certain fixed value of $\lambda$ if for this value of $\lambda$ there exist exactly $r$ non-identically vanishing, linearly independent solutions of equation (2.1) that satisfy conditions (2.2). The method of successive approximations together with a theorem due to Caratheodory* that has recently been isolated and applied by Ettlinger† assures us of the existence of a solution of (2.1) that is continuous in the two variables, $x$ and $\lambda$, over the domain $XL$. By a solution of equation (2.1) we will mean one of the above type. By a fundamental tensor of solutions of equation (2.1) we mean a tensor with non-vanishing determinant such that each of its components is a solution of that equation.

* Vorlesungen über reelle Funktionen, Leipzig, 1918, p. 678.

† Cf. *On continuity in several variables*, Bull. Amer. Math. Soc., vol. 33 (1927), pp. 37–38.

The existence of such tensors appears as a by-product of the above-mentioned existence proof.

THEOREM I. *If for $\lambda = p$ the system* (2.1), (2.2) *has r-fold compatibility and if $k$ and $m$ are any two fixed indices such that $1 \leqq k$, $m < n$, there exists a continuous function of $x$, $g(x)$, such that the equation*

$$N_i = 0 \quad i \neq k, \qquad N_k = g(x) y_m, \tag{2.3}$$

*together with the conditions* (2.2), *is incompatible and furthermore $0 \leqq g(x) < \varepsilon$, where $\varepsilon$ is any arbitrarily assigned positive constant.*

Let $Y$ denote a fundamental tensor of solutions of (2.1). From purely algebraic considerations we arrive at the conclusion that a necessary and sufficient condition that the system (2.1), (2.2) have $r$-fold compatibility when $\lambda = p$ is that the tensor $Q$, where $Q_{ij} = U_i(y_j)$, contain at least one determinant of order $n - r$, and no determinant of higher order than $n - r$, that is different from zero. Let us suppose that for $\lambda = p$ our system has $r$-fold compatibility and let us fix our attention on one determinant of $Q$ that is of order $n - r$ and that is different from zero. We note that any set of sufficiently small variations of the coefficients of the system will change the elements of this determinant by only a slight amount (since the elements of $Y$ are continuous functions of their arguments) so that it will still be different from zero. Hence we have the following lemma:

LEMMA I. *If for $\lambda = p$ the system* (2.1), (2.2) *has just r-fold compatibility, it has no higher order of compatibility after any set of variations of its coefficients that is uniformly sufficiently small on $XL$.*

In particular we get from the above lemma that, if $r = 0$ for $\lambda = p$, it remains zero for a set of sufficiently small variations of the coefficients of the differential system. We will suppose that $r \geqq 1$ and that the rows and columns of $Q$ have been arranged so that the determinant of order $n - r$ that lies in the upper left hand corner of $Q$ is different from zero. The general solution of our system for $\lambda = p$ is $y$, where

$$y_i = \begin{vmatrix} y_{i1} & y_{i2} & \cdots & y_{in-r} & c_1 y_{i,n-r+1} & + \cdots + & c_k y_{in} \\ U_1(y_1) & U_1(y_2) & \cdots & U_1(y_{n-r}) & c_1 U_1(y_{n-r+1}) & + \cdots + & c_k U_1(y_n) \\ U_2(y_1) & U_2(y_2) & \cdots & U_2(y_{n-r}) & c_1 U_2(y_{n-r+2}) & + \cdots + & c_k U_2(y_n) \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ U_{n-r}(y_1) & \cdot & \cdots & U_{n-r}(y_{n-r}) & c_1 U_{n-r}(y_{n-r+1}) & + \cdots + & c_k U_{n-r}(y_n) \end{vmatrix}. \tag{2.4}$$

Lemma I tells us that we can choose a neighborhood, $d(p)$, of $p$ so small that for values of $\lambda$ in $d(p)$ the system (2.1), (2.2) has no higher than

$r$-fold compatibility. We will choose such a neighborhood and suppose that for $\lambda$ in $d(p)$ our system has just $r$-fold compatibility. Then, if we replace $p$ by $\lambda$ in (2.4), we have the general solution of (2.1), (2.2) for values of $\lambda$ in $d(p)$ and furthermore, since $y$ is continuous in $(x, \lambda)$ on $XL$, we have

LEMMA II. *If throughout a certain neighborhood of $\lambda = p$ the system* (2.1), (2.2) *has exactly r-fold compatibility and if $y(x)$ denotes the solution of this system for $\lambda = p$, then there exists a solution $y(x, \lambda)$ of the system such that the limit as $\lambda$ approaches $p$ of $y(x, \lambda)$ is $y(x)$ and this limit is approached uniformly.*

LEMMA III. *Let the adjoint system for system* (2.1), (2.2) *be*

$$M = 0, \qquad V = 0, \tag{2.5}$$

*and let $v$ be any solution of this system; then*

$$\int_a^b g(t)\, v_k \bar{y}_m \, dt = 0, \tag{2.6}$$

*where $\bar{y}$ is any solution of the system* (2.3), (2.2).

*Proof.* We apply Green's formula,

$$\int_a^b (v \cdot N - y \cdot M)\, dt = \sum_{i=1}^{i=2n} U_i V_{2n-i+1},$$

noting that the right hand side reduces to zero while the left hand side reduces to the single term that involves $N_k$. Our lemma follows immediately.

LEMMA IV. *If $\varepsilon$ is a positive constant and if for $\lambda = p$ the system* (2.1), (2.2) *has r-fold compatibility, there exists a continuous function $g(x)$ such that $0 \leqq g(x) < \varepsilon$ and such that the system* (2.3), (2.2) *has less than r-fold compatibility.*

*Proof.* Assume this lemma false. Since the adjoint system (2.5) has the same order of compatibility as the system (2.1), (2.2) we let $y$ be a non-identically vanishing solution of (2.1), (2.2) and let $v$ be the corresponding solution of (2.5). Since these are continuous, non-identically vanishing solutions of our differential systems it follows that neither $y_m$ nor $v_k$ can vanish over an everywhere dense set of points on $X$. We can therefore find a subinterval $(a', b')$ of $X$ such that the product $y_m v_k$ does not vanish on this subinterval. We define $\varphi(x)$ as a continuous function of $x$ satisfying the relations $0 < \varphi(x) < \varepsilon$ on $(a', b')$ and $\varphi(x) \equiv 0$ outside of $(a', b')$. It is easy to see that

$$\int_a^b \varphi(t)\, v_k y_m \, dt \neq 0. \tag{2.7}$$

We write the left hand side of (2.7) as the sum of three integrals taken between the limits $a$ to $a'$, $a'$ to $b'$, and $b'$ to $b$ respectively and note that the second integral is different from zero while the other two are both zero. Let $g(x) = \theta\varphi(x)$, where $\theta$ is a positive constant less than unity. From (2.6) we get

$$\int_a^b \varphi(t)\, v_k \bar{y}_m\, dt = 0. \tag{2.8}$$

Since our lemma is false by hypothesis, we have that for all positive values of $\theta$ that are less than unity the system (2.3), (2.2) has at least $r$-fold compatibility while by Lemma I it can have no higher order of compatibility for $\theta$ sufficiently small. We restrict $\theta$ so that this system has just $r$-fold compatibility and applying Lemma II we get $\bar{y}(x, \theta)$ continuous in $(x, \theta)$ such that $\lim_{\theta \to 0} \bar{y}(x, \theta) = y(x)$ and such that this limit is approached uniformly. Hence

$$\lim_{\theta \to 0} \int_a^b \varphi(t)\, v_k \bar{y}_m\, dt = \int_a^b \varphi(t)\, v_k y_m\, dt.$$

This contradicts equations (2.7) and (2.8) so that our assumption that the lemma is false has lead to a contradiction.

If the compatibility of system (2.3), (2.2) is not zero after one application of Lemma IV, we repeat the operation a finite number of times, each time choosing a different subinterval of $X$, and define $g(x)$ as a combination of the functions obtained in this manner. Since $r \leqq n$, it is easy to see that $g(x)$ can be choosen identically zero except for an arbitrarily small subinterval of $X$.

**3. Green's tensor for a value of $\lambda$ for which system (2.1), (2.2) is incompatible.** Consider the system (2.1), (2.2) for values of $\lambda$ for which it is incompatible. We set up a Green's tensor for this system in the following manner. We choose a point $t$ of $X$ and determine a tensor $G$ in such a way that it has the following properties:

1. $G$ is absolutely continuous in $x$ and absolutely continuous in $t$ on $X$ except at the point $x = t$.
2. At $x = t$
$$G(t+0, t; \lambda) - G(t-0, t; \lambda) = E$$
$$G(x, x-0; \lambda) - G(x, x+0; \lambda) = E$$
where $E$ is the unit tensor.*
3. $G_j$ as a function of $x$ satisfies the boundary conditions (2.2) and satisfies equation (2.1) at every point of $X$ except at $x = t$.

* $E$ is defined such that if $T$ is any tensor, $E \cdot T = T \cdot E = T$ and from this it follows that $E_{ij} = 0$, for $i \neq j$, and $E_{ii} = 1$.

4. If $H$ is the Green's tensor for the adjoint system (2.5)

$$H_{ij}(x, t; \lambda) = -G_{ji}(t, x; \lambda).$$

Let $Y$ be a fundamental tensor of solutions of equation (2.1). We seek to determine tensors $C$ and $D$ in such a way that

$$\begin{aligned} G &= Y \cdot C \text{ on } a \leqq x \leqq t, \\ G &= Y \cdot D \text{ on } t \leqq x \leqq b. \end{aligned} \tag{3.1}$$

Let $F = D - C$. From the conditions on continuity at $x = t$ we have $Y \cdot F = E$ at $x = t$. This gives us $n$ vector equations in the $n$ vectors, $F_j$. The determinant of coefficients of the components in each case is $W(t, \lambda)$ where

$$W(x, \lambda) = |Y| = k e^{\int_a^x A \cdot E ds}$$

and $k$ is a constant different from zero. If we let $d(ij)$ be $W(t, \lambda)$ with the $i$th row and $j$th column deleted we have the following unique value for $F_{ij}$:

$$F_{ij} = (-1)^{i+j} d(ij)/W(t, \lambda).$$

Since $Y$ is continuous in $(x, \lambda)$ on $XL$ and since $W$ is different from zero, and examination of the above equation reveals that $F$ is continuous in $(t, \lambda)$ on $XL$. Let $S_{ij} = \varphi_i[y_j(b)]$ and substitute $G_j$ into the boundary conditions (2.2). When we replace $D$ by $F + C$, we get

$$Q \cdot C = S \cdot F$$

and since $|Q|$ is different from zero for the values of $\lambda$ under consideration, we can solve this equation for the unique value of $C$. Let $W(rj)$ be $W(t, \lambda)$ with the $j$th row replaced by $S_{r1}, S_{r2}, \cdots, S_{rn}$ and let $\Delta_{ij}$ denote $|Q|$ with the $i$th column replaced by $W_{1j}, W_{2j}, \cdots, W_{nj}$ then

$$\begin{aligned} C_{ij} &= \Delta_{ij}/W(t, \lambda)|Q|, \\ D_{ij} &= \Delta_{ij}/W(t, \lambda)|Q| + (-1)^{i+j} d(ij)/W(t, \lambda). \end{aligned}$$

It is easy to see from these formulas that $C$ and $D$ are continuous functions of $(t, \lambda)$ on $XL$. $G$ is uniquely determined by the relations (3.1) so that it has the properties 1, 2, 3, and 4 listed above. The proof that $G$ has property 4 is essentially that given by Bounitzky.*

* Loc. cit., page 80.

Consider the semi-reduced system

$$N = \beta(x, \lambda), \tag{3.2}$$

$$U(y) = 0, \tag{3.3}$$

where $\beta$ is assumed to satisfy the same conditions that are imposed on $A$. We will show that $y$, where

$$y_j = \int_a^b G_j \cdot \beta \, dt, \tag{3.4}$$

is the unique solution of the system (3.2), (3.3). Calculate $\frac{dy_j}{dx}$ by differentiating under the integral sign.*

$$\frac{dy_j}{dx} = \int_a^b \frac{\partial G_j}{\partial x} \cdot \beta_j \, dt + \beta_j(x, \lambda). \tag{3.5}$$

The term $\beta_j(x, \lambda)$ enters when we write the term that involves $G_{jj}$ as

$$\int_a^x G_{jj} \beta_j \, dt + \int_x^b G_{jj} \beta_j \, dt$$

and derive to get

$$\int_a^b \frac{\partial G_{jj}}{\partial x} \beta_j \, dt + \beta_j(x, \lambda)[G(x, x-0; \lambda) - G(x, x+0; \lambda)]$$

and then replace the bracket by its value as given by property 2 of $G$. When we substitute $y_j$ as given by (3.4) along with its derivative as given by (3.5) into equation (3.2) and take advantage of the fact that $G_j$ satisfies (2.1), we see that (3.2) is satisfied. Conditions (3.3) are satisfied since $G_j$ satisfies these conditions. The uniqueness of this solution follows when we note that the difference between $y$ and any other solution $\bar{y}$ must satisfy the system (2.1), (2.2) which is incompatible. Finally we note that $y$ is continuous in $x$ on $X$ for the values of $\lambda$ under consideration.

Finally let us consider the non-homogeneous system

$$N = \beta(x, \lambda), \tag{3.2}$$

$$U = \alpha(\lambda), \tag{3.6}$$

where $\alpha$ is continuous in $\lambda$ on $L$. Let $u$ be the solution of the system (2.1), (3.6) and $y$ be the solution of (3.2), (3.3). The solution of (3.2), (3.6) is $y+u$. The $i$th component of this solution is

$$\int_a^b G_i \cdot \beta \, dt + u_i;$$

* This differentiation is justified by noting that all of the hypotheses of a theorem due to Caratheodory (loc. cit. p. 664) are satisfied by $G_j \cdot \beta$.

$u$ is unique since (2.1), (2.2) is incompatible and it follows that $y+u$ is the unique solution of (3.2), (3.6).

**4. The Green's tensor for characteristic numbers of the differential system.** We suppose now that there exists a set of characteristic numbers for the system (2.1), (2.2). By a characteristic number we mean a value of $\lambda$ for which the system has a solution other than the identically vanishing solution. We will suppose that these characteristic numbers are at most denumerably infinite and have no more than one point of accumulation. If such a point of accumulation exists, we suppose that it is the point $\lambda = L_2$. We order the distinct characteristic numbers, which we will call $p_i$, as follows:

$$L_1 < p_1 < p_2 < \cdots < p_i < \cdots < L_2$$

and proceed to define a reduced system that is incompatible for each of the characteristic numbers of system (2.1), (2.2). We use Theorem I and define our reduced equation by a method that is analogous to that used by the author* in a previous paper. Theorem I gives us much more than is needed in this particular application. Substitute the $i$th characteristic number into the system (2.1), (2.2). The resulting system has compatibility at least as great as unity. By Theorem I we can define a positive continuous function $g(x, p_i)$ in such a way that the system

$$\begin{aligned} N_j(x, p_i) &= 0 \quad \text{for } j \neq k, \\ N_k(x, p_i) &= g(x, p_i)\, y_m(x, p_i), \\ U[y(x, p_i)] &= 0, \end{aligned} \tag{4.1}$$

is incompatible. We define $g(x, p_i)$ so that it has the above properties and we do this for each characteristic number of the system (2.1), (2.2). We define a function $g(x, \lambda)$ by the equation

$$\begin{aligned} g(x, \lambda) &= g(x, p_i) + \frac{\lambda - p_i}{p_{i+1} - p_i}[g(x, p_{i+1}) - g(x, p_i)] \quad \text{on } p_i \leqq \lambda \leqq p_{i+1}, \\ g(x, \lambda) &= g(x, p_1) \quad \text{on } L_1 < \lambda \leqq p_1. \end{aligned} \tag{4.2}$$

$g(x, \lambda)$ is defined completely as a continuous function of $(x, \lambda)$ on $XL$ that is positive and takes on the value $g(x, p_i)$ at $\lambda = p_i$. Hence the system

$$\begin{aligned} N_j = 0, \quad j \neq k; \qquad N_k &= g(x, \lambda)\, y_m, \\ U(y) &= 0, \end{aligned} \tag{4.3}$$

* Whyburn, loc. cit.

is incompatible when the system (2.1), (2.2) is compatible. If we define $\bar{A}$ to be equal to $A$ except that $\bar{A}_{km} = A_{km} - g(x, \lambda)$, equation (4.3) becomes

$$\bar{N} = 0 \tag{4.4}$$

while equation (2.1) takes the form

$$\bar{N}_j = 0, \quad j \neq k; \qquad \bar{N}_k = -g(x, \lambda)\, y_m. \tag{4.5}$$

We proceed now to set up the Green's tensor for the system (4.4), (2.2) knowing that this system is incompatible when the system (2.1), (2.2) is compatible. We obtain a tensor which we call $K$, in exactly the same manner as that by which we obtained $G$ in the preceeding section. $K$ will exist and have the properties 1, 2, 3, and 4 of $G$ for those values of $\lambda$ for which the system (2.1), (2.2) is compatible. We obtain an expression for the solution of system (2.1), (2.2) (taken in the form of (4.5), (2.2)) by the same method that was used in treating system (3.2), (3.3). This gives the vector $y$ where

$$y_j(x, \lambda) = -\int_a^b K_{kj}(x, t; \lambda)\, g(t, \lambda)\, y_m(t, \lambda)\, dt. \tag{4.6}$$

This system of $n$ integral equations in the $n$ unknowns $y_j$ is equivalent to the differential system (2.1), (2.2) in the sense that, when one system is compatible, the other is also compatible and every solution of the one system is also a solution of the other system. We may therefore state that the integral system (4.6) has the same set of characteristic numbers as the differential system (2.1), (2.2).

If we set up the integral system corresponding to the system (3.2), (3.3) using the Green's tensor $K$, we get

$$y_j(x, \lambda) = -\int_a^b K_{kj}(x, t; \lambda)\, g(t, \lambda)\, y_m(t, \lambda)\, dt + f_j(x, \lambda) \tag{4.7}$$

where

$$f_j(x, \lambda) = \int_a^b K_j \cdot \beta\, dt.$$

Finally, if we consider the system (3.2), (3.6), we get an integral system that has the form of (4.7) but in this case $f_j$ contains an additional term, $u_j$.

Although we have assumed the existence of a set of characteristic numbers having certain properties, this ceases to be an assumption in the case where $n$ is equal to two. Sturdivant* has shown that, when certain

* Cf. a paper offered to Trans. Amer. Math. Soc.

simple conditions are satisfied by the coefficients of system (2.1), (2.2), there will exist a set of characteristic numbers for the second order system and this set will have the properties that are assumed in the foregoing discussion. Sturdivant also proves oscillation theorems that carry over without difficulty to the corresponding integral system. The additional power that is contained in Theorem I may be used to get other relations between the differential and integral systems.

5. **The complex case.** Let the variable $x$ and the parameter $\lambda$ be complex and let the coefficients of the system (2.1), (2.2) be analytic in a certain domain $XL$. Let the boundary conditions be related to two points, $a$ and $b$, of $D$. We proceed to define Green's tensors for the system for all values of $\lambda$ for which it is incompatible. We set up two tensors, $G$ and $H$, so that they have the following properties:

1. $G$ and $H$ are analytic in $x$ and analytic in $t$ throughout $D$.
2. As functions of $x$ their components satisfy equation (2.1) throughout $D$.
3. At $x = t$ the following equation is valid:

$$H(t, t; \lambda) - G(t, t; \lambda) = E.$$

4. $\varphi[G_j(a)] - \varphi[H_j(b)] = 0$.
5. $G$ and $H$ are analytic in $\lambda$ except for poles at the characteristic numbers of the system (2.1), (2.2).

We can choose our fundamental tensor, $Y$, so that it will be analytic in $\lambda$. We let $G = Y \cdot C$ and $H = Y \cdot D$ and it is easy to see that, if the same calculations are made here that were made in the previous case, $G$ and $H$ will be uniquely determined so that they have the above properties. That $G$ and $H$ have property 5 follows when we examine the explicit formulas for these quantities and note that the only singularities that enter are poles of some order (depending upon the multiplicity of the characteristic numbers) due to the vanishing of the denominators. But $|Q|$ is the only quantity in the denominators that can vanish and it vanishes only at the characteristic numbers of the system (2.1), (2.2).

The solution of system (3.2), (2.2) is $y$, where

$$y_j = \int_a^x H_j \cdot \beta \, dt - \int_b^x G_j \cdot \beta \, dt. \tag{5.1}$$

This can be verified by substitution.

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# ON LAMÉ FAMILIES OF SURFACES.*

By C. E. Weatherburn.

1. **Introduction.** It is known that, in order that a family of surfaces $\varphi(x, y, z) = \text{const.}$ may form part of a triply orthogonal system, the function $\varphi$ must satisfy a certain differential equation of the third order, represented by the vanishing of a determinant of the sixth order. We here examine the question from a different point of view, and obtain results both simpler in form and more illuminating geometrically. A family of surfaces is determined if the unit normal $\boldsymbol{n}$ is given as a point function in space, satisfying the condition of normality

$$\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n} = 0. \tag{1}$$

For this vector is the unit tangent to a normal congruence of curves which cut the family of surfaces orthogonally. Conversely, if the equation of the family is given in the form $\varphi(x, y, z) = \text{const.}$, the unit normal is

$$\boldsymbol{n} = \frac{\nabla \varphi}{|\nabla \varphi|}.$$

It will first be shown that, if a singly infinite family of surfaces is a Lamé family, that is to say, forms part of a triply orthogonal system, the unit normal $\boldsymbol{n}$ satisfies the differential equation

$$\operatorname{div}\left(\frac{\partial}{\partial n} \operatorname{curl} \boldsymbol{n}\right) = 0 \tag{2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the direction of $\boldsymbol{n}$. This equation really involves only derivatives of $\boldsymbol{n}$ of the first two orders, as the third derivatives vanish identically. Other related forms for the condition are deduced, and certain consequences are noticed which follow very simply from these equations. In terms of the orthogonal trajectories of the family of surfaces, the above condition will be transformed to

$$\operatorname{div}(\varkappa' \boldsymbol{b} - \varkappa \tau \boldsymbol{p}) = 0 \tag{3}$$

where $\varkappa, \tau$ are the curvature and torsion, and $\boldsymbol{p}, \boldsymbol{b}$ the unit principal normal and binormal, the prime denoting differentiation with respect to the arc-length. From this also some important general results are very easily

* Received October 30, 1926.

deduced. Other forms of the condition are then established in terms of the *distance function* for the family of surfaces; and an alternative form to (2) is obtained for a family of surfaces of constant first curvature.

2. **Equation of condition.** Consider the families of surfaces

$$u = \text{const.}, \qquad v = \text{const.}, \qquad w = \text{const.}$$

forming a triply orthogonal system. The position vector $\boldsymbol{r}$, of a point in the space occupied by the surfaces, may be expressed as a function of the three parameters $u, v, w$. Let suffixes 1, 2, 3 denote partial differentiations with respect to $u, v, w$ respectively. Thus

$$\boldsymbol{r}_1 = \frac{\partial \boldsymbol{r}}{\partial u}, \qquad \boldsymbol{r}_2 = \frac{\partial \boldsymbol{r}}{\partial v}, \qquad \boldsymbol{r}_{23} = \frac{\partial^2 \boldsymbol{r}}{\partial v\, \partial w},$$

and so on. Also let $a, b, c$ be defined by

$$a = \boldsymbol{r}_1^2, \qquad b = \boldsymbol{r}_2^2, \qquad c = \boldsymbol{r}_3^2.$$

Then the elements of arc of the parametric curves are $\sqrt{a}\,du$, $\sqrt{b}\,dv$, $\sqrt{c}\,dw$. It will be convenient to introduce the symbol

$$p = \sqrt{abc}.$$

The unit normal $\boldsymbol{n}$ to a surface $w = \text{const.}$ is $\boldsymbol{r}_3/\sqrt{c}$, and its curl is given by*

$$\operatorname{curl} \boldsymbol{n} = \frac{1}{2p\sqrt{c}}(c_2 \boldsymbol{r}_1 - c_1 \boldsymbol{r}_2).$$

The derivative of this in the direction of $\boldsymbol{n}$ is therefore

$$\frac{\partial}{\partial n} \operatorname{curl} \boldsymbol{n} = \frac{1}{2\sqrt{c}} \frac{\partial}{\partial w}\left[\frac{1}{p\sqrt{c}}(c_2 \boldsymbol{r}_1 - c_1 \boldsymbol{r}_2)\right].$$

Then, since

$$\boldsymbol{r}_{13} = \frac{1}{2}\left(\frac{a_3}{a}\boldsymbol{r}_1 + \frac{c_1}{c}\boldsymbol{r}_3\right), \qquad \boldsymbol{r}_{23} = \frac{1}{2}\left(\frac{b_3}{b}\boldsymbol{r}_2 + \frac{c_2}{c}\boldsymbol{r}_3\right),$$

we may express the relation as

$$\frac{\partial}{\partial n} \operatorname{curl} \boldsymbol{n} = \frac{1}{2p}\left[\left(\frac{\partial^2}{\partial v\, \partial w}\log c - \frac{c_2 b_3}{2bc}\right)\boldsymbol{r}_1 - \left(\frac{\partial^2}{\partial u\, \partial w}\log c - \frac{c_1 a_3}{2ac}\right)\boldsymbol{r}_2\right].$$

* Cf §§ 4, 5 of a paper by the author "On triple systems of surfaces etc". Proc. Roy. Soc. Edinburgh, vol. 46, pp. 194–205 (1926).

In the divergence of this function the derivatives of the third order vanish identically, giving

$$\operatorname{div}\left(\frac{\partial}{\partial n}\operatorname{curl}\boldsymbol{n}\right) = \frac{1}{4}\left[\frac{\partial}{\partial v}\left(\frac{c_1 a_3}{ac}\right) - \frac{\partial}{\partial u}\left(\frac{c_2 b_3}{bc}\right)\right].$$

Then using Lamé's relations* between the derivatives of $a, b, c$ of the type

$$\frac{\partial^2}{\partial v\,\partial w}\log a = \frac{a_3 c_2}{2ac} + \frac{a_2 b_3}{2ab} - \frac{a_2 a_3}{2a^2}$$

we find that the above expression vanishes identically. Hence the theorem:

*The unit normal* $\boldsymbol{n}$ *to a Lamé family of surfaces satisfies the differential equation*

$$\operatorname{div}\left(\frac{\partial}{\partial n}\operatorname{curl}\boldsymbol{n}\right) = 0, \tag{2}$$

which may also be written

$$\operatorname{div}(\boldsymbol{n}\cdot\nabla\operatorname{curl}\boldsymbol{n}) = 0. \tag{2'}$$

In particular a *family of parallel surfaces* forms part of a triply orthogonal system. For such a family is characterised by the property† that $\operatorname{curl}\boldsymbol{n}$ vanishes identically.

The equation (2) is really of the second order in the derivatives of $\boldsymbol{n}$. This may be shown by transforming as follows. On taking the divergence of both members of the identity

$$\operatorname{curl}(\boldsymbol{n}\operatorname{div}\boldsymbol{n} + \boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}) = (\nabla\operatorname{div}\boldsymbol{n})\times\boldsymbol{n} + (\operatorname{curl}\boldsymbol{n})\cdot\nabla\boldsymbol{n} - \boldsymbol{n}\cdot\nabla\operatorname{curl}\boldsymbol{n},$$

we obtain the relation

$$\begin{aligned}\operatorname{div}(\boldsymbol{n}\cdot\nabla\operatorname{curl}\boldsymbol{n}) &= \operatorname{div}[(\operatorname{curl}\boldsymbol{n})\cdot\nabla\boldsymbol{n} - \boldsymbol{n}\times\nabla\operatorname{div}\boldsymbol{n}]\\ &= \operatorname{div}[(\operatorname{curl}\boldsymbol{n})\cdot\nabla\boldsymbol{n} - \operatorname{curl}\boldsymbol{n}\cdot(\nabla\operatorname{div}\boldsymbol{n})].\end{aligned}$$

Since the last expression involves no derivatives of $\boldsymbol{n}$ higher than the second, it follows that:

*The unit normal* $\boldsymbol{n}$ *to a Lamé family of surfaces satisfies a differential equation of the second order which may be expressed in the alternative forms*

$$\operatorname{div}[(\operatorname{curl}\boldsymbol{n})\cdot\nabla\boldsymbol{n} - \boldsymbol{n}\times\nabla\operatorname{div}\boldsymbol{n}] = 0, \tag{4}$$

* Cf. the author's *Differential Geometry*, § 111 and p. 257, Ex. 16.

† *Differential Geometry*, p. 261.

or

$$\text{div}\,[(\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n}] = \text{curl}\,\boldsymbol{n}\cdot(\nabla\,\text{div}\,\boldsymbol{n}). \tag{5}$$

Various other alternative forms* may be deduced for the above condition. Thus, from the identity

$$\text{curl}\,(\boldsymbol{n}\times\text{curl}\,\boldsymbol{n}) = (\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n} - \boldsymbol{n}\cdot\nabla\,\text{curl}\,\boldsymbol{n} - \text{div}\,\boldsymbol{n}\,\text{curl}\,\boldsymbol{n},$$

on taking the divergence of both members, we see that the equation (2′) is equivalent to

$$\text{div}\,[(\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n} - \text{div}\,\boldsymbol{n}\,\text{curl}\,\boldsymbol{n}] = 0. \tag{6}$$

3. **Orthogonal trajectories.** Consider the congruence of curves which are the orthogonal trajectories of the family of surfaces $w = \text{const.}$ The unit tangent $\boldsymbol{t}$ to the curve at any point is the unit normal $\boldsymbol{n}$ to the surface at that point. Let $\varkappa, \tau$ be the curvature and torsion of the curve, $s$ the arc-length, and $\boldsymbol{p}, \boldsymbol{b}$ the unit principal normal and binormal respectively. Then, since $\boldsymbol{n}$ is a vector of constant length,†

$$-\boldsymbol{n}\times\text{curl}\,\boldsymbol{n} = \boldsymbol{n}\cdot\nabla\boldsymbol{n} = \frac{d\boldsymbol{t}}{ds} = \varkappa\boldsymbol{p}$$

and therefore

$$\text{curl}\,\boldsymbol{n} = \boldsymbol{n}\times(\varkappa\boldsymbol{p}) = \varkappa\boldsymbol{t}\times\boldsymbol{p} = \varkappa\boldsymbol{b}. \tag{7}$$

Thus the magnitude of curl $\boldsymbol{n}$ is the curvature of the orthogonal trajectory; and its direction is that of the binormal of this curve. The condition (2) for a Lamé family may then be expressed

$$\text{div}\,\frac{d}{ds}(\varkappa\boldsymbol{b}) = 0, \tag{8}$$

that is, in virtue of the Serret-Frenet formulae‡

$$\text{div}\,(\varkappa'\boldsymbol{b} - \varkappa\tau\boldsymbol{p}) = 0. \tag{9}$$

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* Taking the gradient of both member of (1) we obtain

$$0 = (\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n} + \boldsymbol{n}\cdot\nabla\,\text{curl}\,\boldsymbol{n} + \boldsymbol{n}\times\text{curl curl}\,\boldsymbol{n}.$$

Thus the condition (2) is equivalent to

$$\text{div}\,[(\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n} + \boldsymbol{n}\times\text{curl curl}\,\boldsymbol{n}] = 0,$$

or, in virtue of (4) to

$$\text{div}\,[\boldsymbol{n}\times(\nabla\,\text{div}\,\boldsymbol{n} + \text{curl curl}\,\boldsymbol{n})] = 0.$$

† Cf. the author's *Advanced Vector Analysis*, § 8, (19), London, 1924.

‡ *Differential Geometry*, p. 15.

Hence the theorem:

*If a family of surfaces forms part of a triply orthogonal system, its orthogonal trajectories must be such that* $\operatorname{div}(\varkappa' \boldsymbol{b} - \varkappa\tau\boldsymbol{p})$ *vanishes identically.*

The vector $\varkappa' \boldsymbol{b} - \varkappa\tau\boldsymbol{p}$ will itself vanish identically only if the orthogonal trajectories are plane curves and $\varkappa'$ is zero. These curves then constitute a normal congruence of circles, commonly called a "cyclic congruence". Thus:

*Any Lamé family possessing the property that* $\frac{\partial}{\partial n} \operatorname{curl} \boldsymbol{n}$, (*but not* $\operatorname{curl} \boldsymbol{n}$) *vanishes identically, is such that its orthogonal trajectories constitute a cyclic congruence.*

For a family of parallel surfaces $\varkappa$ is everywhere zero, and $\operatorname{curl} \boldsymbol{n}$ vanishes identically as stated above.

More generally, if the orthogonal trajectories of a Lamé family are plane curves, these must satisfy the equation $\operatorname{div} \varkappa' \boldsymbol{b} = 0$, that is,

$$\operatorname{div}\left(\frac{\varkappa'}{\varkappa} \operatorname{curl} \boldsymbol{n}\right) = 0$$

and therefore

$$(\operatorname{curl} \boldsymbol{n}) \cdot \nabla \frac{\varkappa'}{\varkappa} = 0.$$

Thus $\nabla\left(\frac{d}{ds} \log \varkappa\right)$ is parallel to the osculating plane. And conversely:

*If a normal congruence of plane curves is such that* $\nabla\left(\frac{d}{ds} \log \varkappa\right)$ *is everywhere parallel to the osculating plane, the surfaces orthogonal to these constitute a Lamé family.*

**4. Lines of equidistance.** The distance along the normal between consecutive surfaces of the family $w = \text{const.}$ is $\sqrt{c}\, dw$. We may call $\sqrt{c}$ the *distance function,* the curves $c = \text{const.}$ on any surface being the *lines of equidistance.* Since the two-parametric gradient of the function $c$ on this surface has the value*

$$\frac{1}{a} c_1 \boldsymbol{r}_1 + \frac{1}{b} c_2 \boldsymbol{r}_2,$$

this vector is perpendicular to $\operatorname{curl} \boldsymbol{n}$. Thus *the direction of* $\operatorname{curl} \boldsymbol{n}$ *at any point is tangential to the line of equidistance through that point.*

Again it is easily verified that

$$(\nabla \boldsymbol{n}) \cdot \nabla \log c = \frac{1}{2\sqrt{c}}\left[\frac{a_3 c_1}{a^2 c} \boldsymbol{r}_1 + \frac{b_3 c_2}{b^2 c} \boldsymbol{r}_2 - \left(\frac{c_1^2}{a c^2} + \frac{c_2^2}{b c^2}\right) \boldsymbol{r}_3\right]$$

* Cf. a paper by the author "On differential invariants in geometry of surfaces, etc." § 3, Quar. Journ. of Math., vol. 50, p. 233; or *Differential Geometry*, p. 223.

and therefore

$$\boldsymbol{n}\times(\nabla\boldsymbol{n})\cdot\nabla\log c=\frac{1}{2p}\left(\frac{a_3c_1}{ac}\boldsymbol{r}_2-\frac{b_3c_2}{bc}\boldsymbol{r}_1\right).$$

Now, as in § 2, the divergence of this vector vanishes identically in virtue of Lamé's relations between the derivatives of $a, b, c$. Hence:

*The distance function $\sqrt{c}$ for a Lamé family of surfaces satisfies the differential equation*

$$\operatorname{div}\left[\boldsymbol{n}\times(\nabla\boldsymbol{n})\cdot\nabla\log\sqrt{c}\right]=0. \tag{10}$$

For a system of *parallel surfaces* $\nabla\boldsymbol{n}\cdot\nabla\log\sqrt{c}$ vanishes identically; while, for a *family of planes*, $\boldsymbol{n}\times(\nabla\boldsymbol{n})\cdot\log\sqrt{c}$ is zero. Thus the equation (10) is satisfied in both cases.

In this equation the differential invariants are three-parametric invariants for the space occupied by the family of surfaces. We may establish a similar formula in terms of the *two-parametric* operator $\nabla$ for a surface $w=$ const. The parameters on this surface are $u, v$, and with this operator*

$$\begin{aligned}\boldsymbol{n}\times(\nabla\boldsymbol{n})\cdot\nabla\sqrt{c}&=\boldsymbol{n}\times\left(\frac{a_3c_1}{4a^2c}\boldsymbol{r}_1+\frac{b_3c_2}{4b^2c}\boldsymbol{r}_2\right)\\&=\frac{1}{4\sqrt{ab}}\left(\frac{a_3c_1}{ac}\boldsymbol{r}_2-\frac{b_3c_2}{bc}\boldsymbol{r}_1\right)\end{aligned}$$

and the two-parametric divergence† of this vector vanishes for the same reason as before. Now, on any one surface of the family, the operator $(\nabla\boldsymbol{n})\cdot\nabla$ is an "invariant" operator for that surface. We have elsewhere‡ denoted it by $-\bar{\nabla}$. Thus:

*On each surface of a Lamé family the distance function $\sqrt{c}$ satisfies the two-parametric differential equation*

$$\operatorname{div}\left(\boldsymbol{n}\times\bar{\nabla}\sqrt{c}\right)=0, \tag{11}$$

*which may also be expressed*

$$\operatorname{div}\left[\bar{\nabla}\times(\boldsymbol{n}\sqrt{c})\right]=0. \tag{12}$$

We may here prove another theorem dealing with the condition that the lines of equidistance on each surface of the system may constitute a family of *parallels*. We have seen that, if $\boldsymbol{b}$ is the unit tangent to a line

* Cf. *Differential Geometry*, § 110.

† Ibid., § 116.

‡ In a paper "On small deformation of surfaces, etc.", Quar. Journ. of Math., vol. 50 (1925), p. 277.

of equidistance, and $\varkappa$ the curvature of the orthogonal trajectory of the family of surfaces, $\operatorname{curl} \boldsymbol{n} = \varkappa \boldsymbol{b}$. Now the three-parametric divergence of $\operatorname{curl} \boldsymbol{n}$ vanishes identically; and therefore, if div and $\nabla$ denote two-parametric operators for the surface $w = \text{const.}$, we have

$$0 = \operatorname{div}(\sqrt{c} \operatorname{curl} \boldsymbol{n}) = \operatorname{div}(\varkappa \sqrt{c} \boldsymbol{b})$$
$$= \boldsymbol{b} \cdot (\varkappa \nabla \sqrt{c} + \sqrt{c} \nabla \varkappa) + \varkappa \sqrt{c} \operatorname{div} \boldsymbol{b}.$$

The first term vanishes because $\boldsymbol{b}$ is tangential to the curve $c = \text{const.}$ The vanishing of $\operatorname{div} \boldsymbol{b}$ is the condition that the lines of equidistance may be a family of parallels. Hence the theorem:

*A necessary and sufficient condition that the lines of equidistance on any surface of a family may be parallels (and the lines of slope of the distance function therefore geodesics)* is that the curvature of the orthogonal trajectories of the family of surfaces be constant along each line of equidistance.*

We may remark that this theorem is true whether the family be a Lamé family or not; for we have made no use of the relation (2) or its equivalent.

Another theorem also may here be mentioned; viz.

*If one family of a triply orthogonal system is a family of developables, either it consists of planes, or else a second family is also one of developables and the third is one of parallels.*

Suppose, for instance, that the surfaces $u = \text{const.}$ are developables, and that the principal curvature in the direction of $\boldsymbol{r}_2$ is zero. Then $b_1 = 0$,† and consequently in virtue of the relation

$$\frac{\partial}{\partial w}\left(\frac{b_1}{b}\right) = \frac{b_1 a_3}{2ba} + \frac{b_3 c_1}{2bc} - \frac{b_1 b_3}{2b^2}$$

either $c_1 = 0$ or $b_3 = 0$. In the former case both principal curvatures of a surface $u = \text{const.}$ are zero, and it is therefore a plane. In the second case the surfaces $w = \text{const.}$ are developables, and the surfaces $v = \text{const.}$ are parallels, because for this family $\operatorname{curl} \boldsymbol{n}$ then vanishes identically ($b_1 = 0$, $b_3 = 0$).

5. **Surfaces of constant first curvature.** Consider next a family of surfaces, $w = \text{const.}$ for which the first curvature‡ (or mean curvature), $J$, is constant over each surface, but may vary from one to another. Then since $J = -\operatorname{div} \boldsymbol{n}$, and $\nabla J$ is parallel to $\boldsymbol{n}$, we must have

$$\boldsymbol{n} \times \nabla \operatorname{div} \boldsymbol{n} = 0.$$

* *Differential Geometry*, p. 259.

† Ibid., p. 210 (table).

‡ Ibid., p. 264.

The equation (4) thus takes the form

$$\text{div}\,[(\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n}] = 0. \tag{13}$$

That is to say:

*For a Lamé family of surfaces of constant first curvature, the unit normal satisfies the differential equation*

$$\text{div}\,[(\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n}] = 0.$$

For example, in the case of a *family of planes*, the derivative of $\boldsymbol{n}$ is zero for all directions perpendicular to $\boldsymbol{n}$. But curl $\boldsymbol{n}$ is perpendicular to $\boldsymbol{n}$. Hence $(\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n}$ vanishes identically, and the equation (13) is satisfied.

Or again, for a *family of spheres*, if $\boldsymbol{a}$ is any vector perpendicular to $\boldsymbol{n}$,

$$\boldsymbol{a}\cdot\nabla\boldsymbol{n} = F\boldsymbol{a}$$

where $F$ is a function of $w$ only, being the curvature of a great circle of the sphere. In particular

$$(\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n} = F\,\text{curl}\,\boldsymbol{n}$$

and therefore

$$\text{div}\,[(\text{curl}\,\boldsymbol{n})\cdot\nabla\boldsymbol{n}] = \nabla F\cdot\text{curl}\,\boldsymbol{n} = 0$$

since $\nabla F$ is parallel to $\boldsymbol{n}$, and therefore perpendicular to curl $\boldsymbol{n}$. The equation (13) is thus satisfied in the case of a family of spheres.

CANTERBURY COLLEGE, CHRISTCHURCH, N. Z.
November 1926.

# DERIVATION OF THE FREDHOLM THEORY FROM A DIFFERENTIAL EQUATION OF INFINITE ORDER.*

By H. T. Davis.

**1. The reduction method.** The author is aware of two distinct methods for obtaining the solution of Fredholm's integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x, s)\, u(s)\, ds. \tag{1}$$

The first of these is the method which suggested to I. Fredholm his original results† and which was later completely developed by D. Hilbert.‡ In this, equation (1) is shown to be equivalent to an infinite set of linear equations in an infinite number of variables and the solution obtained as the limit of the solution of the set. The second method is due to E. Goursat§ and H. Lebesgue.‖ The former, starting with a kernel of the form $K(x, s) = \sum_{i=1}^{n} U_i(x)\, V_i(s)$, obtained Fredholm's solution for this special kernel by assuming $u(x)$ to be of the form

$$u(x) = f(x) + \sum_{i=1}^{n} C_i\, U_i(x),$$

where the $C_i$ are constants. Lebesgue developed Goursat's theory in the general case where $n$ became infinite.

It is the purpose of this paper to indicate an entirely different way of solving (1) by employing the theory of differential equations of infinite order. Aside from the novelty of the method, the derivation of Fredholm's solution will serve as an example of the application of the theory to a broad class of linear functional equations. The author has elsewhere indicated another example.¶

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* Received October 14, 1926; presented to the American Mathematical Society, April 19, 1924.

† Comptes Rendus, vol. 134 (1902), p. 219; cf. also Acta Math., vol. 27 (1903), pp. 365–390.

‡ Theorie der linearen Integralgleichungen, Leipzig (1912), Chaps. 1, 2.

§ Sur un cas élémentaire de l'équation de Fredholm, Bull. de la soc. math. de France, vol. 35 (1907), pp. 163–173; see also Goursat, Cours d'analyse, vol. 3 (1915), pp. 386–390.

‖ Sur la méthode de M. Goursat pour la résolution de l'équation de Fredholm, ibid, vol. 36 (1908), pp. 3–19.

¶ The Euler differential equation of infinite order. Amer. Math. Monthly, vol. 32 (1925), pp. 223–233.

Let us assume that there has been established the formal equivalence of a linear functional equation,

$$L(u) = f(x), \tag{2}$$

and a differential equation of infinite order,*

$$[1+\varphi_0(x)]\,u(x)+\varphi_1(x)\,u'(x)+\cdots+\varphi_n(x)\,u^{(n)}(x)+\cdots=f(x). \tag{3}$$

Assuming infinite differentiability of the coefficients and of $f(x)$ we can differentiate (3) an infinite number of times and thus arrive at the following infinite system of equations in the infinitely many variables $u(x)$, $u'(x)$, $u''(x)$, $\cdots$:

$$\begin{aligned}
f(x) &= [1+\varphi_0(x)]\,u(x)+\varphi_1(x)\,u'(x)+\varphi_2(x)\,u''(x)+\cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots+\varphi_n(x)\,u^{(n)}(x)+\cdots \\
f'(x) &= [(1+\varphi_0)u]' \quad +(\varphi_1u')' \quad +(\varphi_2u'')' \quad +\cdots+(\varphi_nu^{(n)})' \quad +\cdots \\
f''(x) &= [(1+\varphi_0)u]'' \quad +(\varphi_1u')'' \quad +(\varphi_2u'')'' \quad +\cdots+(\varphi_nu^{(n)})'' \quad +\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
f^{(n-1)}(x) &= [(1+\varphi_0)u]^{(n-1)}+(\varphi_1u')^{(n-1)}+(\varphi_2u'')^{(n-1)}+\cdots+(\varphi_nu^{(n)})^{(n-1)}+\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{4}$$

In the solution of such systems it has been customary to restrict solutions to Hilbert space [i. e., the space of points $\{u^{(i)}\}$ for which $\sum_{i=1}^{\infty}(u^{(i)})^2$ converges] with corresponding conditions on the coefficients although modifications are to be found in F. Riesz: Systèmes d'équations linéaires, Paris (1913), pp. 73–77.†

It will be convenient in the present instance to use the so called "reduction method" (la méthode des réduites) which is contained in a theorem derived independently by H. von Koch in 1913‡ and A. Pellet in 1914.§ Its great usefulness is indicated in the recent application made of

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* For a formal method for deriving (3) from (2) see C. Bourlet: Sur les opérations en général et les équations différentielles linéaires d'ordre infini, Ann. de l'éc. norm. sup., (3), vol. 14 (1897), p. 150 et seq. There we have

$$\varphi_0 = L(1)-1,$$

$$\varphi_n = \frac{1}{n!}\left\{L(x^n)-{}_nC_1\,x\,L(x^{n-1})+{}_nC_2\,x^2\,L(x^{n-2})-\cdots\pm x^n\,L(1)\right\}, \quad n>0.$$

† An example of this method and the difficulties to which it leads can be seen in E. Hilb's solution of the Laplace equation of infinite order. Math. Annalen, vol. 82 (1920–21), pp. 1–39; vol. 84 (1921), pp. 16–30 and pp. 43–52.

‡ Jahresbericht Deutsch. Math.-Ver., vol. 22 (1913), pp. 285–291.

§ Sur la méthode des réduites, Bull. de la soc. math. de France, vol. 42 (1914), pp. 48–53.

it by H. W. March in the problem of the deflection of a rectangular plate fixed at the edges.*

THEOREM: *If in the system*

$$x_i - \psi_i = c_i, \qquad i = 1, 2, 3, \cdots, \infty,$$

*where* $\psi_i = \sum_{k=1}^{\infty} a_{ik} x_k$, *the* $c_i$ *are bounded, and the coefficients are subject to the condition that*

$$\sum_{k=1}^{\infty} |a_{ik}| < 1, \qquad i = 1, 2, 3, \cdots, \infty,$$

*then the* $x_i$ *exist and are equal to the limiting form as* $m \to \infty$, *of* $x_i^{(m)}$ *determined by solving the reduced system*

$$x_i^{(m)} - \sum_{k=1}^{m} a_{ik} x_k^{(m)} = c_i, \qquad i = 1, 2, 3, \cdots, m.$$

A sufficient condition for the existence of a solution of (4) is readily obtained from this theorem. If we make use of the abbreviation, $\sigma = |\varphi_0| + |\varphi_1| + |\varphi_2| + \cdots + |\varphi_n| + \cdots$, it is clear upon expansion of the terms of (4) that the sums $\sum_{k=1}^{\infty} |a_{ik}|$ of the theorem become in this case the sums:

$$\sigma,\ \sigma_1 = \sigma + \sigma' = e^{-x} \frac{d}{dx}(e^x \sigma),$$

$$\sigma_2 = \sigma + 2\sigma' + \sigma'' = e^{-x} \frac{d^2}{dx^2}(e^x \sigma),$$

$$\sigma_n = \sigma + n\sigma' + \frac{n(n-1)}{2!}\sigma'' + \cdots = e^{-x} \frac{d^n}{dx^n}(e^x \sigma).$$

We thus obtain the following result:

*A solution of* (4) *exists and can be obtained by the "reduction method" provided,*

$$e^{-x} \frac{d^n}{dx^n}(e^x \sigma) < 1, \qquad n = 1, 2, \cdots.$$

**2. An existence theorem for the Fredholm problem.** In order to apply the theory of the preceding section to the Fredholm equation, we first reduce (1) formally to a differential equation of infinite order by expanding $u(s)$ into the Taylor's series: $u(s) = u(x) + (s-x)u'(x) + \frac{(s-x)^2}{2!} u''(x) + \cdots$.

* Trans. Amer. Math. Soc., vol. 27 (1925), p. 311 et seq.

When this is substituted in (1) we obtain an equation of form (3) where

$$\varphi_i(x) = -\lambda \int_a^b K(x, s) \frac{(s-x)^i}{i!}\, ds. \tag{5}$$

After these coefficients have been substituted in system (4) the coefficients of the unknowns reduce in a remarkable way and we obtain for $D(n, x)$, i. e., the determinant of the reduced system obtained by suppressing terms of order greater than $n-1$ and all rows beyond the $n$th, the following:

$$D(n,x) = \begin{vmatrix} 1-\lambda\int_a^b K(x,s)\,ds, & -\lambda\int_a^b K(x,s)(s-x)\,ds, & \cdots \\ & \cdots, & -\lambda\int_a^b K(x,s)\frac{(s-x)^{n-1}}{(n-1)!}\,ds \\ -\lambda\int_a^b K'(x,s)\,ds, & 1-\lambda\int_a^b K'(x,s)(s-x)\,ds, & \cdots \\ & \cdots, & -\lambda\int_a^b K'(x,s)\frac{(s-x)^{n-1}}{(n-1)!}\,ds \\ \cdots\cdots\cdots\

This inequality illustrates an interesting feature of the reduction method. Although the limitation on $\lambda$ resembles the one ordinarily derived by the method of successive substitutions, the actual solution found by the reduction method is the meromorphic function of Fredholm.

**3. The Fredholm determinant.** It is evident that the Fredholm determinant, $D(\lambda)$ must be identified with the limit $\lim_{n=\infty} D(n, x)$. For this purpose we use the well known development for the secular determinant:*

$$\Delta = \begin{vmatrix} 1+C_{11}, & C_{12}, & \cdots, & C_{1n} \\ C_{21}, & 1+C_{22}, & \cdots, & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1}, & C_{n2}, & \cdots, & 1+C_{nn} \end{vmatrix}$$

$$= 1+\frac{1}{1!}\sum C_{r_1 r_2}+\frac{1}{2!}\sum \begin{vmatrix} C_{r_1 r_1}, & C_{r_1 r_2} \\ C_{r_2 r_1}, & C_{r_2 r_2} \end{vmatrix} + \cdots \tag{7}$$

$$\cdots + \frac{1}{n!}\sum \begin{vmatrix} C_{r_1 r_1}, & C_{r_1 r_2}, & \cdots, & C_{r_1 r_n} \\ C_{r_2 r_1}, & C_{r_2 r_2}, & \cdots, & C_{r_2 r_n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{r_n r_1}, & C_{r_n r_2}, & \cdots, & C_{r_n r_n} \end{vmatrix}$$

where $r_1, r_2, \cdots, r_n$ run independently over all values from 1 to $n$.

Furthermore it is easily shown† that

$$\begin{vmatrix} \int_a^b \varphi_{11}(s)\,ds, & \int_a^b \varphi_{12}(s)\,ds, & \cdots, & \int_a^b \varphi_{1n}(s)\,ds \\ \int_a^b \varphi_{21}(s)\,ds, & \int_a^b \varphi_{22}(s)\,ds, & \cdots, & \int_a^b \varphi_{2n}(s)\,ds \\ \cdots & \cdots & \cdots & \cdots \\ \int_a^b \varphi_{n1}(s)\,ds, & \int_a^b \varphi_{n2}(s)\,ds, & \cdots, & \int_a^b \varphi_{nn}(s)\,ds \end{vmatrix}$$

$$= \underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} \varphi_{11}(s_1), & \varphi_{12}(s_1), & \cdots, & \varphi_{1n}(s_1) \\ \varphi_{21}(s_2), & \varphi_{22}(s_2), & \cdots, & \varphi_{2n}(s_2) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{n1}(s_n), & \varphi_{n2}(s_n), & \cdots, & \varphi_{nn}(s_n) \end{vmatrix} ds_1\, ds_2 \cdots ds_n.$$

Introducing the notation $K_{ij} = K^{(j-1)}(x, s_i)$, and $S_{ij} = \dfrac{(s_i - x)^{j-1}}{(j-1)!}$, it is clear that $D(n, x)$ can be written in the following form:

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* G. Kowalewski: Einführung in die Determinantentheorie, Leipzig, (1909), pp. 455–456.

† See E. Goursat, Sur un cas élémentaire etc., loc. cit. p. 165–166.

$$D(n,x) = 1 - \frac{\lambda}{1!}\int_a^b \sum \Delta_1\, ds_1 + \frac{\lambda^2}{2!}\int_a^b\int_a^b \sum \Delta_2\, ds_1\, ds_2 + \cdots$$
$$\cdots + (-1)^n \frac{\lambda^n}{n!} \underbrace{\int_a^b \cdots \int_a^b}_{n} \sum \Delta_n\, ds_1\, ds_2 \cdots ds_n, \tag{8}$$

where we have used the abbreviation,

$$\Delta_n = \begin{vmatrix} K_{1r_1}\, S_{1r_1}, & K_{2r_1}\, S_{2r_2}, & \cdots, & K_{nr_1}\, S_{nr_n} \\ K_{1r_2}\, S_{1r_1}, & K_{2r_2}\, S_{2r_2}, & \cdots, & K_{nr_2}\, S_{nr_n} \\ \cdots\cdots\cdots\cdots & & & \\ K_{1r_n}\, S_{1r_1}, & K_{2r_n}\, S_{2r_2}, & \cdots, & K_{nr_n}\, S_{nr_n} \end{vmatrix}$$

and the values of $r_1, r_2, \cdots, r_n$ run independently over all integers from 1 to $n$.

But the $m$-th sum, if we interchange the $K_{ij}$ by row and column as we can do since the $S_{ik}$ are factors of each column, is precisely the definition of the product of the two matrices*

$$P_{mn} = \left\| \begin{matrix} K_{11}, & K_{12}, & \cdots, & K_{1n} \\ K_{21}, & K_{22}, & \cdots, & K_{2n} \\ \cdots\cdots\cdots & & & \\ K_{m1}, & K_{m2}, & \cdots, & K_{mn} \end{matrix} \right\| \cdot \left\| \begin{matrix} S_{11}, & S_{12}, & \cdots, & S_{1n} \\ S_{21}, & S_{22}, & \cdots, & S_{2n} \\ \cdots\cdots\cdots & & & \\ S_{m1}, & S_{m2}, & \cdots, & S_{mn} \end{matrix} \right\|.$$

Making use of the identity $\sum K_{ij}\, S_{kj} = K(s_i, s_k)$, we have the following remarkable reduction:

$$\lim_{n=\infty} P_{mn} = |\, K(s_i, s_j)\,|,$$

which, using the notation of Fredholm, we may designate by the symbol $K\begin{pmatrix} s_1, s_2, \cdots, s_m \\ s_1, s_2, \cdots, s_m \end{pmatrix}$.

In the limiting case of (8) we then get the value of Fredholm's determinant,

$$\lim_{n=\infty} D(n,x) = D(\lambda) = 1 - \lambda\int_a^b K(s_1,s_1)\, ds_1 + \frac{\lambda^2}{2!}\int_a^b\int_a^b K\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} ds_1\, ds_2 + \cdots$$
$$\cdots + \frac{(-1)^n}{n!}\lambda^n \underbrace{\int_a^b \cdots \int_a^b}_{n} K\begin{pmatrix} s_1, s_2, \cdots, s_n \\ s_1, s_2, \cdots, s_n \end{pmatrix} ds_1\, ds_2 \cdots ds_n + \cdots.$$

**4. The Fredholm minors.** We turn next to the computation of the cofactors of the elements of the first column of (6). These cofactors for brevity we may designate by $D_i$ and the solution of the original integral

* Kowalewski, loc. cit., p. 72.

equation as given by the reduction method will be, in terms of them, the limit as $n \to \infty$ of the following sum:

$$(9) \qquad u_n(x) = \frac{D_1 f(x) + D_2 f'(x) + D_3 f''(x) + \cdots + D_n f^{(n-1)}(x)}{D(n, x)}.$$

For our purpose we introduce the determinant

$$\begin{vmatrix} 1+C_{rr}, & C_{rs}, & C_{rk_1}, & \cdots, & C_{rk_{n-2}} \\ C_{sr}, & 1+C_{ss}, & C_{sk_1}, & \cdots, & C_{sk_{n-2}} \\ C_{k_1 r}, & C_{k_1 s}, & 1+C_{k_1 k_1}, & \cdots, & C_{k_1 k_{n-2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{k_{n-2} r}, & C_{k_{n-2} s}, & C_{k_{n-2} k_1}, & \cdots, & 1+C_{k_{n-2} k_{n-2}} \end{vmatrix}.$$

The cofactor of $C_{sr}$, $r \neq s$, can be expanded as follows:*

$$-\Delta_{rs} = C_{rs} + \frac{1}{1!}\sum \begin{vmatrix} C_{rs}, & C_{rr_1} \\ C_{r_1 s}, & C_{r_1 r_1} \end{vmatrix} + \frac{1}{2!}\sum \begin{vmatrix} C_{rs}, & C_{rr_1}, & C_{rr_2} \\ C_{r_1 s}, & C_{r_1 r_1}, & C_{r_1 r_2} \\ C_{r_2 s}, & C_{r_2 r_1}, & C_{r_2 r_2} \end{vmatrix} + \cdots$$

$$\cdots + \frac{1}{(n-2)!}\sum \begin{vmatrix} C_{rs}, & C_{rr_1}, & \cdots, & C_{rr_{n-2}} \\ C_{r_1 s}, & C_{r_1 r_1}, & \cdots, & C_{r_1 r_{n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ C_{r_{n-2} s}, & C_{r_{n-2} r_1}, & \cdots, & C_{r_{n-2} r_{n-2}} \end{vmatrix}.$$

where $r_1, r_2, \cdots, r_{n-1}$ range independently over all the values $k_1, k_2, \cdots, k_{n-2}$.

Applying this formula to the elements of the first column of (6) we have, for $i \neq 1$,

$$D_i = \lambda \int_a^b K_{11} S_{1i}\, ds_1 - \frac{\lambda^2}{1!}\int_a^b\int_a^b \sum_{r_1} \delta_{i1}\, ds_1\, ds_2 + \cdots$$

$$\cdots + (-1)^{n-2}\frac{\lambda^{n-1}}{(n-2)!}\underbrace{\int_a^b \cdots \int_a^b}_{n} \sum_{r_1 r_2 \cdots r_{n-2}} \delta_{i,n-1}\, ds_1\, ds_2 \cdots ds_{n-1}$$

where we use the abbreviation

$$\delta_{i,m-1} = \begin{vmatrix} K_{11} & S_{1i}, & K_{21} & S_{2r_1}, & \cdots, & K_{m_1} & S_{mr_{m-1}} \\ K_{1r_1} & S_{1i}, & K_{2r_1} & S_{2r_1}, & \cdots, & K_{mr_1} & S_{mr_{m-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{1r_{m-1}} & S_{1i}, & K_{2r_{m-1}} & S_{2r_1}, & \cdots, & K_{mr_{m-1}} & S_{mr_{m-1}} \end{vmatrix}.$$

$D_1$, the cofactor of the first element, is easily computed by observing that we have

* Kowalewski, loc. cit. p. 468.

$$\Delta_{11} = \Delta - C_{11}\Delta_{11} - C_{21}\Delta_{21} - \cdots - C_{n1}\Delta_{n1}$$

$$= \Delta - C_{11} - \frac{1}{1!}\sum\begin{vmatrix} C_{11}, & C_{1r_1} \\ C_{r_1 1}, & C_{r_1 r_1} \end{vmatrix} - \frac{1}{2!}\sum\begin{vmatrix} C_{11}, & C_{1r_1}, & C_{1r_2} \\ C_{r_1 1}, & C_{r_1 r_1}, & C_{r_1 r_2} \\ C_{r_2 1}, & C_{r_2 r_1}, & C_{r_2 r_2} \end{vmatrix} - \cdots$$

$$\cdots - \frac{1}{(n-1)!}\sum\begin{vmatrix} C_{11}, & C_{1r_1}, & \cdots, & C_{1r_{n-1}} \\ C_{r_1 1}, & C_{r_1 r_1}, & \cdots, & C_{r_1 r_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ C_{r_{n-1}1}, & C_{r_{n-1}r_1}, & \cdots, & C_{r_{n-1}r_{n-1}} \end{vmatrix},$$

where $r_1, r_2, \cdots, r_{n-1}$ range over the values 2, 3, $\cdots$, $n$.

Replacing the $C_{ij}$ by the elements of (6) we get,

$$D_1 = D(n,x) + \lambda\int_a^b K(x,s_1)\,ds_1 - \frac{\lambda^2}{1!}\int_a^b\int_a^b \sum_{r_1}\delta_{11}\,ds_1\,ds_2 + \cdots$$

$$\cdots + (-1)^{n-1}\frac{\lambda^n}{(n-1)!}\underbrace{\int_a^b\cdots\int_a^b}_{n}\sum_{r_1 r_2\cdots r_{n-1}}\delta_{1,n-1}\,ds_1\,ds_2\cdots ds_n.$$

Substituting the values of $D_i$ in the numerator of (9) we see that it can be written as follows:

$$f(x)D(n,x) + \lambda\int_a^b\sum_i K_{11}S_{1i}f^{(i-1)}(x)\,ds_1 - \frac{\lambda^2}{1!}\int_a^b\int_a^b\sum_{ir_1}\delta_{i1}f^{(i-1)}(x)\,ds_1\,ds_2 + \cdots$$

$$\cdots + (-1)^{n-1}\frac{\lambda^{n-1}}{(n-2)!}\underbrace{\int_a^b\cdots\int_a^b}_{n-1}\sum_{ir_1r_2\cdots r_{n-2}}\delta_{i,n-2}f^{(i-1)}(x)\,ds_1 ds_2\cdots ds_{n-1} \tag{10}$$

$$+ (-1)^{n-1}\frac{\lambda^n}{(n-1)!}\underbrace{\int_a^b\cdots\int_a^b}_{n}\sum_{r_1 r_2\cdots r_{n-1}}\delta_{i,n-1}f(x)\,ds_1\,ds_2\cdots ds_n,$$

where $r_1, r_2, \cdots, r_{n-1}$ run over all values from 2 to $n$ and $i$ runs from 1 to $n$.

In simplifying this expression we note the two following identities:

$$\sum_{i=1}^{\infty} S_{1i}f^{(i-1)}(x) = f(s_1),$$

and

$$\begin{Vmatrix} K(x,s_1), & K'(x,s_1), & \cdots \\ K(x,s_2), & K'(x,s_2), & \cdots \\ \cdots & \cdots & \cdots \\ K(x,s_n), & K'(x,s_n), & \cdots \end{Vmatrix}\begin{Vmatrix} 1, & 0, & 0, & \cdots \\ 1, & s_2 - x, & \frac{(s_2-x)^2}{2!}, & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1, & s_n - x, & \frac{(s_n-x)^2}{2!}, & \cdots \end{Vmatrix}$$

$$= \begin{vmatrix} K(x,s_1), & K(x,s_2), & \cdots, & K(x,s_n) \\ K(s_2,s_1), & K(s_2,s_2), & \cdots, & K(s_2,s_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(s_n,s_1), & K(s_n,s_2), & \cdots, & K(s_n,s_n) \end{vmatrix}.$$

It is convenient to use Fredholm's abbreviation $K\begin{pmatrix} x, & s_2, & \cdots, & s_n \\ s_1, & s_2, & \cdots, & s_n \end{pmatrix}$ for this last determinant.

If now in (10) we let $n$ approach infinity and make use of the two identities just written down, we shall have, since $S_{1i}$ is a factor of each determinant, the following limit:

$$f(x)\,D(\lambda)+\lambda\int_a^b K(x,s)f(s)\,ds-\frac{\lambda^2}{1!}\int_a^b\int_a^b K\begin{pmatrix} x, & s_2 \\ s_1, & s_2 \end{pmatrix} f(s_1)\,ds_1\,ds_2+\cdots$$

$$\cdots+(-1)^{n-1}\frac{\lambda^n}{(n-1)!}\underbrace{\int_a^b\cdots\int_a^b}_{n} K\begin{pmatrix} x, & s_2, & \cdots, & s_n \\ s_1, & s_2, & \cdots, & s_n \end{pmatrix} f(s_1)\,ds_1\,ds_2\cdots ds_n+\cdots.$$

The limiting form of (9) is thus seen to be

$$\lim_{n=\infty} u_n(x) = u(x) = f(x)+\lambda\int_a^b \frac{D\begin{pmatrix} x \\ s \end{pmatrix}}{D(\lambda)} f(s)\,ds, \tag{11}$$

where we employ the usual abbreviation

$$D\begin{pmatrix} x \\ y \end{pmatrix} = K(x,y)-\frac{\lambda}{1!}\int_a^b K\begin{pmatrix} x, & s_2 \\ y, & s_2 \end{pmatrix} ds_2+\frac{\lambda^2}{2!}\int_a^b\int_a^b K\begin{pmatrix} x, & s_2, & s_3 \\ y, & s_2, & s_3 \end{pmatrix} ds_2\,ds_3-\cdots.$$

The formal verification that (11) is actually a solution of equation (1) when $D(\lambda)\neq 0$ can be made in the usual manner by substitution.

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# ASYMPTOTIC EXPRESSION FOR THE PROBABILITY OF TRIALS CONNECTED IN A CHAIN.*

By Echo D. Pepper.

In the theory of probability, trials connected in a chain present a very important case of dependent trials. Markoff† gave a solution based on a most difficult theory of moments created by him and Tchebecheff. The solution does not give the error, a fact which from Markoff's point of view diminished its practical importance.

Here is presented a direct solution of the problem with the order of error. It is even possible to obtain an exact estimation of the error, but to avoid calculation we do not do that. Further, we obtain not only the probability for $m$ (the number of appearances of an event in $n$ trials) to lie in some interval, but also the probability for particular values of $m$.

Consider $n$ trials $T_1, T_2, \cdots, T_n$, each of which results in $E$ the appearance, or $F$ the non-appearance of an event. Let $p$ denote the probability of $E$ in $T_k$ the $k$th trial, when the results of all other trials are not known, and $q = 1 - p$ the probability of $F$ in that trial. Suppose further that the trials $T_1, T_2, \cdots, T_n$ are not independent, so that a knowledge of the results of the previous trials affects the probability of $E$ in the given trial.

Let $p_0$ denote the probability of $E$ in the $k$th trial when $F$ resulted in the $(k-1)$th trial, $q_0 = 1 - p_0$ the probability of $F$ in this case. Let $p_1$ denote the probability of $E$ in the $k$th trial when $E$ resulted in the $(k-1)$th trial, $q_1 = 1 - p_1$ the probability of $F$ in this case.

We suppose further that the probability of $E$ in the $k$th trial when the result of $T_{k-1}$ is known, is independent of the results of all the other previous trials.

The appearance of $E$ at $T_k$ is equivalent to the appearance of one of the two combinations; 1. the appearance of $E$ at $T_{k-1}$ and at $T_k$, 2. the appearance of $F$ at $T_{k-1}$ and of $E$ at $T_k$. These two combinations are incompatible. The probability of the first combination is equal to the probability of $E$ at $T_{k-1}$ multiplied by the probability of $E$ at $T_k$, when it is known that $E$ appeared at $T_{k-1}$, i. e., the probability of the first combination is $pp_1$, in the same way the probability of the second combination is $qp_0$. Then from the addition theorem for probabilities

$$p = pp_1 + qp_0$$

* Received December 16, 1926.

† Bull. de l'Acad. Imp. de Sc. de St. Petersburg 1907.

from which we have

$$pq_1 = qp_0. \tag{1}$$

We seek now an expression for $P_{m,n}$, the probability of exactly $m$ appearances of the event $E$ in $n$ trials, $m$ having a value specified later.

Let $P^{+}_{+m,n}$ denote the probability of exactly $m$ appearances $E$ in $n$ trials, $E$ appearing in both the first and the last trials. Similarly $P^{-}_{+m,n}$ the probability of exactly $m$ appearances in $n$ trials, in which $E$ appears at the first trial and not at the last, $P^{+}_{-m,n}$ the same with $E$ appearing at the last, but not at first trial, $P^{-}_{-m,n}$ the same with $E$ not appearing in both the first and the last trials.

Evidently

$$P_{m,n} = P^{+}_{+m,n} + P^{+}_{-m,n} + P^{-}_{+m,n} + P^{-}_{-m,n}.$$

Consider first $P^{+}_{+m,n}$. Suppose the appearances occur in the following distribution

(A)

| | | |
|---|---|---|
| $e_1$ | appearances | $E$ |
| $f_1$ | non-appearances | $F$ |
| $e_2$ | appearances | $E$ |
| $f_2$ | non-appearances | $F$ |
| . . . . . | | |
| . . . . . | | |
| $e_{k-1}$ | appearances | $E$ |
| $f_{k-1}$ | non-appearances | $F$ |
| $e_k$ | appearances | $E$ |

in which

$$e_1 + e_2 + \cdots + e_k = m, \qquad f_1 + f_2 + \cdots + f_{k-1} = n - m,$$

so that we have $k$ groups of consecutive appearances of $E$ and $k-1$ of $F$.

The probability of this sequence of appearances may be calculated as follows. The probability of $E$ at $T_1$ is $p$, at each of $T_2, \cdots, T_{e_1}$, $E$ appears with the appearance of $E$ at the previous trial, hence the probability of $e_1$ appearances of $E$ at the first $e_1$ trials is

$$p p_1^{e_1 - 1}.$$

At the first trial of the following group of $f_1$ trials (i. e. at $T_{e_1+1}$) $F$ appears after $E$, at all other trials of this group, $F$ appears following $F$. Hence the probability of the results of the first two groups is

$$p p_1^{e_1 - 1} q_1 q_0^{f_1 - 1}.$$

Proceeding in this manner, we find that the probability of the whole sequence (A) is

$$\begin{aligned} &p\,p_1^{e_1-1}\,q_1\,q_0^{f_1-1}\,p_0\,p_1^{e_2-1}\,q_1\,q_0^{f_2-1}\cdots p_0\,p_1^{e_{k-1}-1}\,q_1\,q_0^{f_{k-1}-1}\,p_0\,p_1^{e_k-1} \\ (\mathrm{B}) \qquad &= p\,p_0^{k-1}\,p_1^{e_1+e_2+\cdots+e_k-k}\,q_1^{k-1}\,q_0^{f_1+\cdots+f_{k-1}-k+1} \\ &= p\,p_0^{k-1}\,p_1^{m-k}\,q_1^{k-1}\,q_0^{n-m-k+1}. \end{aligned}$$

Thus we see that the probability depends on the value of $k$, but not on the values of $e_1, \cdots, e_k, f_1, \cdots, f_{k-1}$. Consequently the probability for any sequence for which

$$e_1+e_2+\cdots+e_k = m, \qquad f_1+f_2+\cdots+f_{k-1} = n-m$$

is given by (B).

But we can represent $m$ as the sum of $k$ positive numbers $e$ in $C_{m-1}^{k-1}$ ways, and $n-m$ as the sum of $k-1$ positive numbers $f$ in $C_{n-m-1}^{k-2}$ ways, therefore there are

$$C_{m-1}^{k-1}\,C_{n-m-1}^{k-2}$$

different sequences of type (A), with the condition

$$e_1+e_2+\cdots+e_k = m, \qquad f_1+f_2+\cdots+f_{k-1} = n-m,$$

the probability of each being given by (B). Therefore the probability of exactly $m$ appearences of $E$ in $k$ groups, $E$ appearing at $T_1$ and $T_n$ is

$$C_{m-1}^{k-1}\,C_{n-m-1}^{k-2}\,p\,p_0^{k-1}\,p_1^{m-k}\,q_0^{n-m-k+1}\,q_1^{k-1}.$$

But $k$ may have any value from 2 to $m$, hence by addition of probabilities we have finally

$$P_{+m,n}^{+} = \sum_{k=2}^{m} C_{m-1}^{k-1}\,C_{n-m-1}^{k-2}\,p\,p_0^{k-1}\,p_1^{m-k}\,q_0^{n-m-k+1}\,q_1^{k-1}.$$

In the same way we deduce the expressions

$$\begin{aligned} P_{-m,n}^{+} &= \sum C_{m-1}^{k-1}\,C_{n-m-1}^{k-1}\,q\,p_0^{k}\,p_1^{m-k}\,q_0^{n-m-k}\,q_1^{k-1} \\ P_{+m,n}^{-} &= \sum C_{m-1}^{k-1}\,C_{n-m-1}^{k-1}\,p\,p_0^{k-1}\,p_1^{m-k}\,q_0^{n-m-k}\,q_1^{k} \\ P_{-m,n}^{-} &= \sum C_{m-1}^{k-1}\,C_{n-m-1}^{k}\,q\,p_0^{k}\,p_1^{m-k}\,q_0^{n-m-k-1}\,q_1^{k}. \end{aligned}$$

We calculate the expression

$$A_{m,n} = \sum_{k=1}^{m} C_m^k \, p_1^{m-k} \, q_1^k \, C_{n-m}^k \, p_0^k \, q_0^{n-m-k} \tag{2}$$

analogous to $P^+_{+m,n}$, $P^-_{+m,n}$, $P^+_{-m,n}$, $P^-_{-m,n}$, and then reduce these expressions to it.

Each term of the sum $A_{m,n}$ is the product of the two expressions $C_m^k \, p_1^{m-k} \, q_1^k$ and $C_{n-m}^k \, p_0^k \, q_0^{n-m-k}$, which have well known meanings in the theory of probability. The expression $C_m^k \, p_1^{m-k} \, q_1^k$ is equal to the probability of exactly $k$ appearances of an event in $m$ independent trials, in which the probability of the event at each trial is $q_1$.

We suppose first that

$$m = np + z\sqrt{2npq}, \tag{3}$$

in which $z = o(n^\alpha)$, $\alpha$ being any number $>0$ and $<1/6$. We calculate expression (2) for

$$k = mq_1 + t\sqrt{2mp_1q_1}, \qquad t = o\left(m^{\frac{1}{6}}\right).$$

By Stirling's formula

$$n! = \sqrt{2\pi n}\, n^n \, e^{-n}\left[1 + O\left(\frac{1}{n}\right)\right];$$

so that

$$\frac{m!}{(m-k)!\,k!}\, p_1^{m-k}\, q_1^k = \sqrt{\frac{m}{2\pi(m-k)k}}\; \frac{m^m\, p_1^{m-k}\, q_1^k}{(m-k)^{m-k}\, k^k}\left[1 + O\left(\frac{1}{n}\right)\right].$$

We have

$$\sqrt{\frac{m}{2\pi(m-k)k}} = \frac{1}{\sqrt{2\pi m p_1 q_1}}\left(1 - t\sqrt{\frac{2q_1}{mp_1}}\right)^{-\frac{1}{2}}\left(1 + t\sqrt{\frac{2p_1}{mq_1}}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi m p_1 q_1}}\left[1 + O\left(\frac{t}{n^{1/2}}\right)\right].$$

Further

$$B \equiv \frac{m^m\, p_1^{m-k}\, q_1^k}{(m-k)^{m-k}\, k^k} = \left(\frac{mq_1}{k}\right)^k \left(\frac{mp_1}{m-k}\right)^{m-k}$$

$$= \left(\frac{mq_1}{mq_1 + t\sqrt{2mp_1q_1}}\right)^{mq_1 + t\sqrt{2mp_1q_1}} \left(\frac{mp_1}{mp_1 - t\sqrt{2mp_1q_1}}\right)^{mp_1 - t\sqrt{2mp_1q_1}}$$

$$= \left(1 + t\sqrt{\frac{2p_1}{mq_1}}\right)^{-\left(mq_1 + t\sqrt{2mp_1q_1}\right)} \left(1 - t\sqrt{\frac{2q_1}{mp_1}}\right)^{-\left(mp_1 - t\sqrt{2mp_1q_1}\right)}.$$

$$\lg B = -(mq_1 + t\sqrt{2mp_1q_1})\left[t\sqrt{\frac{2p_1}{mq_1}} - t^2\frac{p_1}{mq_1} + O\left(\frac{t^3}{n^{3/2}}\right)\right]$$

$$+ (mp_1 - t\sqrt{2mp_1q_1})\left[t\sqrt{\frac{2q_1}{mp_1}} + t^2\frac{q_1}{mp_1} + O\left(\frac{t^3}{n^{3/2}}\right)\right]$$

$$= -t^2 + O\left(\frac{t^3}{n^{1/2}}\right),$$

$$B = e^{-t^2}\left[1 + O\left(\frac{t^3}{n^{1/2}}\right)\right].$$

Hence

$$(4) \qquad \frac{m!}{(m-k)!\,k!}\,p_1^{m-k}\,q_1^k = \frac{1}{\sqrt{2\pi mp_1q_1}}\,e^{-t^2}\left[1 + O\left(\frac{t^3}{n^{1/2}}\right)\right].$$

Similarly $C^k_{n-m}\,p_0^k\,q_0^{n-m-k}$ is equal to the probability of exactly $k$ appearances of an event in $n-m$ independent trials in which the probability of the event at each trial is $p_0$. As above, we obtain for

$$k = (n-m)p_0 + u\sqrt{2(n-m)p_0q_0}, \qquad u = o\left(n^{\frac{1}{6}}\right)$$

$$(5) \qquad \frac{(n-m)!}{(n-m-k)!\,k!}\,p_0^k\,q_0^{n-m-k} = \frac{1}{\sqrt{2\pi(n-m)p_0q_0}}\,e^{-u^2}\left[1 + O\left(\frac{u^3}{n^{1/2}}\right)\right].$$

Since each term of (2) is the product of the two expressions

$$C^k_m\,p_1^{m-k}\,q_1^k, \qquad C^k_{n-m}\,p_0^k\,q_0^{n-m-k}$$

for the same value of $k$, if we take in the first of them $k$ in the form $k = mq_1 + t\sqrt{2mp_1q_1}$ and in the second $k$ in the form $k = (n-m)p_0 + u\sqrt{2mp_0q_0}$, we must have

$$(6) \qquad mq_1 + t\sqrt{2mp_1q_1} = (n-m)p_0 + u\sqrt{2mp_0q_0}.$$

Remember

$$(7) \qquad m = np + z\sqrt{2npq}, \qquad z = o(n^\alpha), \qquad \alpha < \frac{1}{6}.$$

From (6), using (1) and (7), we have

$$zq_1\sqrt{2npq} + t\sqrt{2mp_1q_1} = -zp_0\sqrt{2npq} + u\sqrt{2(n-m)p_0q_0}.$$

Hence

$$(8) \qquad u = \frac{1}{\sqrt{2(n-m)p_0q_0}}\left[z(q_1+p_0)\sqrt{2npq} + t\sqrt{2mp_1q_1}\right].$$

Formula (8) shows that, if $t = o\left(n^{\frac{1}{6}}\right)$, then also $u = o\left(n^{\frac{1}{6}}\right)$. This fact permits us to put the above asymptotic expressions [(4), (5)] for both multiples of (2) for $k = mq_1 + t\sqrt{2mp_1q_1}$ with $t = o\left(n^{\frac{1}{6}}\right)$.

In order to arrive at an asymptotic expression for $A_{m,n}$, we separate $A_{m,n}$ into three sums and show that one of these three dominates the other two. Write

$$A_{m,n} = \sum_{k=1}^{m} = \sum_{k=1}^{mq_1 - t_1\sqrt{2mp_1q_1}} + \sum_{k=mq_1 - t_1\sqrt{2mp_1q_1}}^{mq_1 + t_1\sqrt{2mp_1q_1}} + \sum_{k=mq_1 + t_1\sqrt{2mp_1q_1}}^{m},$$

in which $t_1 = n^{\beta}$ with $\frac{1}{12} < \beta < \frac{1}{6}$, $\beta > \alpha$. For brevity, write

$$A_{m,n} = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

The first factor in (2) increases from $k = 1$ to $k = mq_1$, and after decreases, the second factor increases from $k = 1$ to $k = (n-m)p_0$ and after decreases. From our supposition about the value of $t_1$, we see that the greatest values of both factors are included in the sum $\Sigma_2$. The terms of $\Sigma_1$ are increasing and of $\Sigma_3$ decreasing.

Consider

$$\Sigma_1 = \sum_{k=1}^{mq_1 - t_1\sqrt{2mp_1q_1}} C_m^k p_1^{m-k} q_1^k C_{n-m}^k p_0^k q_0^{n-m-k} \tag{9}$$

$$= u_1 + u_2 + \cdots + u_j, \qquad j = mq_1 - t_1\sqrt{2mp_1q_1}.$$

We have $u_i/u_{i+1} < 1$, further

$$\frac{u_i}{u_{i+1}} = \frac{C_m^i p_1^{m-i} q_1^i C_{n-m}^i p_0^i q_0^{n-m-i}}{C_m^{i+1} p_1^{m-i-1} q_1^{i+1} C_{n-m}^{i+1} p_0^{i+1} q_0^{n-m-i-1}}$$

$$= \frac{(i+1)^2}{(m-i)(n-m-i)} \frac{p_1 q_0}{q_1 p_0},$$

thus the ratio $u_i/u_{i+1}$ is increasing.

Replace (9) by the geometric series

$$v_1 + v_2 + \cdots v_{j-1} + v_j$$

in which $v_j = u_j$, $v_{j-1} = u_{j-1}$. Then

$$u_1 + u_2 + \cdots + u_j < v_1 + v_2 + \cdots + v_j.$$

But

$$v_1 + v_2 + \cdots + v_j < \frac{1}{1 - v_{j-1}/v_j} v_j,$$

$$\frac{1}{1-v_{j-1}/v_j} = \frac{(m-j+1)(n-m-j+1)\,p_0\,q_1}{(m-j+1)(n-m-j+1)\,p_0\,q_1 - j^2 p_1\,q_0} = O\left(n^{\frac{1}{2}-\beta}\right),$$

$$v_j = \frac{1}{\pi}\cdot\frac{1}{\sqrt{2\,m\,p_1\,q_1}}\cdot\frac{1}{\sqrt{2\,(n-m)\,p_0\,q_0}}\,e^{-t_1^2-u_1^2}\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right]$$

$$= O\left(\frac{e^{-t_1^2}}{n}\right) = O\left(\frac{e^{-n^{2\beta}}}{n}\right).$$

Consequently

$$\Sigma_1 = O\left(\frac{e^{-n^{2\beta}}}{n^{1/2+\beta}}\right).$$

Similarly

$$\Sigma_3 = O\left(\frac{e^{-n^{2\beta}}}{n^{1/2+\beta}}\right).$$

To compute

$$\Sigma_2 = \sum_{k=mq_1-t_1\sqrt{2mp_1q_1}}^{mq_1+t_1\sqrt{2mp_1q_1}},$$

using (4) and (5), we have

$$\Sigma_2 = \left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right]\sum_{k=mq_1-t_1\sqrt{2mp_1q_1}}^{mq_1+t_1\sqrt{2mp_1q_1}}\frac{1}{\pi}\,e^{-t^2-u^2}\,\frac{1}{\sqrt{2\,m\,p_1\,q_1}}\cdot\frac{1}{\sqrt{2\,(n-m)\,p_0\,q_0}}$$

in which the value of $u$ is given in (8).

Since $k = m\,q_1 + t\sqrt{2\,m\,p_1\,q_1}$, the consecutive values of $t$ in $\Sigma_2$ differ by

$$\Delta t = \frac{1}{\sqrt{2\,m\,p_1\,q_1}};$$

and, since we sum from $-t_1$ to $+t_1$, we may write

$$\Sigma_2 = \left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right]\frac{1}{\pi}\,\frac{1}{\sqrt{2(n-m)\,p_0\,q_0}}\sum_{-t_1}^{+t_1} e^{-t^2-u^2}\,\Delta t.$$

Now

$$\sum_{-t_1}^{+t_1} e^{-t^2-u^2}\,\Delta t = \left[1+O\left(\frac{1}{n^{1/2}}\right)\right]\int_{-t_1}^{+t_1} e^{-t^2-u^2}\,dt$$

$$= \left[1+O\left(\frac{1}{n^{1/2}}\right)\right]\int_{-\infty}^{+\infty} e^{-t^2-u^2}\,dt.$$

Hence

$$\Sigma_2 = \left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right]\frac{1}{\pi}\,\frac{1}{\sqrt{2\,(n-m)\,p_0\,q_0}}\int_{-\infty}^{+\infty} e^{-t^2-u^2}\,dt.$$

To compute the integral, we have

$$t^2+u^2 = \frac{m\,p_1\,q_1+(n-m)\,p_0\,q_0}{(n-m)\,p_0\,q_0}\left[t+\frac{z(q_1+p_0)\sqrt{n\,p\,q\,m\,p_1\,q_1}}{m\,p_1\,q_1+(n-m)\,p_0\,q_0}\right]^2 + \frac{z^2(q_1+p_0)^2\,n\,p\,q}{m\,p_1\,q_1+(n-m)\,p_0\,q_0};$$

hence

$$\int_{-\infty}^{+\infty} e^{-(t^2+u^2)}\,dt = \sqrt{\pi}\sqrt{\frac{(n-m)\,p_0\,q_0}{m\,p_1\,q_1+(n-m)\,p_0\,q_0}}\;e^{-\frac{z^2(q_1+p_0)^2 npq}{mp_1q_1+(n-m)p_0q_0}}.$$

Consequently

$$\Sigma_2 = \left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right]\frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{2\,[m\,p_1\,q_1+(n-m)\,p_0\,q_0]}}\;e^{-\frac{z^2(q_1+p_0)^2 npq}{mp_1q_1+(n-m)p_0q_0}}.$$

Therefore we have finally for $m = np+z\sqrt{2npq}$

$$A_{m,n} = \left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right]\frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{2\,[m\,p_1\,q_1+(n-m)\,p_0\,q_0}}\;e^{-\frac{z^2(q_1+p_0)^2 npq}{mp_1q_1+(n-m)p_0q_0}}.$$

Suppose now

$$m = np+z\sqrt{\frac{2\,[m\,p_1\,q_1+(n-m)\,p_0\,q_0]}{(q_1+p_0)^2}}. \tag{10}$$

Then

$$A_{m,n} = \frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{2\,[m\,p_1\,q_1+(n-m)\,p_0\,q_0]}}\;e^{-z^2}\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right].$$

By (10) the consecutive values of $z$ differ by

$$\Delta z = \frac{q_1+p_0}{\sqrt{2\,[m\,p_1\,q_1+(n-m)\,p_0\,q_0]}}.$$

Consequently

$$A_{m,n} = \frac{1}{q_1+p_0}\,\frac{1}{\sqrt{\pi}}\;e^{-z^2}\,\Delta z\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right].$$

It is clear that in a similar way we obtain

$$P^+_{+m,n} = p\,p_0\,\frac{1}{q_1+p_0}\,\frac{1}{\sqrt{\pi}}\,e^{-z^2}\,\Delta z\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right],$$

$$P^+_{-m,n} = q\,p_0\,\frac{1}{q_1+p_0}\,\frac{1}{\sqrt{\pi}}\,e^{-z^2}\,\Delta z\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right],$$

$$P^-_{+m,n} = p\,q_1\,\frac{1}{q_1+p_0}\,\frac{1}{\sqrt{\pi}}\,e^{-z^2}\,\Delta z\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right],$$

$$P^-_{-m,n} = q\,q_1\,\frac{1}{q_1+p_0}\,\frac{1}{\sqrt{\pi}}\,e^{-z^2}\,\Delta z\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right].$$

Hence

$$P_{m,n} = (pp_0+qp_0+pq_1+qq_1)\cdot\frac{1}{q_1+p_0}\,\frac{1}{\sqrt{\pi}}e^{-z^2}\Delta z\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right]$$

(11)

$$= \frac{1}{\sqrt{\pi}}e^{-z^2}\Delta z\left[1+O\left(\frac{t_1^3}{n^{1/2}}\right)\right];$$

viz. the probability that

$$m = np+z\sqrt{\frac{2\,[mp_1q_1+(n-m)p_0q_0]}{(q_1+p_0)^2}}.$$

But by (1)

$$pp_1q_1+qp_0q_0 = qp_0(p_1+q_0); \qquad qp_0 = pq_1; \qquad p_0 = p(q_1+p_0);$$

$$qp_0 = pq(q_1+p_0),$$

thus

$$pp_1q_1+qp_0q_0 = pq(p_1+q_0)(q_1+p_0),$$

so that we have

$$mp_1q_1+(n-m)p_0q_0 = n(pp_1q_1+qp_0q_0)+O(z\sqrt{n})$$

$$= npq(p_1+q_0)(q_1+p_0)+O(z\sqrt{n}).$$

Therefore (11) is also the probability that

$$m = np+z\sqrt{2npq\frac{p_1+q_0}{q_1+p_0}}.$$

Further, if $z$ is finite, we can suppose $t_1 = n^{\beta}$, in which $\beta$ is as small as we please, so that

$$P_{m,n} = \frac{1}{\sqrt{\pi}}e^{-z^2}\Delta z\left[1+O\left(n^{-\frac{1}{2}+\varepsilon}\right)\right]. \tag{12}$$

From (12) we obtain immediately that the probability for the inequality

$$np+z_1\sqrt{2npq\frac{p_1+q_0}{q_1+p_0}} < m < np+z_2\sqrt{2npq\frac{p_1+q_0}{q_1+p_0}},$$

in which $z_1$ and $z_2$ are constants, is

$$\left[1+O\left(n^{-\frac{1}{2}+\varepsilon}\right)\right]\frac{1}{\sqrt{\pi}}\int_{z_1}^{z_2}e^{-z^2}\,dz. \tag{13}$$

# ON CERTAIN INDEFINITE QUATERNARY FORMS REPRESENTING ALL INTEGERS.*

By Claiborne G. Latimer.

A number of papers have been published on those definite forms, $ax^2+by^2+cz^2+dw^2$, which represent all positive integers. Finally, Ramanujan† showed that there were fifty-five such forms and no more. There seems to be little if anything on the corresponding problem for indefinite forms. The object of this paper is to prove the following‡ theorem.

Theorem. *If $\alpha \leqq 163$ is a positive integer in the form $4k+3$ or the double of such an integer, and contains no square factor $>1$, then the form*

$$s^2+t^2-\alpha u^2-\alpha v^2 \tag{1}$$

*represents all integers.*

Consider the congruence

$$\xi^2+\eta^2-\alpha\zeta^2-\alpha\lambda^2 \equiv 0 \pmod{n_0} \tag{2}$$

where $\alpha$ is one of the integers mentioned in the theorem and $n_0>1$. If $(\xi_0, \eta_0, \zeta_0, \lambda_0)$ is a solution, by adding arbitrary multiples of $n_0$ to this solution we obtain a second, $(\xi_1, \eta_1, \zeta_1, \lambda_1)$. Suppose $0 \neq \xi_1^2+\eta_1^2-\alpha\zeta_1^2-\alpha\lambda_1^2 = n_0 n_1 \neq n_0^2$. Then this second solution is also a solution of (2) with the modulus $n_1$. By repeating this process we obtain a sequence of congruences which shall be said to be derived from the first congruence and solution $(\xi_0, \eta_0, \zeta_0, \lambda_0)$. Suppose $(\xi_0, \eta_0, 1, 0)$ is a solution of (2). If $n_0$ satisfies one of the conditions below, by dropping proper multiples of $n_0$ from $\xi_0$ and $\eta_0$ we obtain integers, $\xi_1$ and $\eta_1$, satisfying the conditions indicated.

$$n_0 > \sqrt{\alpha}, \qquad \frac{n_0}{2} \geqq |\xi_1|, \qquad \frac{n_0}{2} \geqq |\eta_1|,$$

$$\sqrt{\alpha} > n_0 > \sqrt{\frac{2}{3}\alpha}; \qquad n_0 \geqq |\xi_1| \geqq \frac{n_0}{2}, \qquad n_0 \geqq |\eta_1| \geqq \frac{n_0}{2},$$

$$\sqrt{\frac{2}{3}\alpha} > n_0 > \sqrt{\frac{\alpha}{2}}; \qquad \frac{3}{2}n_0 \geqq |\xi_1| \geqq n_0, \qquad \frac{n_0}{2} \geqq |\eta_1|.$$

* Received February 23, 1926.

† Dickson, *History of the theory of numbers*, vol. 3, p. 233; Proc. Cambr. Phil. Soc., 19, I, 1916, 11–21.

‡ The method used is a generalization of one due to Dickson. See Amer. Jour. of Math., vol. 46 (1924), p. 2.

It is readily shown in each case that $|\xi_1^2+\eta_1^2-\alpha|<n_0^2$. If $n_0=\sqrt{\frac{2}{3}\alpha}$, then $n_0=2$, $\alpha=6$. Therefore *if* $n_0>\sqrt{\frac{\alpha}{2}}$, $n_0>2$ *and if* $(\xi_0, \eta_0, 1, 0)$ *is a solution of* (2), *there exists a sequence of congruences, derived from this congruence and solution, terminating with one with a modulus* $n_j$ *such that* $|n_j|<\sqrt{\frac{\alpha}{2}}$. For every $\alpha$ given in the theorem, it may be verified that, with the exceptions stated below, if $|n_j|<\sqrt{\frac{\alpha}{2}}$ and if $(\xi_j, \eta_j, 1, 0)$ is a solution of (2) with the modulus $n_j$, then there exists a sequence of congruences, in the form (2), derived from this congruence and solution, which terminates with one such that the left member is equal to the modulus or its negative. The exceptions are the cases where $|n_j|=2$, $\xi_j$ is odd, and $\alpha$ is even. However such a congruence cannot occur in a sequence of congruences with solutions $(\xi_i, \eta_i, 1, 0)$, derived from a congruence with an odd modulus and such a solution. For suppose it did. Let $n_{j-1}$ be the modulus preceding $n_j$ in the sequence. Then $|\xi_j^2+\eta_j^2-\alpha|=|2n_{j-1}|$. Since $\xi_j$ and $\eta_j$ are odd and since $\alpha=8k+6$, it follows that $n_{j-1}$ is the double of an odd integer. Repetition of the argument shows that every modulus in the sequence is such an integer, contrary to hypothesis that $n_0$ is odd. It is known* that for every prime modulus, (2) has a solution with $\zeta_0=1$, $\lambda_0=0$. Hence we have the

LEMMA. *If* $\alpha$ *is one of the integers given in the Theorem, if* $n_0$ *is an odd prime and if* $(\xi_0, \eta_0, 1, 0)$ *is a solution of* (2); *then there exists a sequence of congruences, derived from this congruence and solution, which terminates with a congruence with modulus* $n_{k-1}$ *and solution* $\xi_k, \eta_k, \zeta_k, \lambda_k$ *such that* $\xi_k^2+\eta_k^2-\alpha\zeta_k^2-\alpha\lambda_k^2=n_{k-1}n_k$ *where* $n_k=\pm 1$.

Since the form (1) is self-reproductive under multiplication and since it may be verified that, for every $\alpha$ given in the theorem, (1) represents $-1$, to prove the theorem it is sufficient to show that for each of these values of $\alpha$, (1) represents every odd prime. Let $n_0$ be an odd prime and let $\alpha$ be one of integers given in the theorem. Let $(\xi_0, \eta_0, 1, 0)$ be a solution of (2) and let the congruences of the derived sequence, shown to exist by the Lemma, have the moduli $n_i (i=0, 1, \cdots, k-1)$. Let the corresponding solutions be $\xi_i, \eta_i, \zeta_i, \lambda_i$ and let $n_k=\pm 1$, $\zeta_0=1$, $\lambda_0=0$. We shall determine integers $x_0, y_0, z_0, w_0, X_0, Y_0, Z_0, W_0$, such that if

$$\text{(3a)} \qquad \begin{aligned} s &= n_0x_0+\xi_0X_0+\eta_0Y_0-\alpha\lambda_0Z_0-\alpha\zeta_0W_0 \\ t &= n_0y_0+\eta_0X_0-\xi_0Y_0-\alpha\zeta_0Z_0+\alpha\lambda_0W_0 \end{aligned}$$

* Dickson, *History of the theory of numbers*, vol. 2, p. 288.

$$(3b)\qquad \begin{aligned} u &= n_0 w_0 + \zeta_0 X_0 - \lambda_0 Y_0 - \xi_0 Z_0 + \eta_0 W_0 \\ v &= n_0 z_0 + \lambda_0 X_0 + \zeta_0 Y_0 - \eta_0 Z_0 - \xi_0 W_0 \end{aligned}$$

then (1) is equal to $n_0$. Since $\xi_1, \eta_1, \zeta_1, \lambda_1$, differ from $\xi_0, \eta_0, \zeta_0, \lambda_0$, respectively by multiples of $n_0$, if the latter are replaced by the former, $x_0, y_0, z_0, w_0$ are replaced by other unknowns, $x_1, y_1, z_1, w_1$. Make these replacements and substitute the resulting expressions in $s^2+t^2-\alpha u^2-\alpha v^2=n_0$. We obtain on expanding

$$\begin{aligned} n_0 = {} & n_0^2(x_1^2+y_1^2-\alpha z_1^2-\alpha w_1^2)+n_0 n_1(X_0^2+Y_0^2-\alpha Z_0^2-\alpha W_0^2) \\ & +2n_0[x_1(\xi_1 X_0+\eta_1 Y_0-\alpha\lambda_1 Z_0-\alpha\zeta_1 W_0)+y_1(\eta_1 X_0-\xi_1 Y_0-\alpha\zeta_1 Z_0+\alpha\lambda_1 W_0) \\ & -\alpha z_1(\lambda_1 X_0+\zeta_1 Y_0-\eta_1 Z_0-\xi_1 W_0)-\alpha w_1(\zeta_1 X_0-\lambda_1 Y_0-\xi_1 Z_0+\eta_1 W_0)]. \end{aligned} \tag{4}$$

Multiply this by $n_1/n_0$. We obtain an equation identical with (4) except that $n_0$ and $n_1$, and the corresponding large and small Roman letters are interchanged. Hence our problem reduces to the finding of integers $x_1, y_1, z_1, w_1, X_0, Y_0, Z_0, W_0$, such that if

$$\begin{aligned} S &= n_1 X_0 + \xi_1 x_1 + \eta_1 y_1 - \alpha\lambda_1 z_1 - \alpha\zeta_1 w_1 \\ T &= n_1 Y_0 + \eta_1 x_1 - \xi_1 y_1 - \alpha\zeta_1 z_1 + \alpha\lambda_1 w_1 \\ U &= n_1 W_0 + \zeta_1 x_1 - \lambda_1 y_1 - \xi_1 z_1 + \eta_1 w_1 \\ V &= n_1 Z_0 + \lambda_1 x_1 + \zeta_1 y_1 - \eta_1 z_1 - \xi_1 w_1 \end{aligned}$$

then $S^2+T^2-\alpha U^2-\alpha V^2 = n_1$. If $n_1 \neq \pm 1$, repeat the process following (3). Repetition of this process $k-1$ times leads to an equation similar to (4). If $k$ is odd, this equation may also be obtained from (4) by adding $k-1$ to the subscripts of the $n$'s and of the Greek letters and adding $i = \frac{1}{2}(k-1)$ to the subscripts of the other Roman letters. Then choose $x_{i+1} = y_{i+1} = z_{i+1} = w_{i+1} = 0$. If $n_k = 1$, choose $X_i = 1$, $Y_i = Z_i = W_i = 0$. If $n_k = -1$, since as noted before, (1) represents $-1$, there exist values of $X_i, Y_i, Z_i, W_i$ which satisfy the equation. If $k$ is even, the equation resulting from these $k-1$ operations may also be obtained from (4) by changing the subscripts on the $n$'s and the Greek letters as before; interchanging the large and small Roman letters, and advancing the subscripts of these letters to $\frac{1}{2}k$. A solution of this equation is obtained as before. In either case, by retracing these steps a solution of (4) is obtained. Hence the theorem is proved.

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# APPLICATION OF ALGEBRAIC NUMBER THEORY TO CONGRUENCES INVOLVING BINOMIAL COEFFICIENTS.*

By H. S. Vandiver.

In statements concerning the scope of the theory of algebraic numbers the applications of this subject to Diophantine Analysis have been emphasized by many writers, so much so that many readers of these statements are likely to acquire the impression that such applications are the only important ways in which the theory has been used to derive results involving rational integers only. Let me then direct attention to some theorems concerning congruences which were obtained at about the time the theory of cyclotomic algebraic integers was founded, for example the laws of biquadratic and cubic reciprocity, each of which involves said theory in its simplest statement and yet yield an algorithm for determining the solvability of $x^4 \equiv a \pmod{p}$ and $x^3 \equiv b \pmod{q}$ where $p$ is a prime of the form 1 (mod 4) and $q$ is of the form 1 (mod 3). Also it has been shown by means of cyclotomic integers that, if $p = 4n+1 = a^2+b^2$ is a prime, then

$$a \equiv \pm \frac{(2n)!}{2(n!)^2} \pmod{p}.$$

As far as I know this result has never been established by essentially different means.

In the present note I give an application of a somewhat different type. Let $l$ and $p$ be odd primes, $\alpha = e^{2i\pi/l}$ and let $p$ belong to the exponent $f$, modulo $l$. Set $p^b - 1 = cl$. Hence in the field $K$ defined by $\alpha$, we see that $p^b$ is the norm of $\mathfrak{p}$, a prime ideal divisor of $(p)$ in the field $K$.† Now using $l$-th power characters‡ we have for $k$ any positive integer or zero,

$$\left\{\frac{\alpha^k+1}{\mathfrak{p}}\right\} = \left\{\frac{\alpha^{\frac{k}{2}}}{\mathfrak{p}}\right\}\left\{\frac{\alpha^{\frac{k}{2}}+\alpha^{-\frac{k}{2}}}{\mathfrak{p}}\right\}.$$

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* Presented to the American Mathematical Society at Philadelphia, December 29, 1926; received December 27, 1926.

† Hilbert, Die Theorie der algebraischen Zahlkörper, Jahresbericht der Deutschen Math. Vereinig., 1894, p. 333.

‡ Hilbert, l. c., p. 365.

Let $f$ be even, the $\mathfrak{p}$ is unaltered by the substitution $(\alpha/\alpha^{-1})$, and if

$$\left\{\frac{\alpha^{\frac{k}{2}}+\alpha^{-\frac{k}{2}}}{\mathfrak{p}}\right\} = \alpha^t,$$

then

$$\left\{\frac{\alpha^{\frac{k}{2}}+\alpha^{-\frac{k}{2}}}{\mathfrak{p}}\right\} = \alpha^{-t},$$

so that $t \equiv 0 \pmod{p}$. Hence

$$\left\{\frac{\alpha^k+1}{\mathfrak{p}}\right\} = \left\{\frac{\alpha^{\frac{k}{2}}}{\mathfrak{p}}\right\}$$

and

$$(\alpha^k+1)^c \equiv \alpha^{\frac{ck}{2}} \pmod{\mathfrak{p}}.$$

Expanding by the binomial theorem and letting $k$ range over the values $0, 1, \cdots, l-1$, we obtain congruences which may be put in the form

$$\begin{aligned}
&A_{p-1} &&+ A_{p-2} &&+ \cdots + A_0 \equiv 0 \pmod{\mathfrak{p}},\\
&A_{p-1}\,\alpha^{p-1} &&+ A_{p-2}\,\alpha^{p-2} &&+ \cdots + A_0 \equiv 0 \pmod{\mathfrak{p}},\\
&\cdots\cdots\cdots

This result may be extended in various ways, for example, we may consider

$$\left\{\frac{1+\alpha\beta}{\mathfrak{P}}\right\}$$

where $\beta$ is an $m$th root of unity and $\mathfrak{P}$ is unaltered by the substitution $(\alpha\beta/(\alpha\beta)^{-1})$.

Theorem I does not include, however, the known analogous relations of Hermite,* Glaisher† and Bachmann.‡

A result which does include these theorems, however, may be obtained by taking a primitive root $\varrho$ in a Galois field of order $p^n$ and noting that

$$(\varrho^k-1)^{a+b(p^n-1)} = (\varrho^k-1)^a$$

in the field, $0<a\leqq p^n-1$, for $k=0, 1, \cdots, p^n-2$. Expanding this by the binomial theorem and eliminating the $\varrho$'s from these $(p^n-1)$ congruences we have the

THEOREM II. *We have*

$$\binom{a}{i} \equiv \binom{d}{i} + \binom{d}{i+p^n-1} + \binom{d}{i+2(p^n-1)} + \cdots$$

*modulo* $p$, *where* $d \equiv a \pmod{p^n-1}$, $0<a\leqq p^n-1$, $0<i\leqq p^n-1$.

Setting $n=1$ gives the known results mentioned above.

The expansion of $(1+\varrho+\varrho^2+\cdots+\varrho^k)^d$ gives a result involving multinomial coefficients. For $n=1$, a result of this type was given by Dickson.§

* Journ. für Math. (Crelle), 81, 1876, 94.

† Quart. Journ. Math. 30, 1899, 150–156, 349–366.

‡ Niedere Zahlentheorie, II, 1910, 46.

§ Quart. Journ. Math. 33, 1902, 381–384.

# TERNARY QUADRATIC FORMS AND CONGRUENCES.*

By L. E. Dickson.

**1. Examples of regular and irregular forms.** It is well known that $f = x^2+y^2+z^2$ represents every positive integer not of the form $4^k(8n+7)$, but represents no integer of that form. We shall say briefly that $f$ represents exclusively all $\neq 4^k(8n+7)$. Again, $3x^2+4y^2+12z^2$ represents exclusively all $\neq 4n+1$, $4n+2$, $9^k(12n+8)$, $9^k(12n+11)$. We shall call these two forms *regular*. In general, all the integers not represented by a regular form $f$ coincide with all the positive integers given by certain arithmetical progressions; the same is then evidently true of the integers represented by $f$. The writer has recently proved that various forms are regular.† Regular forms are important in the determination of quaternary quadratic forms which represent all positive integers and in the proof of Waring's theorem on sums of cubes, fourth powers, etc.

On the contrary, consider $g = x^2+y^2+10z^2$. By § 9, the positive even integers represented by $g$ are exclusively all $\neq 4^k(16n+6)$. Ramanujan computed a list

3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, ···

of the odd integers not represented by $g$, and remarked that they do not seem to obey any simple law. It will be proved that they do not coincide with all the positive integers given by any arithmetical progressions whatever. We shall therefore call $g$ *irregular* (with respect to odd representations).

We shall determine all regular forms $x^2+by^2+cz^2$ in which $b$ and $c$ are relatively prime.

**2. Quadratic congruences.** Write $f = ax^2+by^2+cz^2$.

LEMMA 1. *If $p$ is an odd prime dividing neither $a$ nor $b$ and if $k$ is any integer, $ax^2+by^2 \equiv k \pmod p$ has integral solutions.*

For, $x^2$ takes $(p+1)/2$ values incongruent modulo $p$. The same is therefore true of $ax^2-k$ and of $-by^2$. One of the values of the former is congruent to one of the values of the latter, since otherwise there would be $p+1$ integers incongruent modulo $p$.

LEMMA 2. *If no one of $a$, $b$, $c$ is divisible by the odd prime $p$, $f \equiv k \pmod p$ has solutions with $x$ and $y$ not both divisible by $p$.*

* Received November 17, 1926.

† Bull. Amer. Math. Soc., 33 (1927), p. 63, and Amer. Jour. Math., 49 (1927), p. 39. Ramanujan had found empirically several regular forms, Proc. Cambridge Phil. Soc., 19 (1916–9), pp. 11–21.

According as $k$ is not or is divisible by $p$, take $z \equiv 0$ or $z \equiv 1 \pmod p$. Then $l = k - cz^2 \not\equiv 0 \pmod p$. By Lemma 1, $ax^2 + by^2 \equiv l \pmod p$ has solutions $x, y$, which are evidently not both divisible by $p$.

THEOREM 1. *If $p$ is an odd prime not dividing $abc$, $f \equiv k \pmod{p^n}$ has solutions when $k$ and $n$ are arbitrary.*

By Lemma 2, there are solutions when $n = 1$ such that $x$ and $y$ are not both divisible by $p$. To proceed by induction on $n$, let $f \equiv k \pmod{p^m}$ have solutions $\xi, \eta, \zeta$ such that $\xi$ and $\eta$ are not both divisible by $p$. Write

$$a\xi^2 + b\eta^2 + c\zeta^2 = k + p^m q, \quad x = \xi + p^m X, \quad y = \eta + p^m Y, \quad z = \zeta + p^m Z.$$

Then

$$f \equiv k + p^m q + 2p^m L \pmod{p^{m+1}}, \qquad L = a\xi X + b\eta Y + c\zeta Z.$$

We may choose $X, Y, Z$ so that $q + 2L \equiv 0 \pmod p$. For example if $\xi$ is not divisible by $p$, take $Y = Z = 0$. Then $f \equiv k \pmod{p^{m+1}}$, and the induction is complete.

THEOREM 2. *If an odd prime $p$ divides $c$, but not $ab$, and if $k$ is prime to $p$, $f \equiv k \pmod{p^n}$ is solvable.*

By Lemma 1, $f \equiv k \pmod p$ has solutions. Evidently $x$ and $y$ are not both divisibe by $p$. The proof of Theorem 1 applies also here.

THEOREM 3. *If $k$ is odd and if $f \equiv k \pmod 8$ is solvable, then $f \equiv k \pmod{2^n}$ is solvable when $n$ is arbitrary.*

To proceed by induction on $n$, let $f \equiv k \pmod{2^m}$ have solutions $\xi, \eta, \zeta$ for $m \geqq 3$. Write

$$a\xi^2 + b\eta^2 + c\zeta^2 = k + 2^m q, \quad x = \xi + 2^{m-1} X, \cdots, z = \zeta + 2^{m-1} Z.$$

Then

$$f \equiv k + 2^m q + 2^m L \pmod{2^{m+1}}, \qquad L = a\xi X + b\eta Y + c\zeta Z.$$

Since $a\xi, \cdots, c\zeta$ are not all even ($k$ being odd), there exist integral solutions $X, Y, Z$ of $q + L \equiv 0 \pmod 2$. Hence $f \equiv k \pmod{2^{m+1}}$ and the induction is complete.

3. Let $b$ and $c$ be divisible by a prime $p$ not dividing $a$. Then $f \equiv 0 \pmod p$ implies

$$x = pX, \quad f = pF, \quad F = apX^2 + \frac{b}{p}y^2 + \frac{c}{p}z^2.$$

If $F$ is irregular, $f$ is irregular with respect to multiples of $p$. The determinant $abc$ of $f$ is the product of that of $F$ by $p$. The problem for $f$ therefore reduces to that of a form of smaller determinant.

In what follows we assume* that $a = 1$, and that $b$ and $c$ are relatively prime. We may evidently take $c \geqq b$.

* The case $a > 1$ is being investigated by one of my students.

4. **Theorem 4.** *$f = x^2+by^2+cz^2$ is irregular if there exists an odd integer $k$ relatively prime to both $b$ and $c$, such that $k$ is not represented by $f$ and such that $f \equiv k \pmod 8$ is solvable.*

By Theorems 1, 2, 3, $f \equiv k \pmod{q^n}$ is solvable when $q$ is any prime and $n$ is any positive integer. Likewise, $f \equiv k \pmod{r^m}$ has solutions $X, Y, Z$ if $r$ is any prime $\neq q$. By a well known theorem we can determine integers $\xi, \eta, \zeta$ which are congruent to $x, y, z$ respectively modulo $r^m$. Since $\xi^2+b\eta^2+c\zeta^2-k$ is divisible by both $q^n$ and $r^m$, it is divisible by their product. Similarly for any product of powers of distinct primes. Hence $f \equiv k \pmod N$ is solvable when $N$ is any integer.

We shall prove that no arithmetical progression contains exclusively integers not represented by $f$, whence $f$ is irregular by definition. For, let the progression $Nu+M (u = 0, \pm 1, \pm 2, \cdots)$ contain $k$, which is not represented by $f$. By the preceding result, $f$ represents

$$k+Nv = Nu+M+Nv = N(u+v)+M,$$

for certain integral values of $u, v$.

5. **Theorem 5.** *If $b \equiv 3 \pmod 4$, $f$ is irregular except when $b = 3$, $c = 4$ or 10.*

The squares have the residues 0, 1, 4 modulo 8. Thus $x^2+by^2 \equiv 0, 1, 3, 4, 5, 7 \pmod 8$. In particular, if $k$ is any odd integer, $f \equiv k \pmod 8$ is solvable.

First, let $b = 3$, whence $c \geqq 4$. If $c = 4$, $f$ is regular, since* it represents exclusively all integers $\neq 4n+2$, $9^k(9n+6)$. Since $-3$ is a quadratic non-residue of every prime $p = 5, 11, 17, 23, 29, 41, \cdots$ of the form $6n-1$, $p$ is not represented by $x^2+3y^2$. If $c$ is not divisible by 5, we may apply Theorem 4 with $k = 5$, since 5 is not represented by $f$ with $c>5$. If $c$ is divisible by 5 but not by 11, and if† $c \neq 10$, we may take $k = 11$, since 11 is not represented by $f$ with $c>11$ or $c = 5$. If $c$ is divisible by both 5 and 11, but not by 17, we may take $k = 17$. If $c$ is divisible by 5, 11, and 17, but not by 23, we may take $k = 23$; etc.

Next, let $b \geqq 7$. Let $p_1, \cdots, p_r$ denote the odd primes $<b$ which do not divide $b$. No one of them is represented by $f$ since $p_i<b \leqq c$. If $p_i$ does not divide $c$, it may be taken as $k$. There remains the case in which $c$ is divisible by the product $P = p_1 \cdots p_r$. Let $q_1, q_2, \cdots$ denote the

* Amer. Jour. Math., 49 (1927), p. 44.

† When $c = 10$, we find by discussing the cases in which $f$ is divisible by 3 or 5 that $f$ represents no integer $9^k(9n+6)$ or $25^k(25n\pm 5)$. Also, no $4^k(16n+2)$. By trial, $f$ represents all further integers $\leqq 200$, and is doubtless regular.

primes in order which exceed $b$, do not divide $b$, and are not represented by $x^2+by^2$. Then* $q_1<P\leqq c$, whence $q_1$ is not represented by $f$. If $q_1$ is not a divisor of $c$, it may be taken as $k$. Hence let $c$ be divisible by $Pq_1$. Then $q_2<c$, whence $q_2$ is not represented by $f$. If $q_2$ is not a divisor of $c$, it may be taken as $k$; etc. For example, if $b=19$, $P=3\cdot5\cdot7\cdot11\cdot13\cdot17$, $q_1=29$, $q_2=31$, $q_3=37, \cdots$.

6. **Theorem 6.** *If $b\equiv 2\,(mod\,4)$, $f$ is irregular except when $b=2$, $c=3$ or 5.*

Since $c$ is prime to $b$, it is odd and $c\cdot 2^2\equiv 4\,(\mathrm{mod}\,8)$. But

$$x^2+2y^2\equiv 0,1,2,3,4,6; \qquad x^2+6y^2\equiv 0,1,2,4,6,7\ (\mathrm{mod}\,8).$$

Hence $f\equiv k\,(\mathrm{mod}\,8)$ is solvable when $k$ is arbitrary.

Let $b=2$. Then $f$ is regular if† $c=3$ or 5. Let $c>5$. Since $-2$ is a quadratic non-residue of every prime $p=5,7,13,23,29,31,\cdots$ of the form $8n-1$ or $8n-3$, $p$ is not represented by $x^2+2y^2$. If $c$ is not divisible by 5, we may take $k=5$. If $c$ is divisible by 5, but not by 7, we may take $k=7$. If $c$ is divisible by 5 and 7, but not by 13, take $k=13$, etc.

THEOREM 7. *If $b\equiv 1$, $c\equiv 2$ or $3\,(mod\,4)$, $f$ is irregular except when $b=1$, $c=2$, 3 or 6.*

For $k$ arbitrary, $f\equiv k\,(\mathrm{mod}\,8)$ is solvable. For $b=1$, $c\geqq 7$, use $k=3$ if $c$ is prime to 3; $k=7$ if $c$ is divisible by 3, but not by 7; $k=11$ if $c$ is divisible by 3 and 7, but not by 11; $k=19$, etc. For $b=5$, $c=6$, use $k=13$.

THEOREM 8. *If $b\equiv 1$, $c\equiv 1$ or $5\,(mod\,8)$, $f$ is irregular except when $b=1$, $c=1,5,9,21$.*

For $b=1$, $c=1$ or 5, $f$ is regular (§ 1, § 9). We next prove that $f=x^2+y^2+9z^2$ *represents exclusively all positive integers except* $9n\pm3$, $N=4^k(8n+7)$, and hence is regular. Any number of the form $N$ is not a sum of three squares (§ 1) and hence is not represented by $f$. Let $m$ be a positive integer not of the form $N$. By § 1, $m$ is represented by $\varphi=X^2+Y^2+Z^2$. If one of $X, Y, Z$, is divisible by 3, $\varphi$ is of type $f$. Let no one of the three be divisible by 3. Then each is $\equiv\pm1\,(\mathrm{mod}\,3)$, and $\varphi\equiv 0\,(\mathrm{mod}\,3)$. Thus $m\equiv 0\,(\mathrm{mod}\,3)$. But if $f\equiv 0$, then $x\equiv y\equiv 0$ $(\mathrm{mod}\,3)$, and $f$ is divisible by 9. Hence $9n\pm3$ is not represented by $f$. Consider therefore $m=9\mu\neq N$. Hence $\mu$ is not of the like form $N$, whence $\mu=x^2+y^2+z^2$. Thus $m$ is represented by $f$.

* We shall not give here a formal proof of this intuitively evident fact. See the footnote to § 3.

† Amer. Math. Soc. Bull., vol. 33 (1927), pp. 67–70.

First, let $c \equiv 1 \pmod 8$. Then $f \equiv k \pmod 8$ is solvable except when $k \equiv 7 \pmod 8$. Such values of $k$ must be excluded in applying Theorem 4.

For $b = 1$, $c \neq 1$ or 9, we have $c \geqq 17$. The available $k$'s are 3, 11, 19, 43, $\cdots$. For $b = 9$, the available $k$'s are 5, 11, 17, 19, 29, $\cdots$. For $b = 17$, they are 3, 5, 11, 13, 19, 29, $\cdots$.

Next, let $c \equiv 5 \pmod 8$. Then $f \equiv k \pmod 8$ is solvable except when $k \equiv 3 \pmod 8$. Let $b = 1$, $c > 5$. If $c$ is prime to 7, we may take $k = 7$. For $c$ divisible by 7, $c = 21 + 56n$. If* $c > 21$, the available $k$'s are 23, 31, 47, $\cdots$. For $b = 9$, the available $k$'s are 5, 7, 17, 23, 29, $\cdots$.

THEOREM 9. *If $b \equiv 5$, $c \equiv 1$ or $5 \pmod 8$, $f$ is irregular.*

According as $c \equiv 1$ or 5, $f \equiv k \pmod 8$ is solvable unless $k \equiv 3$ or 7 (mod 8), respectively.

THEOREM 10. *If $b \equiv 0$, $c \equiv 3 \pmod 4$, $f$ is irregular.*

If $k$ is odd, $f \equiv k \pmod 8$ is solvable.

## 7. Remaining cases reduced to earlier ones.

THEOREM 11. *If $b \equiv 1 \pmod 4$, $c \equiv 4 \pmod 8$, $f$ is irregular except when $b = 1$, $c = 4$ or 12.*

We prove that $f = x^2 + y^2 + 12z^2$ represents exclusively all integers except $4n + 3$, $N = 9^k(9n + 6)$. If $f$ is even, $x + y = 2X$, $x - y = 2Y$, $f = 2F$, $F = X^2 + Y^2 + 6z^2$. But† $F$ represents exclusively all $\neq 9^k(9n + 3)$. Conversely, $2F$ is of the form $f$. If $f$ is divisible by 3, $x$ and $y$ are, and $f/3 = 3X^2 + 3Y^2 + 4z^2 \equiv 0$ or $1 \pmod 3$; hence $f$ represents no $N$. Finally, let $f$ be odd. Permuting $x$, $y$ if necessary, take $x$ odd, $y$ even. Then $f$ becomes $g = x^2 + 4y^2 + 12z^2$, $g \equiv 1 \pmod 4$. Any integer $4n + 1$, not of type $N$, is represented† by $x^2 + 3y^2 + 4z^2$. Then $y$ is even and $4n + 1$ is represented by $g$ and hence by $f$.

When $b = 1$, it remains to consider $c = 8n + 4 \geqq 20$. If $f$ is even, we see as before that $f = 2F$, $F = X^2 + Y^2 + (4n + 2)z^2$. By Theorem 7, $F$ is irregular, whence $f$ is irregular at least as regards even integers.

Finally, consider $f = x^2 + by^2 + (8n + 4)z^2$ for $b > 1$. If $f$ is a multiple of 4,

$$x = 2X, \quad y = 2Y, \quad f = 4F, \quad F = X^2 + bY^2 + Cz^2, \quad C = 2n + 1.$$

If $C \geqq b$, $F$ is irregular by Theorems 7–9. If $C < b$, interchange $Y$ with $z$, $b$ with $C$. We get $F$, having now $b < C$, $b$ odd, $b \geqq 3$, $C \equiv 1 \pmod 4$; it is irregular by Theorems 5, 8, 9.

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* $f = x^2 + y^2 + 21z^2$ is readily proved to represent no integer of the forms $4^k(8n + 3)$, $9^k(9n + 6)$, $49^k(49n + 7r)$, where $r = 1$, 2 or 4. It was verified that $f$ represents all remaining positive integers $\leqq 200$.

† Amer. Jour. Math., 1927, pp. 39–45.

THEOREM 12. *If* $b = 8n+4$, $c \equiv 1 \pmod 4$, $f$ *is irregular. However, if* $b = 4$, $c = 5, 9$ *or* $21$, $f$ *is irregular only as to certain doubles of odd number, these being* 2, 26 *and* 38 *alone when* $c = 5$, *and* 2 *alone when* $c = 9$.

If $f$ is divisible by 4,

$$x = 2X, \quad z = 2Z, \quad f = 4F, \quad F = X^2+(2n+1)y^2+cZ^2,$$

and $F$ is irregular by Theorems 5, 8, 9 unless $2n+1 = 1$, $c = 5, 9, 21$.

It remains to consider $f_c = x^2+4y^2+cz^2$ for $c = 5, 9, 21$. It was just seen to be regular as to all multiples of 4. Evidently $f$ represents no integer $4m+3$. If $4m+1$ is represented by $f_c$, it is obviously represented by $g_c = x^2+y^2+cz^2$. The converse is true, since $x^2+y^2+z^2 \equiv 1 \pmod 4$ implies that two of $x, y, z$ are even, whence $x$ or $y$ is even. In other words, $f_c$ and $g_c$ represent the same numbers of the form $4m+1$. Since $g_c$ is regular by Theorem 8, $f_c$ is regular as to all odd mumbers. Irregularity can arise only as to doubles of odd integers; we need the following new test.

THEOREM 13. *If* $b$ *is divisible by* 4 *and if* $c$ *is odd,* $f = x^2+by^2+cz^2$ *is irregular if there exists an odd integer* $h$ *relatively prime to both* $b$ *and* $c$, *such that* $k = 2h$ *is not represented by* $f$ *and such that* $f \equiv k \pmod 8$ *is solvable.*

Then $f \equiv k \pmod{2^n}$ is solvable. To prove this by induction let $f \equiv k \pmod{2^m}$ have solutions $\xi, \eta, \zeta$ when $m \geq 3$. Then $\xi$ and $\zeta$ are not both even, since otherwise $f$ is divisible by 4, while $k = 2h$ is not. The proof now proceeds as for Theorem 3, the coefficients $\xi$ and $c\zeta$ of $X$ and $Z$ in $L$ being now not both even. Theorem 13 now follows exactly as in § 4.

When $f$ is $f_c$, the final condition in Theorem 13 is satisfied. For $x = z = 1$, $y = 0$ or 1, we have $f_c \equiv 1+c$ or $5+c \pmod 8$. But $c \equiv 1$ or $5 \pmod 8$. In each case $f_c$ is congruent to 2 or 6 and hence to the double of any odd integer.

Hence $f_c$ is irregular with respect to $2h$ if $h$ is an odd integer prime to $c$ such that $2h$ is not represented by $f_c$.

Evidently $f_c$ is double an odd integer if and only if $x$ and $z$ are odd. A short computation yields a list of all integers $N \leq 400$ such that $N$ is double an odd integer and $N$ is not represented by $f_c$. For $c = 5$, $N = 2, 26, 38$. For $c = 9$, $N$ is 2 or a multiple of 3. But if $f_9$ is divisible by 3, it is the product of 9 by the regular form $X^2+4Y^2+z^2$.

Finally, consider $f = f_c$ for $c = 21$. For $p = 3$ or 7, $f \equiv 0$ implies $x \equiv y \equiv 0 \pmod p$, whence

$$f = pF, \quad F = pX^2+4pY^2+21p^{-1}z^2.$$

Hence $f$ represents no integer of either of the forms $9^k(9n+6)$, $49^k(49n+7e)$, where $e = 1, 2$ or $4$. Excluding these, we find that the remaining $N$'s are

$$2,\quad 9\cdot 2,\quad 9^2\cdot 2,\quad 49\cdot 2,\quad 10,\quad 9\cdot 10,\quad 182 = 7\cdot 26,\quad 318 = 3\cdot 106.$$

Of these, only 2 and 10 are prime to 21 and serve as $k$ in Theorem 13. Doubtless $f$ is irregular with respect to many doubles of odd numbers divisible by 3 or 7.

8. **Theorem 14.** *If $b \equiv 1 \pmod 4$ and $c = 8n$, $f$ is irregular except for $\sigma = x^2+5y^2+8z^2$ and $f(c) = x^2+y^2+cz^2$, $c = 8, 16, 24$.*

If $f$ is divisible by 4,

$$x = 2X,\quad y = 2Y,\quad f = 4F,\quad F = X^2+bY^2+2nz^2.$$

(I) Let $2n \geqq b$. If $2n \equiv 2 \pmod 4$, $F$ is irregular by Theorem 7 unless $b = 1$, $2n = 2$ or 6. If $2n \equiv 4 \pmod 8$, $F$ is irregular by Theorem 11 unless $b = 1$, $2n = 4$ or 12. We postpone the case of $2n \equiv 0 \pmod 8$.

When $c$ is a multiple of 4, $f(c)$ evidently represents no integer of the form $4n+3$. By known theorems on $f(2)$ and $f(4)$, we readily prove that $f(8)$ represents exclusively all integers except $4n+3$, $16n+6$, $4^k(16n+14)$, and hence is regular.

To prove that $f(24)$ is regular, note that, if it is even, $x+y = 2X$, $x-y = 2Y$, and $f$ is the double of a form $f(12)$, which is regular by Theorem 11. We there noted that $f(6)$ represents all integers $\neq 9^k(9n+3)$. Let $4m+1$ be not of that form; then it is represented by $f(6)$, and $x+y$ is odd, $x^2+y^2 \equiv 1 \pmod 4$, whence $z$ is even. Hence $4m+1$ is represented by $f(24)$.

Next, $f(16)$ is regular. If it is even, it is the double of $f(8)$. Any $N = 8n+5$ is a sum of three squares, two of which are even and one odd, whence $N = x^2+4y^2+4z^2$. Then $4 \equiv 4y^2+4z^2 \pmod 8$, and one of $y$, $z$ is even. Thus $N$ is represented by $f(16)$. The same is true also of any $8n+1$; but a proof seems difficult.

If $f(48)$ is even, it is the double of $f(24)$. By the usual proof, $f(48) \neq M = 9^k(9n+6)$. It was verified that $f(48)$ represents every $4m+1 < 300$ except 21, $9\cdot 21$, and numbers $M$. We apply

LEMMA 3. *$f(48)$ is irregular as to 21.*

Since $f(48) \equiv 21 \pmod 8$ for $x = 2$, $y = 1$, $f(48) \equiv 21 \pmod{2^n}$ is solvable by Theorem 3. We next prove by induction that $f(48) \equiv 21 \pmod{3^n}$ is solvable. We may take $x = y = 0$. For $m \geqq 2$, let $48\zeta^2 = 21+3^m q$, whence $\zeta^2 \equiv 1 \pmod 3$. Take $z = \zeta+3^{m-1}Z$. Then

$$48z^2 = 21+3^m q+32\cdot 3^m \zeta Z+16\cdot 3^{2m-1}Z^2 \equiv 21 \pmod{3^{m+1}}$$

if $q+32\zeta Z \equiv 0 \pmod 3$. The latter is satisfied by choice of $Z$, since $\zeta$ is not divisible by 3. If $p$ is a prime $>3$, Theorem 1 shows that $f(48) \equiv 21 \pmod{p^n}$ is solvable. Lemma 3 now follows as in § 4.

(II) Let $2n<b$. In $F$ interchange $Y$ with $z$ and write $c$ for $b$. We get $\varphi = X^2+2nY^2+cz^2$, where $2n<c$, $c \equiv 1 \pmod 4$. If $2n \equiv 2 \pmod 4$, $\varphi$ is irregular by Theorem 6 unless $2n=2$, $c=5$. In the latter case, $f$ is $\sigma$. If $\sigma$ is divisible by 5, the quotient is $5X^2+y^2+40Z^2 \equiv 0, \pm 1 \pmod 5$, whence $\sigma$ represents no $N=25^k(25n\pm 10)$. By the Bulletin paper, $g = x^2+5y^2+2z^2$ represents all integers $\neq N$. Thus $g$ represents every $4m+1 \neq N$; then $x+y$ is odd, $x^2+5y^2 \equiv 1 \pmod 4$, whence $z$ is even, and $4m+1$ is represented by $\sigma$. Next, if $\sigma \equiv 2 \pmod 4$, $x$ and $y$ are odd and $\sigma \equiv 6 \pmod 8$. Now $\sigma$ represents every $8n+6<200$ except 110 and 190, which are of the form $N$ already excluded.

If $2n \equiv 4 \pmod 8$, $\varphi$ is irregular by Theorem 12.

It remains to treat the case $2n = 8m$ in (I) and (II). The argument made for $f$ is now repeated for $F$. If $F$ is divisible by 4, the quotient is $\psi = \xi^2+b\eta^2+2m\zeta^2$. First, let $2m \geqq b$, $2m \equiv 2 \pmod 4$, $\psi$ (and hence $F$ and $f$) is irregular unless $b=1, 2m=2$ or 6. Then $f$ is $f(32)$ or $f(96)$. Evidently $f \neq 3$, 6 or 7 (mod 8). We find that $f(32)$ represents all $8n+2<200$, and all $4n+1<300$ except 21 and 161; the further numbers represented are all multiples of 4, and $f$ was seen to be regular for them. Next, $f=f(96)$ evidently represents no $N=9^k(9n+3)$; $f$ represents all $8n+2<200$ except 42, $66=N$ and $138=N$, and all $4n+1<200$ except 33, 69, 77 and integers $N$.

Next, if $2m \equiv 4 \pmod 8$, $\psi$ and hence $f$ is irregular unless $b=1$, $2m=4$ or 12. Then $f$ is either $f(64)$ or $f(192)$. These are irregular as to numbers $4n+1$; the first represents all $<400$ except 21, 33, 57, 133, 141, 253, 385; the second represents all except 77, 133, 161 and multiples of 3.

For $2m<b$, it remains by (II) to consider the case $2m=2$, $c=5$, whence $f$ is $x^2+5y^2+32z^2$. The integers $<100$ not represented by $f$ for which $f$ is irregular are 13, 22, 17, 97, with doubt as to the multiples 65 and 85 of 5.

Further repetions of the process evidently lead to irregular forms.

There remains only the case $b=8n$, $c \equiv 1 \pmod 4$, $c>8n$. If $f$ is divisible by 4, the quotient is $\varphi$ in (II). The only regular $\varphi$ has $2n=2$, $c=5$, which contradict $c>8n$.

9. **Theorem 15.** *A positive integer is represented by $f=x^2+y^2+5z^2$ if and only if it is not of the form $4^k(8n+3)$.*

If $f \equiv 3 \pmod 4$ then $x, y, z$ are all odd and $f \equiv 7 \pmod 8$. Hence $f$ represents no $8n+3$. If $f$ is divisible by 4, $x, y, z$ are all even; hence $f$

represents $4m$ if and only if $f$ represents $m$. The theorem will therefore follow if we prove that $f$ represents every positive $8N+7$ and positive $a \equiv 1$ or $2 \pmod 4$.

We first prove that $f$ represents every such $a$. The determinant of the halves of the second partial derivatives will be called the Hessian. The only reduced positive forms of Hessian 5 are $f$ and $g = x^2+2y^2+3z^2+2yz$. Since $g \equiv x^2+2(y-2z)^2 \pmod 5$, $g \equiv 0$ implies $x = 5X$, $y = 2z+5W$, $g = 5G$, $G = 5X^2+3z^2+10zw+10w^2$, $G \equiv 0$ or $\pm 3 \pmod 5$. Thus $g$ represents no $5(5n \pm 1)$. We shall prove the existence of a positive form

$$F = ax^2+by^2+cz^2+2ryz+2xz, \qquad b = 5(5n-1),$$

of Hessian 5. Evidently $F$ represents $a$ and $b$. Since $F$ is therefore not equivalent to $g$, it is equivalent to $f$, whence also $f$ represents $a$.

The Hessian of $F$ is 5 if $b = a\Delta - 5$, where $\Delta = bc-r^2$. Then if $\Delta > 0$, $F$ is a positive form. Take $\Delta = 50L$, $L = 4\lambda+1$. Then $b = 5\beta$, $\beta = 10aL-1$. Then $\beta = 40a\lambda+10a-1$ is a linear function of $\lambda$ with relatively prime coefficients and hence is a prime for infinitely many values of $\lambda$. We have

$$\beta \equiv 10a-1 \equiv 1 \text{ or } 3 \pmod 8 \text{ according as } a \equiv 1 \text{ or } 2 \pmod 4.$$

Hence

$$\left(\frac{-2}{\beta}\right) = +1, \qquad \left(\frac{L}{\beta}\right) = \left(\frac{\beta}{L}\right) = \left(\frac{-1}{L}\right) = +1,$$

so that $-2L = R^2-\beta C$ is solvable for integers $R$ and $C$. Write $c = 5C$, $r = 5R$. Then $50L = \Delta = bc-r^2$, as desired.

Next, let $a = 8N+7$. We now take $\Delta = 25l$, $l = 4t+1$, $t$ odd. Then $b = 10d$, $d = 10at+20N+17$. Then $d$ is a prime by choice of $t$. Since $d \equiv 3 \pmod 4$, $(-1/d) = -1$, $(-l/d) = 1$. Hence $-l = x^2-dC$ is solvable for integers $x$, $C$. Since we may add to $x$ a multiple of the odd integer $d$, we may choose an odd value $R$ for $x$. Then $C$ is even. Write $C = 2\gamma$, $c = 5\gamma$, $r = 5R$. Hence $\Delta = bc-r^2$.

COROLLARY. *$g = X^2+Y^2+10z^2$ represents all positive even integers except $4^k(16n+6)$.*

For if $g$ is even, $X+Y = 2x$, $X-Y = 2y$, $g = 2f$. Conversely, $2f = (x+y)^2+(x-y)^2+10z^2$ is of the form $g$. By Theorem 7, $g$ is irregular with respect to odd integers.

# CORRESPONDENCES BETWEEN ALGEBRAIC CURVES.*

By S. Lefschetz.

1. The chief object of this paper is to show what can be accomplished with the aid of some simple notions of Analysis Situs in the theory of algebraic correspondences between algebraic curves. Most of the material has been outlined elsewhere [4], [7], p. 49.† Necessities of the case, however, must have made the prerequisite knowledge of Analysis Situs appear more formidable than it actually is as we shall show. Furthermore one can take advantage of the fact that the topological situation is rather elementary to simplify many proofs. As we have already given a rigorous treatment in [2] we shall not hesitate to make a rather large appeal to intuition. The preliminary topological notions are considered first so as not to interrupt the dicussion of the main topic. It may be stated that a full résumé of the theory of correspondences and related topics will appear as part of the Report on Rational Transformations to be issued shortly by the National Research Council.

2. **Algebraic curves and surfaces.** From our point of view an irreducible algebraic curve $C$ is a multiply-sheeted, connected, oriented Riemann surface. Orientation can be determined in various equivalent ways. The most convenient mode is to derive it from a choice of a local system of curvilinear coördinates $\xi, \eta$, for a point $A$. Congruent systems are then defined for all points of $C$ and to change the orientation means to interchange $\xi, \eta$. At the same point if we pass from $\xi, \eta$, to $\xi', \eta'$, the orientation is preserved if $\dfrac{D(\xi, \eta)}{D(\xi', \eta')} > 0$, changed if it is $< 0$. We shall have repeated occasion to consider *cycles* and their *homologies* also the related connectivity indices, [3], Ch. 1; [4]. Homologies, usually defined in terms of boundary relationships, can intuitively best be apprehended thus: Let for example $\gamma_1, \cdots, \gamma_r$ be a set of linear cycles on $C$ and suppose that there exist integers $t_i$ not all zero such that $\sum t_i \gamma_i$ is continuously deformable over $C$ into a sum of pairs of opposite arcs. We express the fact by the homology

$$\sum t_i \gamma_i \sim 0, \quad \text{mod.}\, C$$

("mod. $C$" is often omitted) and state that they are *dependent*. The maximum number of independent linear cycles is precisely the linear index $R_1$ and we recall that $\frac{1}{2} R_1 = p$ the *genus* of $C$.

* Received January 12, 1927.

† The square brackets refer to the bibliography at the end of the paper.

It will be convenient to consider a signed point as a definite zero-cycle and since any point or its negative can be brought into coincidence with any other signed point, every zero-cycle is a multiple of a single point and $R_0 = 1$. Every two-cycle on $C$ is $\sim tC$, and therefore $R_2 = 1$. This will be found more amply justified in [14], p. 110.

All this can be extended to an algebraic surface $F$ fulfilling the following conditions. It is irreducible on an $S_n$ and such that for every point $A$ of $F$ there is a cartesian coördinate representation of its neighborhood by power series in two variables $u, v$, whose Jacobian matrix is of rank two. This is merely the analytical expression of the fact that $F$ is non-singular, but happens to be more convenient with the particular surface that will concern us. We can always choose two cartesian coördinates $x, y$ such that at any point not on a certain algebraic curve of $F$ we can take $u = x - x_0$, $v = y - y_0$ ($x_0$, $y_0$ values of those coördinates at the point).

We recall that $F$ is point-wise homogeneous, representable as a multiple sheeted complex $S_2$ and orientable, [3], Ch. 2.

3. **Kronecker-indices.** We are interested in these only in two special cases.

(a) *Index of two oriented arcs $l, l'$ on $C$.* We assume them analytical and intersecting at a finite number of points, not end points. Let $A$ be one of them, $\xi, \eta$ its local coördinate system as in No. 2, $u(\xi, \eta) = 0$, $v(\xi, \eta) = 0$ the equations of $l, l'$. We assume $u, v$ so chosen that following $l$ positively from $A$ on $v$ is increasing and similarly for $l'$. Then the functions $u, v$ define a new local coördinate system. Let $k$ be the number of points where it is congruent to the system $\xi, \eta$, i. e. where $\dfrac{D(u, v)}{D(\xi, \eta)} > 0$, and $k'$ the number of remaining points. The difference $k - k'$ is the *Kronecker index* of $l, l'$ and is denoted by $(l \cdot l')$. The extension to sums of arcs $\sum l, \sum l'$ is by imposing the distributive law

$$(\sum l \cdot \sum l') = \sum (l \cdot l'). \tag{3.1}$$

There is no difficulty in identifying our index with the numbers of [3], [8]. The process here used determines a small triangle $ABB'$, where $AB$, $AB'$ are positive arcs of $l, l'$ and gives a way of finding whether it is an indicatrix or not.

(b) *Index of two surface elements $L, L'$ by $F$.* The procedure is exactly the same. The elements will be represented about an intersection $A$ by two pairs of equations $u = v = 0$, $w = t = 0$ in the local coördinates $\xi, \eta, \zeta, \theta$. $u, v$ define a coördinate system on $L'$ and are so chosen as to correspond to the orientation on $L'$, and similarly on $L$, etc. The points

are again separated into two categories by the sign of the Jacobian of $u, \cdots, t$ and the rest is as before.

*Remark.* Suppose that $l, l'$ on $C$ or $L, L'$ on $F$ are continuously deformed in such manner that as $A$ varies the elements of each pair do not become tangent at any time. Then the Jacobians involved will vary continuously and never vanish, hence the contribution from $A$ to the particular index involved will remain constant.

For a similar reason we can replace each element by a tangent element. To be precise for example $L$ may be replaced by an element $\bar{L}$: $\bar{u} = \bar{v} = 0$, in which $\bar{u}, \bar{v}$ coincide with $u, v$ in their first degree terms, since the Jacobian considered depends only on these terms. The orientation of one of the elements $L, L'$ defines that of the other.

4. We now give a series of propositions which follow almost directly from the definitions, and in any case are proved at length in [8], [3], Ch. 1. We use throughout $\gamma$ for the linear cycles on $C$, $\Gamma$ for the two-cycles on $F$.

I. $$(\gamma \cdot \gamma') = -(\gamma' \cdot \gamma); \quad (\Gamma \cdot \Gamma') = (\Gamma' \cdot \Gamma).$$

For both types of indices:

II. When one of the cycles is reversed the sign of the index is changed.

III. The distributive law holds.

IV. When one of the cycles is $\sim 0$, the index is zero.

V. When one of the cycles is replaced by a homologous cycle, the index is unchanged.

Finally we have two propositions that are special.

VI. For the retrosections $g_1, \cdots, g_{2p}$ of $C$, properly numbered and oriented all indices are zero except:

$$(g_i \cdot g_{p+i}) = -(g_{p+i} \cdot g_i) = +1.$$

In other terms the matrix

$$\| (g_i \cdot g_j) \| = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

where in the second matrix each term stands for an obvious square matrix of order $p$.

VII. The algebraic curves of $F$ are also two-cycles of the surface. Let $\Gamma, \Gamma'$ be two of them. There is an associated index $(\Gamma \cdot \Gamma')$. Then if their intersections are all simple each contributes the same amount to the index, and curves and surface can be so oriented as to make this amount positive. The following orientation scheme will fulfill the purpose: Let $A$ be any point of $\Gamma$, $u, v$ the corresponding variables of No. 2, $\varphi(u, v) = 0$ the

equation of $\Gamma$ (satisfied by its points of the neighborhood of $A$). Let $u = u' + iu''$, $v = v' + iv''$, $\varphi = \varphi_1(u', \cdots, v'') + i\varphi_2(u', \cdots, v'')$ where all new terms introduced are real. Let similarly the equation of $\Gamma'$ be $\psi = \psi_1 + i\psi_2$. $\psi_1$ and $\psi_2$ can be used as coördinates on $\Gamma$, in the order named, and (Nr. 3, Remark) the orientation thus assigned is fixed and unique not only for $A$ but throughout $\Gamma$ and independent of $\Gamma'$. Similarly for $\Gamma'$ and $\varphi_1, \varphi_2$ also for $F$ and $u', u'', v', v''$.

Under the circumstances the contribution of $A$, equal to one in absolute value, has the sign of $\dfrac{D(\varphi_1, \varphi_2, \psi_1, \psi_2)}{D(u', u'', v', v'')}$. By means of the Cauchy-Riemann differential equations we may actually verify that the Jacobian is equal to $\left|\dfrac{D(\varphi, \psi)}{D(u, v)}\right|^2 > 0$. A more rapid procedure is to apply again the Remark of Nr. 3 and observe that it allows us to replace $\varphi, \psi$ by $u, v$. Then the Jacobian is $+1$ and therefore always positive. As we have thus shown that $A$ always contributes $+1$, VII is in effect proved. The orientations as defined here will henceforth be adopted throughout.

When $\Gamma, \Gamma'$ intersect in multiple points, etc., each intersection must be counted with its multiplicity. This is proved by slightly changing the coefficients in $\varphi$ or $\psi$. For all this see [3], Ch. 2.

5. **Product of two configurations.** By product $E \times F$ of two configurations $E$, $F$ we understand a homeomorph (continuous 1, 1 image) of the set of all point pairs $e, f$, where $e$ belongs to $E$ and $f$ to $F$. This product can be thought of as a point-set where each individual point is representable as $e \times f$. If $E$ and $F$ are continua of $h$ and $k$ dimensions $E \times F$ is a continuum of $h + k$ dimensions.

When $E$ is a point and $F$ an oriented configuration $E \times F$ can be represented by $F$ itself and is to be oriented like it. For example if $F$ is a surface element with a local coördinate system $\xi, \eta$ at the point $A$, $E \times F$ is a surface element with a similar coördinate system and the same variables at $E \times A$.

When $E$, $F$ are both oriented arcs we impose upon the surface element which is their product an orientation such that

$$(E \times f \cdot e \times F) = +1. \tag{5.1}$$

For reasons of continuity if this is true at $e \times f$ it will be true over the whole surface element.

These definitions are extended to sums of oriented configurations by imposing the distributive law

$$\sum E \times \sum F = \sum E \times F. \tag{5.2}$$

6. Of basic interest for the sequel is the product of two algebraic curves $A$, $B$. Call their general $p$, $q$, and let them be non-singular curves of certain spaces $S_m$, $S'_m$. Let $a$, $(x_0, \cdots, x_m)$ be the running point on $A$, $b$, $(y_0, y_1, \cdots, y_m)$ the running point on $B$. It is a simple matter to show that the surface of $S_{m(m+2)}$ whose parametric equations are

$$\varrho X_i = x_j y_k; \qquad (j, k = 0, 1, \cdots, m) \tag{6.1}$$

is a non-singular surface of that space, and furthermore behaves in accordance with No. 2. As it is homeomorphic to $A \times B$ we henceforth identify it with this product.

The characteristic feature of $A \times B$ from the birational standpoint, is that it carries two algebraic curve pencils of genera $p$, $q$, whose images are $B$ and $A$, and which are composed of the curves $A \times b$ and $a \times B$; one of them passes through every point $a \times b$ of $A \times B$. The natural convention for orienting $A \times B$ is by imposing

$$(A \times b \cdot a \times B) = 1 \tag{6.2}$$

analogous to (5.1). It will be observed that our orientation conventions are in agreement with [8], p. 30. From (6.2) and VII, No. 4, follows that the index of two algebraic curves on $A \times B$ is always equal to the number of their intersections.

We have this important theorem. Every $k$-cycle of $A \times B$ is a sum of products of $h$ and $k - h$ cycles of $A$ and $B$. Let $\gamma_1, \cdots, \gamma_{2p}$ and $\delta_1, \cdots, \delta_{2q}$ be fundamental sets for $A$, $B$. The cycles $a \times \delta_i$, $\gamma_j \times b$ constitute a fundamental set for the linear cycles of $A \times B$ and as they are independent its $R_1 = 2p + 2q$. For the two cycles we also have a fundamental set composed of independent cycles in the set $a \times B$, $A \times b$, $\gamma_i \times \delta_j$. Hence $R_2 = 4pq + 2$. For the proofs which are very simple, see [3], p. 52; [8], p. 52; [7], p. 362.

Concerning Kronecker indices there is the important formula

$$(\gamma \times \delta \cdot \gamma' \times \delta') = -(\gamma \cdot \gamma')(\delta \cdot \delta'). \tag{6.3}$$

Owing to V, No. 4, to the distributive laws and to our continuity considerations, everything comes down to examining the contributions at points $a \times b$, $a$, $b$, of intersections of the respective cycle pairs. We can choose our local coördinates such that, with the same notations as in No. 4, $u' = u'' = 0$ represents $A \times b$, $v' = v'' = 0$, $a \times B$. Also, No. 3 Remark, we may take $\gamma$, $\gamma'$, $\delta$, $\delta'$, as being in order the $u''$, $u'$, $v''$, $v'$ axes. Then $(\gamma \cdot \gamma') = (\delta \cdot \delta') = +1$. The coördinates for $\gamma \times \delta$ and $\gamma' \times \delta'$ are respectively $(u'', v'')$, $(u', v')$. Hence

$$(\gamma \times \delta \cdot \gamma' \times \delta') = \frac{D(u', v', u'', v'')}{D(u', u'', v', v'')} = -1 = -(\gamma \cdot \gamma')(\delta \cdot \delta')$$

as was to be proved.

7. The first geometer to consider products of curves would seem to have been Segre. They were also investigated by Picard, Maroni, de Franchis, and especially Severi, in whose beautiful paper [13] (1903) will be found ample references. Somewhat later (1907) general products were introduced by E. Steinitz and also investigated by his student Künneth. For references to their work see [8] p. 29. In that paper products were made the basis for solving the general problem of coincidences of transformations of manifolds without boundary. See also [9], vol. 1, p. 92; [4]; [6]; [3], p. 52; [8], p. 52.

8. **Algebraic correspondences between algebraic curves.** Let $T$ be such a correspondence between our two curves $A, B$. We consider $T$ as an algebraic transformation of $A$ into $B$. To each point $a$ of $A$ correspond $\beta$ transform points $b_1, \cdots, b_\beta$ which constitute the group $Ta$. Similarly to each $b$ of $B$ correspond by $T^{-1}$ points $a_1, \cdots, a_\alpha$, of the group $T^{-1}b$. The numbers $\alpha, \beta$, are the *indices* of $T$.

If $b$ is any point of $Ta$, the set of all points $a \times b$, when $a$ varies, is an algebraic curve $\Gamma$ on $A \times B$, since it is an algebraic locus $\infty^2$ of which no isolated component is a point or a surface. $\Gamma$ as a cycle must be oriented so as to conform with VII, Nr. 4. Conversely every algebraic curve $\Gamma$ not a sum of curves $a \times B$ or $A \times b$, defines a correspondence $T$. The study of correspondences is then identical with that of the curves on $A \times B$. In particular birational invariant integers associated with the curves will yield similar invariants for the correspondences. Thus the genus of $\Gamma$, and its *degree* (Kronecker index $(\Gamma, \Gamma)$, number of intersections with a neighboring curve of the same continuous system) are invariants of $T$, as well brought out by the Italian authors who dealt with the question. However, our concern will be in a distinctly different direction.

The indices of $T$ are

$$(\Gamma \cdot A \times b) = \alpha; \quad (\Gamma \cdot a \times B) = \beta, \tag{8.1}$$

By I, No. 4, we may also interchange the cycles in (8.1) without affecting the indices. And while we are considering these indices we may as well observe that

$$(a \times B \cdot \gamma \times \delta) = (A \times b \cdot \gamma \times \delta) = 0 \tag{8.2}$$

and likewise when the cycles are interchanged. For when $a$ or $b$ vary $a \times B$, $A \times b$ are replaced by homologous cycles and therefore the indices

are unchanged. Take $a$ not on $\gamma$, $b$ not on $\delta$. Then the cycles in each pair will not intersect and the indices are zero. Similarly

$$(8.3) \qquad (a\times B\cdot a\times B) = (A\times b\cdot A\times b) = 0.$$

9. Since $\Gamma$ is a cycle we have by No. 6

$$(9.1) \qquad \Gamma\sim \varepsilon a\times B+\varepsilon' A\times b+\sum \varepsilon_{ij}\gamma_i\times\delta_j.$$

By (8.1), (8.2), (8.3) and (6.2)

$$(\Gamma\cdot A\times b) = \varepsilon = \alpha;\quad (\Gamma\cdot a\times B) = \varepsilon' = \beta.$$

Hence in place of (9.1)

$$(9.2) \qquad \Gamma\sim \alpha a\times B+\beta A\times b+\sum \varepsilon_{ij}\gamma_i\times\delta_j.$$

10. To proceed further we need to bring into play the transforms of the cycles of $A$ by $T$. Since $T$ is a transformation on points, strictly speaking we are quite free to define arbitrarily $T\gamma$. But the only reasonable definition is this: Let first $\gamma$ be a simple circuit. As the point $a$ describes it positively the points of the group $T\cdot a$ on $B$ will describe paths whose sum is a cycle, that we consider by definition as $T\gamma$. If $\gamma$ is an arbitrary cycle we replace it by a homologous sum of simple circuits and define $T\gamma$ as the sum of their transforms.

We next propose to prove the fundamental relation

$$(10.1) \qquad (\Gamma\cdot\gamma\times\delta) = -(T\gamma\cdot\delta)$$

where the first index is measured on the product surface and the second on $B$.

As in No. 6 we can reduce everything to the special case of a single intersection $a\times b$ for the cycles at the left and choose the local coördinates such that $A\times b$, $a\times B$, $\Gamma$, be represented by $u=0$, $v=0$, $u-v=0$. We can besides also assume that $\gamma$, $\delta$, are the real $u''$, $v'$ axes.

Let us compute $(\Gamma\cdot\gamma\times\delta)$. The equations of $\gamma\times\delta$ are $u'=v''=0$. On $\Gamma$ the variables $u'$, $v''$, reduce to $v'$, $v''$. This is the same set of local $\Gamma$ coördinates as derived for it from the condition $(\Gamma\cdot a\times B) = 1$, and corresponds then to the correct orientation of the cycle. On the other hand the equations of $\Gamma$: $u'-v'=0$, $u''-v''=0$ are such that their left sides become $-v'$, $u''$ on $\gamma\times\delta$. They vanish respectively on $\gamma$ and $\delta$ and correspond to $\gamma$ and $-\delta$, hence the equations of $\Gamma$ must now be written as $v'-u'=u''-v''=0$.

$$\therefore\ (\Gamma\cdot\gamma\times\delta) = \frac{D(u',\ v'',\ v'-u',\ u''-v'')}{D(u',\ u'',\ v',\ v'')} = -1.$$

It is at once verified that $T\gamma$ coincides with the $v''$ axis, hence $(T\gamma \cdot \delta) = 1$, from which (10.1) follows.

The formula just proved is a special case of the analogue of formula (59.2) of [8], for transformations between two distinct manifolds, derived in exactly the same way. Its proof as there given is much more general than the above but also far more laborious.

11. The application is immediate. The effect of $T$ upon the cycles is represented by a series of equations:

$$T\gamma_h \sim \sum c_{hk}\, \delta_k. \tag{11.1}$$

From (9.2), (6.2) and (10.1) we find

$$\sum \varepsilon_{ij}\, (\gamma_i \cdot \gamma_j)\, (\delta_j \cdot \delta_k) = \sum c_{hj}\, (\delta_j \cdot \delta_k). \tag{11.2}$$

At this point matrix notations come in very conveniently. We designate by $\varepsilon$, $c$, the matrices whose elements are $\varepsilon_{ij}$, $c_{ij}$, and set

$$C = \| (\gamma_i \cdot \gamma_j) \|, \quad D = \| (\delta_i \cdot \delta_j) \|.$$

From I, No. 4, we conclude that $C$ is an alternate matric, i. e., $C' = -C$, and also of course $D' = -D$. Their respective orders are $2p$, $2q$. When the fundamental sets are changed $C$ and $D$ are multiplied by matrices whose determinants are $\pm 1$. When the fundamental sets are retrosections $|C| = |D| = \pm 1$. Hence these determinants are $\pm 1$ in all cases, and $C^{-1}$ and $D^{-1}$ are well defined. From (11.2) then follows

$$C' \varepsilon D = -C \varepsilon D = cD, \tag{11.3}$$

and since $D^{-1}$ exists,

$$-C\varepsilon = c; \qquad \varepsilon = -C^{-1} c. \tag{11.4}$$

This completely determines the cycle $\Gamma$ attached to $T$.

REMARKS. I. An interesting application of our methods is found in the calculation of the transformation matrix $\tilde{c}$ corresponding to $T^{-1}$ like $c$ to $T$. For $T^{-1}$ and in virtue of the readily verified relation $\gamma \times \delta = -\delta \times \gamma$, we must write in place of (9.2):

$$\Gamma \sim \alpha B \times a + \beta b \times A - \sum \varepsilon_{ij}\, \delta_j \times \gamma_i, \text{ mod. } B \times A. \tag{11.5}$$

Hence $\varepsilon$ is now replaced by $-\varepsilon'$, and by (11.4) we find

$$\tilde{c} = -D' \varepsilon' = Dc' C^{-1}. \tag{11.6}$$

When the fundamental sets are the retrosections, $D$ and $C$ have the form given in VI, No. 4 and $C^{-1} = -C$. Let us write $c = \left\| \begin{matrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{matrix} \right\|$, where the terms are all matrices with $p$ rows and $q$ columns. We have

$$\tilde{c} = -\left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\| \cdot \left\| \begin{matrix} c'_{11} & c'_{21} \\ c'_{12} & c'_{22} \end{matrix} \right\| \cdot \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\| = \left\| \begin{matrix} c'_{22} & -c'_{12} \\ -c'_{21} & c'_{11} \end{matrix} \right\|.$$

This result, except for the notations, has already been derived by Rosati [12], p. 41, by a lengthy calculation involving theta functions.

II. The fact that $T$ is algebraic enters solely in the proofs of several index formulas. These have been independently established in [8]. Hence all our results are valid for general correspondences.

12. Let $T^*$ be another algebraic correspondence between $A$, $B$ and denote all its elements by starring those of $T$. We are particularly interested in the number of coincidences of the two correspondences, i. e., in the number of pairs $a$, $b$ such that $b = Ta = T^* a$. That number is manifestly equal to $(\Gamma \cdot \Gamma^*)$. At once

$$(12.1) \qquad (\Gamma \cdot \Gamma^*) = \alpha\beta^* + \alpha^*\beta + \sum \varepsilon_{ij}\, \varepsilon^*_{hk} (\gamma_i \times \delta_j \cdot \gamma_h \times \delta_k),$$

which by (6.2) becomes

$$(12.2) \qquad \begin{aligned} (\Gamma \cdot \Gamma^*) &= \alpha\beta^* + \alpha^*\beta - \sum \varepsilon_{ij}\, \varepsilon^*_{hk} (\gamma_i \cdot \gamma_h)\, (\delta_j \cdot \delta_k) \\ &= \alpha\beta^* + \alpha^*\beta - \operatorname{trace} \varepsilon' C \varepsilon^* D'. \end{aligned}$$

By (12.4) we find $\varepsilon' C \varepsilon^* D' = c' C'^{-1} c^* D'$. The trace of the latter matrix is the same as that of its transverse $D c'^* C^{-1} c$. For reasons of symmetry $c$ and $c^*$ may be interchanged, hence finally

$$(12.3) \qquad (\Gamma \cdot \Gamma^*) = \alpha\beta^* + \alpha^*\beta - \operatorname{trace} D c'\, C^{-1} c^*.$$

The degree of $\Gamma$ is obtained by assuming $\Gamma^* \sim \Gamma$. It is therefore

$$(12.4) \qquad (\Gamma \cdot \Gamma) = 2\alpha\beta - \operatorname{trace} D c'\, C^{-1} c.$$

13. **Non-singular correspondences.** The correspondence $T$ is said to be *singular* if the corresponding cycle transformation matrix $c$ is $\neq 0$. A non-singular correspondence between the two curves $A$, $B$ is then represented by a curve

$$(13.1) \qquad \Gamma \sim \beta A \times b + \alpha a \times B$$

and its coincidences with $T$, whether singular or not, are given by

$$(13.2) \qquad (\Gamma \cdot \Gamma^*) = \alpha\beta^* + \alpha^*\beta.$$

REMARK. In view of the equivalence between the relations $\sim$ and $=$ (Severi algebraic equivalence), [3], between curves on a surface, (13.1) can be replaced by the Severi relation

$$\Gamma = \beta A \times b + \alpha a \times B. \tag{13.3}$$

This is essentially his result [13], p. 22.

14. **Correspondences on a single algebraic curve** $A$. This time $B$ must be taken as merely another copy of $A$ and the only change throughout is that $D = C$, so that in place of (12.3) we now have the coincidence formula

$$(\Gamma \cdot \Gamma^*) = \alpha\beta^* + \alpha^*\beta - \text{trace } Cc'C^{-1}c^*. \tag{14.1}$$

As a matter of fact there is nothing to prevent us from using on $B$ a fundamental set not made up of the images of the $\gamma$'s and then (12.3) is unmodified.

The interesting problem, however, lies now in connection with the relations of $T$ to the identical transformation. Let us denote all the elements of the latter by the superscript zero. Then $c^0 = 1$, hence $\varepsilon^0 = -C^{-1}$; also $\alpha_0 = \beta_0 = 1$, which determines its cycle $\Gamma^0$. Hence the number of *fixed points* of $T$, i. e., of points $a = Ta$ is now given by

$$(\Gamma \cdot \Gamma^0) = \alpha + \beta - \text{trace } Cc'C^{-1}. \tag{14.2}$$

But by a well known property

$$\text{trace } Cc'C^{-1} = \text{trace } c' = \text{trace } c. \tag{14.3}$$

Hence the number of fixed points is

$$(\Gamma \cdot \Gamma^0) = \alpha + \beta - \text{trace } c. \tag{14.4}$$

Of particular interest are the *valence correspondences* or correspondences whose effect on the cycles is merely to multiply them by one and the same integer $-g$, where $g$ is the *valence* of $T$.† Then $c = -g \cdot I_{2p}$, where $I_{2p}$ is the identity matrix of order $2p$. Here trace $c = -2gp$, wherefore for the number of fixed points:

$$(\Gamma \cdot \Gamma^0) = \alpha + \beta + 2gp \tag{14.5}$$

which is the noted Cayley-Brill formula. Similarly the other coincidence and fixed points formulas generalise known results due to Hurwitz, [2].

† Usually $\gamma$, but the change in notation is necessary to avoid confusion with our notation for cycles.

He assumes throughout that the fundamental sets are retrosections and obtains his results by means of theta functions. See also [4]; [8] p. 47, a topological derivation by Chisini, [1], and a related paper by Alexander, [15].

REMARK. An interesting special type consists of *symmetrical* correspondences or correspondences which coincide with their inverse. From No. 11 follows that for such a $T$, $\varepsilon' = -\varepsilon$, i. e., the matrix $\varepsilon$ is skew symmetrical. Furthermore also $c = Cc'C^{-1}$, and when the retrosections are the fundamental set, $c = -Cc'C$.

15. **Existence of correspondences.** We return to correspondences between $A$ and $B \neq A$. We ask if, given the cycle (9.2) a priori, there exists a related algebraic correspondence $T$? The problem is equivalent to obtaining the conditions under which (9.2) represents an effective curve of $A \times B$. The answer is provided by this theorem which I first proved in [7], p. 346; see also [3], p. 72: In order that $\Gamma$ be homologous to a curve, effective or virtual (difference of two effective curves), it is necessary and sufficient that the corresponding period of every double integral of the first kind be zero. Let $u_1, \cdots, u_p$ and $v_1, \cdots, v_q$ be independent sets of integrals of the first kind of $A$ and $B$. Then every Picard integral of the first kind of $A \times B$ is a linear combination of the $u$'s and $v$'s, [9] vol. 1, p. 129, hence $\frac{1}{2}R_1 = p + q$ in conformity with No. 6. Similarly every double integral of the first kind is a linear combination of the integrals $\int\int du_j\, dv_k$. Hence, according to the above theorem we must have $\int\int_\Gamma du_j\, dv_k = 0$. Let $\omega_1 = \|\omega^1_{j\mu}\|$, $\omega_2 = \|\omega^2_{j\nu}\|$ be the period matrices of the $u$'s and $v$'s as to the $\gamma$'s and $\delta$'s, the rows corresponding to the integrals and the columns to the cycles. Since $u_j$ is constant on $a \times B$ and $v_k$ on $A \times b$, our condition reduces to

$$\sum_{\mu,\nu} \varepsilon_{\mu\nu}\, \omega^1_{j\mu}\, \omega^2_{k\nu} = 0, \tag{15.1}$$

which is the same as

$$\omega_1\, \varepsilon\, \omega_2' = 0. \tag{15.2}$$

The necessary condition involves then, when $\varepsilon \neq 0$, that is, when $T$ is not an ordinary correspondence, a relation between the two period matrices. It is of the type called "simultaneous relation" by Scorza [10], p. 10. The number $\lambda$ of linearly independent relations of that type is his *simultaneity* index for $\omega_1$ and $\omega_2$. Thus a necessary condition for a singular correspondence is $\lambda > 0$.

Conversely let $\lambda > 0$. Then there exists an $\varepsilon \neq 0$ with a $\Gamma \sim$ an effective or virtual curve. We shall show in a moment that in that case there

exist integers $t_1$, $t_2$ such that a curve $\bar{\Gamma} \sim \Gamma + t_1 A \times b + t_2 a \times B$ is *effective*. This curve satisfies the homology

$$\bar{\Gamma} \sim (\alpha + t_2)\, a \times B + (\beta + t_1)\, A \times b + \sum \varepsilon_{ij}\, \gamma_i \times \delta_j,$$

that is corresponds to the same matrix $\varepsilon$, Hence this final theorem: In order that there exist a singular correspondence between $A$ and $B$ it is necessary and sufficient that the simultaneity index of their period matrices be positive. If (15.2) is the relation between the matrices there is a $T$ determined by (9.2) corresponding to the same $\varepsilon$ as in (15.2).

16. To prove the existence of an effective $\bar{\Gamma}$, let us set $\Gamma = \Gamma_1 - \Gamma_2$ where $\Gamma_1$, $\Gamma_2$ are effective. It is sufficient to show that for $t_1$, $t_2$ properly chosen there is a curve $\sim t_1 \cdot A \times b + t_2\, a \times B$ containing $\Gamma_2$ as component.

In the notations of No. 6, observe that the surfaces, $\prod X_i^{n_i} = 0$, $\sum n_i = n$, cut out on $A \times B$ curves $\sim n(m_1\, a \times B + m_2\, A \times b)$ where $m_1$ and $m_2$ are the orders of $A$ and $B$. If follows that this is also the case for hypersurfaces of degree $n$ of $S_{m(m+2)}$. For $n$ sufficiently high there exists such a hypersurface passing through $\Gamma_2$, which proves our assertion.

17. Hurwitz has defined in [8], correspondences $T^i$ as dependent if there are integers $\lambda_i$ not all zero such that on $B$ the abelian sums for the $v$'s and the group $\sum \lambda_i T^i a$ are constant. This means that the periods of the $v$'s as to $\sum \lambda_i T^i \gamma$ are all zero, hence $\sum \lambda_i T^i \gamma \sim 0$ mod. $B$, whatever $\gamma$. Let $c^i$, $e^i$, $\Gamma^i$, be the $c$, $e$, $\Gamma$ for $T^i$. Then

$$\sum \lambda_i\, c^i = 0. \tag{17.1}$$

$$\sum \lambda_i\, \Gamma^i \sim \lambda A \times b + \mu\, a \times B. \tag{17.2}$$

The last condition shows that the cycle $\sum \lambda_i \Gamma^i$ corresponds to an ordinary correspondence and gives the topological interpretation of the relation of dependence. From (17.1) we derive at once by (11.4)

$$\sum \lambda_i\, \varepsilon_i = 0. \tag{17.3}$$

Conversely from (17.3) follows (17.1). Hence the maximum number of independent correspondences is equal to the maximum number of independent relations (15.2), that is, to the simultaneity index $\lambda$ of $\omega_1$ and $\omega_2$. This is precisely the result obtained, loc. cit., by Hurwitz if we keep in mind Scorza's theorem [10], p. 17, according to which the number of independent relations (15.2) is the same as that of relations $\eta\, \omega_1 = \omega_2\, c$, where $\eta$, c are square, of orders $p$, $2q$ and $c$ has its terms all rational.

18. Similar results hold for correspondences on a single $C$. The terminology should be kept clearly in mind. The term *singular* correspondence

is applied only to a $T$ whose effect upon the cycles is not merely to multiply them by a constant integer $-g$, i. e., to non-valence correspondences. To a valence correspondence belongs a matrix $c = -g I_{2p}$, (No. 14); hence by (11.4) its $\varepsilon = gC^{-1}$. The correspondences which change every $\gamma$ into a circle $\sim 0$ are the correspondences with zero-valence. The number of independent correspondences is the number of linearly independent matrices $\varepsilon$ such that $\omega \varepsilon \omega' = 0$ where $\omega$ is the period matrix of $A$. Similarly the number of independent symmetric correspondences is the analogous number obtained when $\varepsilon$ is skew-symmetric. The same results have already been obtained by Rosati, [11], p. 14.

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# ON THE EXISTENCE AND PROPERTIES OF THE SOLUTIONS OF A CERTAIN DIFFERENTIAL EQUATION OF THE SECOND ORDER.*

By T. H. Gronwall.

**1. Introduction.** The differential equation to be considered is

$$\frac{d}{dx}\left[g(x, y)\frac{dy}{dx}\right] = f(x, y) \tag{1}$$

with one of the following boundary conditions:

$$\begin{aligned}
&(A) && y = y_0 \text{ at } x = x_0, && y = y_1 \text{ at } x = x_1;\\
&(B) && g(x, y)y' = c \text{ at } x = x_0, && y = y_1 \text{ at } x = x_1;\\
&(C) && y = y_0 \text{ at } x = x_0, && y \to 0 \text{ as } x \to \infty;\\
&(D) && g(x, y)y' = c \text{ at } x = x_0, && y \to 0 \text{ as } x \to \infty.
\end{aligned}$$

The existence and uniqueness of the solution† will be shown under the following conditions on $f(x, y)$ and $g(x, y)$, where condition (5) is used only with (C) and (D).

First, given any finite $x_1 > x_0$ and any finite $b > 0$, $f(x, y)$ is bounded and satisfies a Lipschitz condition for $x_0 \leqq x \leqq x_1$ and $|y| \leqq b$, and the same is true for $1/g(x, y)$ for $x_0 \leqq x \leqq x_1$ and all finite $y$, so that

$$\left.\begin{aligned} |f(x, y)| &< M_1,\\ |f(x, z) - f(x, y)| &< M_2|z - y|, \end{aligned}\right\} \quad \begin{aligned} &x_0 \leqq x \leqq x_1,\\ &|y| \leqq b, \quad |z| \leqq b. \end{aligned}$$

$$\left.\begin{aligned} \left|\frac{1}{g(x, y)}\right| &< M_3,\\ \left|\frac{1}{g(x, z)} - \frac{1}{g(x, y)}\right| &< M_4|z - y|, \end{aligned}\right\} \quad \begin{aligned} &x_0 \leqq x \leqq x_1,\\ &\text{all } y \text{ and } z. \end{aligned} \tag{2}$$

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* Presented to the American Mathematical Society, January 2, 1926, and February 26, 1927; received January 21, 1927.

† P. Debye and E. Hückel, Zur Theorie der Elektrolyte, Physik. Zeitschr., vol. 24 (1923), p. 185–206, assume without proof the existence of a unique solution of (1, D) in the particular case $g(x, y) = x^2$, $f(x, y) = x^2 f(y)$, where $f(y) = \sum_{\nu=1}^{n} \alpha_\nu \beta_\nu e^{\beta_\nu y}$, the constants $\alpha_\nu, \beta_\nu$ being real, all $\alpha_\nu$ positive and $\Sigma \alpha_\nu \beta_\nu = 0$, so that $f(0) = 0$. Since $f'(y) = \Sigma \alpha_\nu \beta_\nu^2 e^{\beta_\nu y} > 0$, it follows that $f(x, y)$ and $g(x, y)$ satisfy conditions (2) to (7).

where the $M$'s may increase indefinitely with $x_1$ or $b$. Second, for any finite $x \geqq x_0$, $f(x, y)$ increases with $y$, and $f(x, 0) = 0$, so that

$$f(x, z) > f(x, y) \quad \text{when } z > y, \quad (x \geqq x_0), \tag{3}$$

$$f(x, y) \gtreqless 0 \quad \text{according as } y \gtreqless 0, \quad (x \geqq x_0). \tag{4}$$

Third, for any $y_1 \neq 0$,

$$\int_{x_0}^{\infty} f(x, y_1)\, dx \text{ is divergent.} \tag{5}$$

Fourth, $g(x, y)$ has the properties

$$g(x, y) > 0, \quad (x \geqq x_0 \text{ and any finite } y), \tag{6}$$

$$g(x, z) \geqq g(x, y) \quad \text{when } |z| > |y|. \tag{7}$$

In the particular case $g(x, y) = 1$, the boundary problem (A) was treated by Picard* under the sole assumption (2), provided that $x_1 - x_0$ is sufficiently small. With $g(x, y) = 1$, the boundary problems (A) and (C) were solved by Kneser† under the conditions (2) to (7) and the additional assumption that $f(x, y)$ is such that for every pair of values $y_0$, $y_0'$, equation (1) has a solution which is finite throughout the interval $(x_0, x_1)$ or $(x_0, \infty)$ and satisfies the initial conditions

$$y = y_0, \quad y' = y_0' \quad \text{at} \quad x = x_0. \tag{E}$$

The nature of this assumption on $f(x, y)$ is not investigated;‡ it is an essential result of the present paper that this assumption is superfluous.

2. **Existence and uniqueness of the solution of an auxiliary equation with boundary conditions** (E). Let $b > 0$ be fixed arbitrarily and define $F(x, y)$ as follows:

$$F(x, y) = \begin{cases} f(x, b) & \text{when } y \geqq b, \\ f(x, y) & \quad\ \ \text{"} \quad b \geqq y \geqq -b, \\ f(x, -b) & \quad\ \ \text{"} \quad y \leqq -b. \end{cases} \tag{8}$$

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* E. Picard, Traité d'Analyse, vol. 3, chapter 5.

† A. Kneser, Untersuchung und asymptotische Darstellung der Integrale gewisser Differentialgleichungen bei großen reellen Werten des Arguments, Journ. für Math., vol. 116 (1896), p. 178–212.

‡ The existence of a solution of (1, E) finite in $(x_0, x_1)$ is no consequence of conditions (2) to (7), as is shown by the example $y'' = 2y^3$. We find $y'^2 = y^4 + y_0'^2 - y_0^4$, so that when $y_0'^2 - y_0^4 \neq 0$, $x$ is expressed by an elliptic integral of the first kind and consequently has an upper bound $x_2$ for all real values of $y$, so that no solution of (1, E) exists in the interval $(x_0, x_1)$ when $x_1 > x_2$. If, however, $y_0'^2 - y_0^4 = 0$, the solution which is finite for all $x > x_0$ is $y = y_0/(1 + |y_0|(x - x_0))$ with $y_0' = y_0|y_0|$.

Then $F(x, y)$ evidently satisfies those among the conditions (2) to (7) which are satisfied by $f(x, y)$, except that the condition $|y| \leqq b$, $|z| \leqq b$ in (2) is now removed. With the sole assumption (2), we shall now prove the existence of a unique solution for $x > x_0$ of the differential equation

$$\frac{d}{dx}\left[g(x, y)\frac{dy}{dx}\right] = F(x, y) \tag{9}$$

with the boundary conditions (E).

Integrating and using (E), we obtain from (9)

$$y(x) = y_0 + \int_{x_0}^{x}\left[g(x_0, y_0)y_0' + \int_{x_0}^{s} F(t, y(t))\,dt\right]\frac{ds}{g(s, y(s))}, \tag{10}$$

and this is solved by the following successive approximations

$$\begin{aligned} y_1(x) &= y_0 + y_0'(x - x_0), \\ &\cdots\cdots\cdots \\ y_n(x) &= y_0 + \int_{x_0}^{x}\left[g(x_0, y_0)y_0' + \int_{x_0}^{s} F(t, y_{n-1}(t))\,dt\right]\frac{ds}{g(s, y_{n-1}(s))} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{11}$$

We have

$$\begin{aligned} y_{n+1}(x) - y_n(x) = &\int_{x_0}^{x}\left[g(x_0, y_0)y_0' + \int_{x_0}^{s} F(t, y_n(t))\,dt\right] \\ &\times\left[\frac{1}{g(s, y_n(s))} - \frac{1}{g(s, y_{n-1}(s))}\right]ds \\ &+ \int_{x_0}^{x}\int_{x_0}^{s}[F(t, y_n(t)) - F(t, y_{n-1}(t))]\,dt\,\frac{ds}{g(s, y_{n-1}(s))}, \end{aligned}$$

and using (2), we find for $x_0 \leqq x \leqq x_1$

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| < &M_4\,[|g(x_0, y_0)y_0'| + M_1(x_1 - x_0)]\int_{x_0}^{x}|y_n(s) - y_{n-1}(s)|\,ds \\ &+ M_2 M_3 \int_{x_0}^{x}\int_{x_0}^{s}|y_n(t) - y_{n-1}(t)|\,dt\,ds. \end{aligned} \tag{12}$$

But

$$\begin{aligned} \int_{x_0}^{x}\int_{x_0}^{s}|y_n(t) - y_{n-1}(t)|\,dt\,ds &< \int_{x_0}^{x}\int_{x_0}^{x}|y_n(t) - y_{n-1}(t)|\,dt\,ds \\ &\leqq (x_1 - x_0)\int_{x_0}^{x}|y_n(t) - y_{n-1}(t)|\,dt. \end{aligned} \tag{13}$$

When $x_0$, $x_1$, $y_0$ and $y_0'$ are bounded, $\bar{x}_0 \leqq x_0 \leqq x_1 \leqq \bar{x}_1$, $|y_0| \leqq \bar{y}_0$, $|y_0'| \leqq \bar{y}_0'$, the inequalities (12) and (13) give

$$|y_{n+1}(x) - y_n(x)| < M\int_{x_0}^{x}|y_n(s) - y_{n-1}(s)|\,ds, \tag{14}$$

where $M$ is a constant, and the inequality

$$|y_n(x)-y_{n-1}(x)| \leqq \bar{y}'_0 M^{n-1}\frac{(x-x_0)^n}{n!},$$

which holds for $n=1$ by (11), now follows by complete induction from (14), so that $y_n(x)$ converges to a limit $y(x)$ as $n\to\infty$, the convergence being uniform with respect to $x$, $x_0$, $x_1$, $y_0$ and $y'_0$ when $\bar{x}_0 \leqq x_0 \leqq x \leqq x_1 \leqq \bar{x}_1$, $|y_0| \leqq \bar{y}_0$, $|y'_0| \leqq \bar{y}'_0$, and every $y_n(x)$ being a continuous function of $x$, $x_0$, $y_0$ and $y'_0$, the same is true of $y(x)$. Moreover, it follows from (11) that $y(x)$ satisfies (10), so that it is a solution of (9, E), and by (10) $y'(x)$ is a continuous function of $x$, $x_0$, $y_0$ and $y'_0$. This solution is unique, for let $y(x)$ be any solution of (9, E); then it follows from (10) and (2) that $y(x)$ is bounded, and the inequality

$$|y(x)-y_n(x)| < M\int_{x_0}^{x}|y(s)-y_{n-1}(s)|\,ds$$

is derived in exactly the same way as (14). Taking $M'$ so large that $|y(x)-y_1(x)|<M'$ for $x_0 \leqq x \leqq x_1$ it now follows that

$$|y(x)-y_n(x)| < M' M^{n-1}\frac{(x-x_0)^n}{n!} \to 0 \quad \text{as} \quad n\to\infty.$$

We thus have

THEOREM I. *Under condition* (2), *there exists a unique solution* $y(x)$ *of* (9, E); $y(x)$ *and* $y'(x)$ *are finite for all finite* $x \geqq x_0$, *and are continuous functions of* $x_0$, $y_0$ *and* $y'_0$.

**3. Existence and properties of the solution of the differential equation** (1) **with boundary conditions** (E). That a unique solution of (1, E) exists for values of $x$ sufficiently close to $x_0$ is seen in the following manner. Choose $b>|y_0|$ and let $\bar{y}$ be the solution of (9, E); since $\bar{y}$ is continuous, we have $|\bar{y}|<b$ for $x$ sufficiently close to $x_0$ and consequently, there either exists a smallest $x=\xi(b)$ for which $|\bar{y}|=b$, or $|\bar{y}|<b$ for all $x>x_0$, in which case we write $\xi(b)=\infty$. For $x_0 \leqq x<\xi(b)$, we have $F(x,\bar{y})=f(x,\bar{y})$ by (8), so that $\bar{y}$ is also a solution of (1, E). Conversely, when $y$ is a solution of (1, E), $|y|<b$ for $x_0 \leqq x \leqq x_0+\delta$, where $\delta$ is sufficiently small, and hence $y$ is a solution of (9, E) in this interval, so that $y=\bar{y}$, and consequently the solution $y$ of (1, E) is unique and exists for $x_0 \leqq x<\xi(b)$. When $b$ increases, it is evident from the definition of $\xi(b)$ that the latter also increases, and writing $\xi=\lim_{b\to\infty}\xi(b)$, it follows that the solution $y$ of (1, E) exists and is unique for $x_0 \leqq x<\xi$. When $\xi$ is finite, we must have $\limsup_{x\to\xi}|y|=\infty$, for if $|y|$ were bounded

$\leqq \eta$ for $x_0 \leqq x < \xi$, we would only have to make $b > \eta$ in order to obtain $\xi(b) > b$, which contradicts the definition of $\xi$.

We shall now establish some properties of the solution of (1, E); the proofs apply word for word also to (9, E). Consider a solution $y$ of (1, E) for $x_0 \leqq x \leqq x_1$, and let it have an extreme $y_2$ at $x = x_2$ interior to this interval; then $y_2' = 0$ and (1) gives

$$g(x, y(x))\, y'(x) = \int_{x_2}^{x} f(s, y(s))\, ds.$$

For $x_2 < x < x_2 + \delta$ and $\delta$ sufficiently small, the right hand member has the same sign as $y_2$.* Similarly, the sign is opposite that of $y_2$ for $x_2 - \delta < x < x_2$. Since $g(x, y)$ is positive, it follows that $y_2$ is a minimum when positive, a maximum when negative. Of two consecutive extremes, the maximum must therefore be less than the minimum, and we have

THEOREM II. *Under conditions* (4) *and* (6), *any solution of* (1) *has at most one extreme, which is a minimum when positive, a maximum when negative.*

In the case when $\xi$ is finite, we may now make the statement $\limsup_{x \to \xi} |y| = \infty$ more precise. For $x_3 \leqq x < \xi$ and $x_3$ sufficiently close to $\xi$, $y$ is monotone by Theorem II, so that either $y \to +\infty$ or $y \to -\infty$ as $x \to \xi$. Assume, for instance, that $y \to +\infty$; then we have $y' > 0$ and $y > 0$ for $x_3 \leqq x < \xi$. By (1),

$$g(x, y)\, y' - g(x_3, y_3)\, y_3' = \int_{x_3}^{x} f(s, y(s))\, ds,$$

so that, by (4), $g(x, y)\, y'$ increases monotonely with $x$ toward a limit $\eta' > 0$. Assume $\eta'$ finite; then

$$y - y_3 = \int_{x_3}^{x} y'\, dx < \eta' \int_{x_3}^{x} \frac{dx}{g(x, y)} < \eta' M_3 (x - x_3)$$

by (2) and (6), whence it follows that $y$ is bounded as $x \to \xi$. This contradiction shows that $\eta' = +\infty$, and a similar argument holds when $y \to -\infty$, so that we have

THEOREM III. *Under conditions* (2), (4) *and* (6), *there corresponds to every set of values* $y_0$, $y_0'$, *a* $\xi$ (*which may be* $+\infty$) *such that the solution of* (1, *E*) *exists and is unique for* $x_0 \leqq x < \xi$, *and if* $\xi$ *is finite, then* $y$ *and* $g(x, y)\, y'$ *tend monotonely to infinity with the same sign as* $x \to \xi$.

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* If $y_2 = y_2' = 0$, then $y = 0$ identically, since 0 is a solution of (1) by reason of (4), and the solution $y$ is uniquely determined by the values of $y$ and $y'$ at $x = x_2$. The existence theorem as given above yields this conclusion for $x_1 > x \geqq x_2$ only, but the case $x_0 \leqq x \leqq x_2$ is reduced to the former upon replacing $x$ by $x_0 + x_1 - x$.

We shall finally prove the following comparison theorem:

THEOREM IV. *Let $y$ be a solution of* (1, $E$). *and $z$ a solution of*

$$\frac{d}{dx}\left[g(x,z)\frac{dz}{dx}\right] = f_1(x,z) \tag{15}$$

*with the initial conditions $z = z_0$, $z' = z_0'$ for $x = x_0$, where*

$$z_0 \geqq y_0 \quad \text{and} \quad g(x_0, z_0)\, z_0' \geqq g(x_0, y_0)\, y_0'. \tag{16}$$

*Assuming that $f(x, y)$, $f_1(x, y)$ and $g(x, y)$ have the properties* (3), (4) *and* (6), *and that*

$$f_1(x,z) \geqq f(x,z) \tag{17}$$

*for all $x \geqq x_0$ and all $z$, then*

$$z \geqq y \text{ and } g(x,z)\,z' - g(x,y)\,y' \geqq g(x_0,z_0)\,z_0' - g(x_0,y_0)\,y_0' \tag{18}$$

*for $x_0 \leqq x < \xi$, and the equality sign holds in either place in* (18) *only when the equality signs hold simultaneously in both parts of* (16), *as well as in* (17).

Consider first the case where both equality signs do not hold in (16). When $z_0 > y_0$, then evidently $z > y$ for $x_0 < x < x_0 + \delta$, where $\delta$ is sufficiently small. When $z_0 = y_0$ but $g(x_0, z_0)\, z_0' > g(x_0, y_0)\, y_0'$, the latter inequality reduces to $z_0' > y_0'$, and consequently $z' > y'$ for $x_0 < x < x_0 + \delta$, so that $z > y$ in the same interval. If $z < y$ for some $x > x_0$, there exists a smallest $x_1 > x_0$ such that $z = y$ at $x = x_1$, so that, for $|h|$ sufficiently small, $y(x_1 + h) - z(x_1 + h)$ has the same sign as $h$, and consequently $y' \geqq z'$ at $x = x_1$. From (1) and (15) we obtain

$$\begin{aligned} g(x,z)\,z' - g(x,y)\,y' &= g(x_0,z_0)\,z_0' - g(x_0,y_0)\,y_0' \\ &\quad + \int_{x_0}^{x} [f_1(x,z) - f(x,z)]\,dx, \end{aligned} \tag{19}$$

and for $x = x_1$, where $z_1 = y_1$, we find by (16) and (17)

$$g(x_1, y_1)\,(z_1' - y_1') \geqq \int_{x_0}^{x_1} [f(x,z) - f(x,y)]\,dx.$$

But since $z > y$ for $x_0 < x < x_1$, (4) shows that the integral to the right is positive, so that $z' > y'$ at $x = x_1$, whence contradiction. Hence $z > y$ for $x > x_0$, and the second inequality (18) now follows immediately from (19) and (17). In the remaining case, where the equality sign holds in both places in (16), replace $z_0$ by $z_0 + \varepsilon$, where $\varepsilon > 0$. The corresponding $z$ is then greater than $y$ for $x > x_0$, according to what has just been proved,

and letting $\epsilon \to 0$, it follows that $z \geqq y$ for $x > x_0$. Consequently (19) and (3) give

$$g(x, z)\, z' - g(x, y)\, y' = \int_{x_0}^{x} [f_1(x, z) - f(x, y)]\, dx$$
$$\geqq \int_{x_0}^{x} [f_1(x, z) - f(x, z)]\, dx,$$

and the last expression is positive for $x > x_0$ when the equality sign in (17) does not hold everywhere. At any point $x_1 > x_0$, the initial values corresponding to (16) therefore satisfy the conditions $z_1 \geqq y_1$, and $g(x_1, z_1)\, z_1' > g(x_1, y_1)\, y_1'$, so that we are now in the first case, and consequently $z > y$ for $x > x_1$. Letting $x_1$ approach $x_0$, the theorem is therefore proved.

**4. Existence and uniqueness of the solution of (9) with the boundary conditions (A) to (D).** Let $y = y(x, y_0, y_0')$ be the solution of (9, E).

To prove the existence of a unique solution of (9, A), we observe that, by (2) and (8), $|F(x, y)| < M_1$ for $x_0 \leqq x \leqq x_1$ and any $y$. From (9) it now follows by integration that for $x_0 \leqq x \leqq x_1$

$$g(x_0, y_0)\, y_0' - M_1 (x_1 - x_0) \leqq g(x, y)\, y' \leqq g(x_0, y_0)\, y_0' + M_1 (x_1 - x_0).$$

When $y_0'$ is so large that the expression to the left is positive, (6) shows that $y' > 0$; assuming that $y(x_1, y_0, y_0')$ remains bounded $< \eta$ as $y_0' \to +\infty$, it follows from (7) that $g(x, y) \leqq g(x, \eta)$, and

$$y_1 - y_0 \geqq [g(x_0, y_0)\, y_0' - M_1 (x_1 - x_0)] \int_{x_0}^{x_1} \frac{dx}{g(x, \eta)},$$

so that $y(x_1, y_0, y_0') \to +\infty$ as $y_0' \to \infty$. This contradiction shows that $\limsup y(x_1, y_0, y_0') = +\infty$. Similarly, when $g(x_0, y_0)\, y_0' + M_1 (x_1 - x_0) < 0$, we find $y' < 0$ and $\liminf y(x_1, y_0, y_0') = -\infty$ as $y_0' \to -\infty$. Making $z_0 = y_0$ and $f_1(x, z) = f(x, z)$ in Theorem IV, it follows from the first inequality (18) that $y(x_1, y_0, y_0')$ increases monotonely with $y_0'$. Since $y(x_1, y_0, y_0')$ is moreover a continuous function of $y_0'$, it follows that $y(x_1, y_0, y_0') = y_1$ for one and only one value of $y_0'$, so that (9, A) has a unique solution. In particular, it is seen that any two solutions of (9) cannot intersect at more than one point without being identical.

Passing to (9, B), we observe that by (2), (8) and (10),

$$|y(x_1, y_0, y_0') - y_0| < [|g(x_0, y_0)\, y_0'| + M_1 (x_1 - x_0)] \cdot M_2 (x_1 - x_0),$$

so that, when $g(x_0, y_0)\, y_0'$ is kept constant and $y_0$ varies, $y(x_1, y_0, y_0') \to \pm\infty$ according as $y_0 \to \pm\infty$. By Theorem IV, $y(x_1, y_0, y_0')$ is a monotone increasing

function of $y_0$ when $g(x_0, y_0)\,y_0'$ is kept constant, and moreover, this function is continuous. Hence (9, B) has a solution which is unique.

To solve (9, C), it is sufficient to consider the case $y_0 > 0$, since upon replacing $y$ by $-y$, (9) becomes

$$\frac{d}{dx}\left[g(x, -y)\frac{dy}{dx}\right] = -F(x, -y),$$

and $g(x, -y)$ and $-F(x, -y)$ evidently also have the properties (2) to (7).* Consider the solution $y(x, y_0, y_0')$ of (9, E) with $y_0$ fixed and $y_0'$ variable, and assign $y_0'$ to class I or class II according as $y(x, y_0, y_0')$ is positive for all $x \geqq x_0$, or passes through zero for some $x > x_0$. Neither class is empty, since any $y_0 > 0$ belongs to class I by Theorem II, and any solution of (9, A) with $y_1 < 0$ evidently gives an $y_0'$ in class II. Moreover, Theorem IV with $z_0 = y_0$ and $f_1(x, z) = f(x, z)$ shows that when any $y_0'$ belongs to class I, so does any greater $y_0'$, and when an $y_0'$ is in class II, so is also any smaller $y_0'$. Thus classes I and II define a Dedekind cut $y_0'$, and the corresponding $y = y(x, y_0, y_0')$ is a solution of (9, C). In fact, $y$ cannot have a minimum, for suppose a minimum $y_2 > 0$ to occur at $x = x_2$. Let $x_1 > x_2$, and denote by $y_1$ the corresponding value of $y$; take a $z_1$ such that $y_1 > z_1 > y_2$, and form the solution $z$ of (9, A) passing through the two points $x_0, y_0$ and $x_1, z_1$. Since $z_1 < y_1$, and the solutions $z$ and $y$ do not intersect except at $x_0, y_0$, it follows that $z < y$ for $x_0 < x \leqq x_1$, and consequently $z_0' < y_0'$ (as $z_0' = y_0'$ would make $z$ and $y$ identical). But since $z(x_2) < y_2 < y_1$, $z$ must have a (positive) minimum, and thus $z_0'$ belongs to class I, which is contrary to the definition of $y_0'$. Hence $y$ decreases monotonely as $x$ increases. On the other hand, $y$ cannot become negative, for let $y = y_1 < 0$ at $x = x_1$, choose $z_1$ such that $y_1 < z_1 < 0$, and form the solution $z$ of (9, A) passing through $x_0, y_0$ and $x_1, z_1$. It is than seen as above that $z_0' > y_0'$ must belong to class II, whence contradiction. Thus $y$ decreases monotonely to a limit $y_1 \geqq 0$ as $x \to \infty$, and if $y_1 > 0$, then it follows from (9) and (3) that

$$g(x, y)y' \geqq g(x_0, y_0)y_0' + \int_{x_0}^{x} F(x, y_1)\,dx,$$

so that, by (5), $y' > 0$ for $x$ sufficiently large, and $y$ cannot decrease monotonely. Hence $y_1 = 0$, and $y$ is a solution of (9, C). Let $y$ and $z$ be two solutions of (9, C), and assume $z_0' > y_0'$. Then $z > y$ by Theorem IV, and the second inequality (18) gives $g(x, z)z' - g(x, y)y' > g(x_0, y_0)(z_0' - y_0')$, or since $y' < 0$ and $g(x, y) \leqq g(x, z)$ by (7),

* When $y_0 = 0$, Theorem II shows that $y$ increases or decreases monotonely according as $y_0' \gtrless 0$, so that we must make $y_0' = 0$ to obtain a solution of (9, C), and in this case, $y = 0$ identically.

$$g(x, z)(z'-y') > g(x_0, y_0)(z_0'-y_0'), \tag{20}$$

whence $z'-y'>0$, so that the difference $z-y$, which is positive for $x>x_0$, increases with $x$, while (C) requires that $z\to 0$, $y\to 0$ as $x\to\infty$, and this contradiction proves the uniqueness of the solution of (9, C).

To solve (9, D), we may assume $y_0'<0$ (replacing $y$ by $-y$ when $y_0'>0$).* Consider the solutions $y=y(x, y_0, y_0')$ of (9, E) in which $y_0$ is variable but $g(x_0, y_0)y_0'$ constant, and assign $y_0$ to class I or class II according as $y>0$ for $x\geqq x_0$ or $y$ takes a negative value for some $x>x_0$. Class II is not empty, since every $y_0\leqq 0$ belongs to it. That class I is not empty, is seen by constructing a solution $y$ of (9) which has a minimum $y_2>0$ at $x=x_2$, where $x_2>x_0$ (this is done by replacing $x$ by $x_0+x_2-x$ and solving (9, E) with the initial values $y=y_2$ and $y'=0$). Then (9) gives

$$-g(x_0, y_0)y_0' = \int_{x_0}^{x_2} F(x, y)\,dx. \tag{21}$$

We have $0<F(x, y)<M_1$ for $x_0\leqq x\leqq x_2$, so that by taking $x_2$ sufficiently near $x_0$, the expression to the right in (21) becomes smaller than the given value of $-g(x_0, y_0)y_0'$. On the other hand, $y>y_2$ for $x_0\leqq x<x_2$, so that

$$\int_{x_0}^{x_2} F(x, y)\,dx > \int_{x_0}^{x_2} F(x, y_2)\,dx \to \infty \text{ as } x_2\to\infty$$

by (5), and taking $x_2$ sufficiently large with $y_2$ fixed, the expression to the right in (21) may therefore be made greater than the given value of $-g(x_0, y_0)y_0'$. The expression in question being a continuous function of $x_2$ for $y_2$ fixed, we may therefore choose $x_2$ in such a manner as to obtain a solution of (9) having the given value of $g(x_0, y_0)y_0'$ at $x=x_0$ and possessing a minimum $y_2$ at $x=x_2$. The $y_0$ corresponding to this solution evidently belongs to class I.

Using Theorem IV, it is now shown exactly as in the case (9, C) that classes I and II define a Dedekind cut $y_0$, and that the solution of (9, E) with the initial values $y_0$ and $y_0'$ (the latter being calculated from the given values of $y_0$ and $g(x_0, y_0)y_0'$) decreases monotonely toward zero as $x$ increases indefinitely. We have thus a solution of (9, D). Let $z$ be another, and assume $z_0>y_0$; Theorem IV leads, exactly as before, to the inequality (20), and the uniqueness proof is completed just as in the case (9, C).

5. **Existence and uniqueness of the solution of (1) with the boundary conditions (A) to (D).** The passage from (9) to (1) depends

* When $y_0'=0$, Theorem II shows that $y$ increases or decreases monotonely with $x$ according as $y_0\gtrless 0$, so that to obtain a solution of (9, D) we must make $y_0=0$, and $y=0$ identically.

on the circumstance that when any solution of (9) is such that $|y| \leqq b$, then it is also a solution of (1), since $F(x, y) = f(x, y)$ for $|y| \leqq b$. Conversely, if $b$ is greater than the maximum absolute value of a solution $y$ of (1) in a given interval, then this $y$ also satisfies (9) in the interval considered.

A solution of (1, A) or of (9, A) has either an extreme of the same sign as $y_0$, but less in absolute value, or else it varies monotonely from $y_0$ to $y_1$ (Theorem II). It is thus evident that $|y| \leqq |y_0| + |y_1|$ for $x_0 \leqq x \leqq x_1$, and making $b = |y_0| + |y_1|$, the solutions of (1, A) and (9, A) will coincide, and the existence and uniqueness of the solution of (1, A) is thus established. Making $y_1 = 0$, the same argument applies word for word to establish the existence and uniqueness of the solution of (1, C).

In the case of (1, B) or (1, D), assume $b > |y_1|$ but otherwise arbitrary, and with this value of $b$, form the solution $z$ of (9, B) or (9, D) respectively. Let $z_0$ be the value of this $z$ at $x = x_0$, make $b_1 = |z_0| + |y_1|$ and define $F_1(x, y)$ by (8) where $b$ is now replaced by $b_1$. Then we obviously have

$$F_1(x, y) \geqq F(x, y)$$

(the equality sign holding when $|y| \leqq b$). Form the solution $y$ of the equation

$$\frac{d}{dx}\left[g(x, y)\frac{dy}{dx}\right] = F_1(x, y) \tag{22}$$

with boundary conditions (B) or (D) respectively, and let $y_0$ be the value of this $y$ for $x = x_0$.

By eventually changing the sign of $y$, we may assume that $y_0 > 0$. Assume $y_0 > z_0$; then Theorem IV and the argument leading to (20) show that $y' - z' > 0$, so that $y - z$ increases monotonely, and $y$ cannot satisfy $y(x) = y_1$, or $y \to 0$ as $x \to \infty$, respectively. From this contradiction it follows that $y_0 \leqq z_0$, so that, by Theorem II, $|y| \leqq z_0 + |y_1|$ or $|y| \leqq b_1$ everywhere. Hence the solutions of (1) and (22) with boundary conditions (B) or (D) coincide, which proves the existence and uniqueness of the solution of (1, B) and (1, D).

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# A NEW DEFINITION OF ALMOST PERIODIC FUNCTIONS.

By Norbert Wiener.*

According to the definition of Harald Bohr,† a function $f(x)$ (possibly complex valued) of the real argument $x(-\infty < x < \infty)$ is said to have the translation number $\tau_\varepsilon$ pertaining to $\varepsilon$ if

$$|f(x+\tau)-f(x)| \leqq \varepsilon$$

for all real $x$. If for every positive $\varepsilon$ there is a finite number $L_\varepsilon$ such that every interval $(A, A+L_\varepsilon)$ contains at least one translation number $\tau_\varepsilon$ pertaining to $\varepsilon$, the function $f(x)$ is said to be *almost periodic.* Bohr has shown‡ that the class of almost periodic functions is identical with the class of functions to which we may approximate to any desired degree of accuracy by a trigonometrical polynomial

$$\sum_1^N A_n e^{i\lambda_n x}.$$

Bochner§ has given an equivalent but distinct definition of the class of almost periodic functions. The present paper gives yet another definition of this class.

We shall call a real function $\varphi(x)$ *regular* if

$$\lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} \varphi(x)\varphi(x-\tau)\,dx$$

exists for every real $x$. A complex function is regular if its real and pure imaginary parts are regular taken separately. Our theorem is: *if $f(x)$ is regular, and if for every $\varepsilon$ there is a regular real function $\varphi_\varepsilon(x)$ such that*

$$g_\varepsilon(x) = \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} f(x-\xi)\,\varphi_\varepsilon(\xi)\,d\xi$$

*exists for every $\varepsilon$ while*

$$|f(x)-g_\varepsilon(x)| < \varepsilon, \quad (-\infty < x < \infty)$$

* Received January 24, 1927.

† H. Bohr, Acta Math., vol. 45 (1924), pp. 29–127.

‡ H. Bohr, Acta Math., vol. 46 (1925), pp. 101–214.

§ S. Bochner, Math. Annalen, vol. 96 (1926), p. 143.

*then $f(x)$ is almost periodic. Conversely, if $f(x)$ is almost periodic, it is regular, and there exists for every $\varepsilon$ a function $\varphi_\varepsilon(x)$ with the property just indicated.*

We may manifestly confine ourselves to the case where $f(x)$ is real. Our hypotheses concerning $f$ and $\varphi_\varepsilon$ demand the regularity of $f(x)+\varphi_\varepsilon(x)$ and $f(x)-\varphi_\varepsilon(x)$; hence $g_\varepsilon(\tau)$ is the difference of two expressions of the form

$$\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}\psi(x)\,\psi(x-\tau)\,dx.$$

The author has shown* that every such function is the sum of an almost periodic function and a function which is $o(1)$ at $\pm\infty$. Hence we may write

$$g_\varepsilon(x) = g_\varepsilon^{(1)}(x)+g_\varepsilon^{(2)}(x),$$

where $g_\varepsilon^{(1)}(x)$ is almost periodic and $g_\varepsilon^{(2)}(x) = o(1)$ at $\pm\infty$.

We have

$$g_\varepsilon(x)-g_\eta(x) = g_\varepsilon^{(1)}(x)-g_\eta^{(1)}(x)+(g_\varepsilon^{(2)}(x)-g_\eta^{(2)}(x)).$$

The pair of terms in the parenthesis is $o(1)$; the preceding pair is almost periodic. An almost periodic function comes arbitrarily near to its maximum value in any half line; hence

$$\max|g_\varepsilon^{(1)}(x)-g_\eta^{(2)}(x)| \leqq \max|g_\varepsilon(x)-g_\eta(x)|.$$

Hence

$$\lim_{\varepsilon\to 0} g_\varepsilon^{(1)}(x) = f^{(1)}(x)$$

exists as a uniform limit and is almost periodic.† Similarly

$$\lim_{\varepsilon\to 0} g_\varepsilon^{(2)}(x) = f^{(2)}(x)$$

exists as a uniform limit and is $o(1)$ at $\pm\infty$. Clearly

$$f(x) = f^{(1)}(x)+f^{(2)}(x).$$

Let us consider

$$\frac{1}{2\tau}\int_{-\tau}^{\tau} f^{(2)}(x-\xi)\,\varphi_\varepsilon(\xi)\,d\xi.$$

By the Schwarz inequality this cannot exceed

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* In a paper soon forthcoming in the Proc. Lond. Math. Soc. Cf. N. Wiener, Math. Zeit., vol. 24, p. 581 (Theorem IX).

† Cf. the first cited paper of Harald Bohr.

$$\frac{1}{2\tau}\left[\int_{-\tau}^{\tau}|f^2(x-\xi)|^2\,d\xi\int_{-\tau}^{\tau}|\varphi_\epsilon(\xi)|^2\,d\xi\right]^{\frac{1}{2}},$$

and, since

$$\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}|f^2(x-\xi)|^2\,d\xi=0,\qquad \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}|\varphi_\epsilon(\xi)|^2\,d\xi<\infty,$$

we have

$$\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}f^{(2)}(x-\xi)\varphi_\epsilon(\xi)\,d\xi=0,$$

and

$$g_\epsilon(x)=\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}f^{(1)}(x-\xi)\varphi_\epsilon(\xi)\,d\xi.$$

Since $f^{(1)}(x)$ is almost periodic, $g_\epsilon(x)$ must be almost periodic, $g_\epsilon^{(2)}(x)$ is identically 0, $f^{(2)}(x)$ vanishes, and $f(x)$ is almost periodic.

On the other hand, if $f(x)$ is almost periodic, it is clearly regular. By a theorem of Bochner,* $\varphi(x)=\max_\tau|f(\tau+x)-f(\tau)|$ is almost periodic. Let $F_\xi(u)$ be a continuous non-negative function, defined for positive and zero $u$, not identically zero, and vanishing for $u>\epsilon/2$. Let

$$\varphi_\epsilon(u)=F_\epsilon(\varphi(x))/\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}F_\epsilon(\varphi(x))\,dx.$$

The author has proved elsewhere† the existence and almost periodic character of $\varphi_\epsilon(u)$ as well as the fact that

$$\left|\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}f(x-\xi)\varphi_\epsilon(\xi)\,d\xi-f(x)\right|<\epsilon.$$

The converse part of our theorem is thus proved.

* Bochner, loc. cit., p. 136.

† In the forthcoming paper in the Proc. Lond. Math. Soc.

# A THEORY OF INTEGERS, IN RELATION TO THE ITERATION OF ALGEBRAIC FUNCTIONS.*

By O. E. Glenn.

The content and methods of this paper are presented in the form of original algorithms on the number system. The object is to develop a theory of indices in which $\Delta$ of the fundamental relation,

$$\Delta^r \equiv \Delta \pmod{n}, \tag{1}$$

instead of being an integer, is a functional operation upon an integer and $\Delta^r$ is its $r$th iteration.

1. **Introduction.** The well-known congruence,

$$a^{\varphi(n)+1} \equiv a \pmod{n}, \tag{2}$$

where $n$ is any prime or composite number and $\varphi(n)$ is Euler's function, may be considered as a form of iteration of a function $\xi \equiv a\eta \pmod{n}$, $a$ being an integer prime to $n$, such that this function is recurrent upon being iterated $\varphi(n)+1$ times.

Suppose that,

$$F(\xi) = \sum_{j=0}^{m} (a_{j0}\,\xi^{m-j} + a_{j1}\,\xi^{m-j-1}\eta + \cdots + a_{jm-j}\,\eta^{m-j}), \tag{3}$$

is a polynomial in two indeterminates $\xi, \eta$ with integral coefficients $a_{jk}$ and assume that the congruence,

$$F(\xi) \equiv 0 \pmod{n}, \tag{4}$$

has at least one integral solution, $\xi \equiv \beta \pmod{n}$, for each residue $\alpha$ of $n$ which we assign† to $\eta$.

Assign $\alpha$ to $\eta$ and suppose that it determines $\xi \equiv \beta_1 \pmod{n}$; assign $\beta_1$ to $\eta$ and assume that $\xi \equiv \beta_2$ is determined; assign $\beta_2$ to $\eta$ and continue the process.

2. **Postulate for single-valued determinations.** The function $F(\xi)$ is chosen to satisfy an arbitrary set of specific conditions which render the successive determinations of the numbers $\beta_i$ single-valued.

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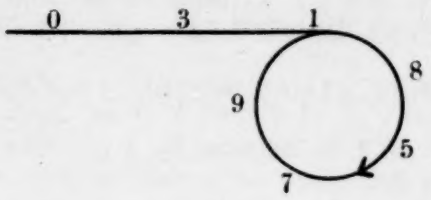
* Presented to the American Mathematical Society, December 1924; February 1927; received January 21, 1927.

† Dickson, Annals of Math., 11 (1896–7), 73, § 11.

These conditions may be any set which defines a single branch of $F(\xi)$. For example a branch of successive determinations, modulo 10, of numbers $\beta$ from the function,

$$F(\xi) = \xi^2 + 3\xi\eta + \eta^2 + 1 \equiv 0 \pmod{10},$$

can be arranged on a line and a circle as follows:




If we consider only the branch represented by this figure and choose for $\alpha$ any integer of the set 1, 5, 7, 8, 9, $F(\xi)$ is periodic with the period 5, this fact being illustrated by the arrangement of the five numbers $\beta$ on the circle in the order of their determination. In comparison, the successive powers of $a$ in (2) form a circular branch of $\varphi(n)$ residues.

LEMMA. *Every branch of $F(\xi)$ terminates in a periodic segment (circle).*

This follows from the fact that only $n$ different residues of $n$, at most, can be determined by the successive solution for $\xi$. When a residue is repeated among the numbers $\beta$ it closes the circular branch. The number of residues ($w$) on the circle is the period. If $w = 1$ the branch will be called unitary.

LEMMA. *An alternating function has n branches of period* 1, *and, except in cases $F(\xi) \equiv a(\xi - \eta)^m$, branches of period* 2. *A symmetric function has branches of period* 2.

Alternating functions are here defined as polynomials (3) written, more explicitly, $F = F(\xi, \eta)$, which satisfy the conditions ($\alpha, \beta$ arbitrary residues),

$$\text{(a)} \qquad F(\alpha, \alpha) \equiv 0, \qquad F(\alpha, \beta) \equiv -F(\beta, \alpha) \pmod{n}.$$

A function is symmetric if

$$\text{(b)} \qquad F(\alpha, \beta) \equiv F(\beta, \alpha) \pmod{n}.$$

If $F$ is alternating $F(\alpha, \alpha) \equiv 0$. Hence one value of $\beta_1$ (cf. § 1), is $\alpha$, and, for this value, $F$ is of period unity. Also $F(\beta_1, \alpha) \equiv -F(\alpha, \beta_1) \equiv 0 \pmod{n}$, from (a); hence $\alpha$ determines $\beta_1$ and $\beta_1$ determines $\alpha$. In this case $w = 2$.

When $F$ is symmetric, condition (b) gives $w = 2$.

The exception stated for the alternating functions represents the instances for which no number $\alpha$, substituted for $\eta$, determines a residue different from $\alpha$.

**3. Formulas for iterated algebraic functions.** If we represent $\beta_1$, which, if considered analytically, would be an algebraic function of $\alpha$, as an explicit function of $\alpha$, symbolically, we may write,

$$\beta_1 \equiv \Delta(\alpha) \pmod{n},$$

then,

$$\beta_2 \equiv \Delta(\Delta(\alpha)) \pmod{n}, \quad (= \Delta^2(\alpha)).$$

Let the $r$th iteration of $\Delta$ be indicated by $\Delta^r(\alpha)$, then

$$\beta_r \equiv \Delta^r(\alpha) \pmod{n}. \tag{5}$$

Upon any branch of $F(\xi)$, if $\alpha$ is any residue and $r$ is the least number for which,

$$\Delta^r(\alpha) \equiv \Delta(\alpha) \pmod{n}, \tag{6}$$

$\Delta$ will be said to appertain to the exponent (period) $w = r-1$. Note that $\alpha$ is the last integer on the straight line or any residue on the circle in the graphical representation of the branch. Thus, in the example of paragraph 2,

$$\Delta^6(5) \equiv \Delta(5) \pmod{10}.$$

The non-unitary branches of a function $F$, of least period, will be called characteristic and, if their period is $n$, $F$ will be said to be primitive. Any branch of $F$, of the maximum period $n$, will be called primitive. The function,

$$F(\xi) \equiv 3\xi + 7\eta + 1 \pmod{10},$$

has a primitive branch.

**4. Determination of all functions which have unitary branches.** In case of a branch of period unity, if $\alpha$ is substituted for $\eta$ in $F(\xi, \eta)$, one value of $\xi$ determined is $\alpha$. Thus,

$$F(\alpha, \alpha) = \sum_{j=0}^{m} (a_{j0} + a_{j1} + \cdots + a_{jm-j})\, \alpha^{m-j} \equiv 0 \pmod{n}. \tag{7}$$

In fact there immediately follows:

THEOREM 1. *A necessary and sufficient condition for a branch of period* $w = 1$ *of* $F(\xi, \eta)$, *is* $F(\alpha, \alpha) \equiv 0 \pmod{n}$ *for some residue* $\alpha$ *of* $n$.

This condition may be satisfied identically or as a congruence of order $\leqq m$ having $\alpha$ as one of its solutions. If it vanishes identically i. e. if,

$$(8) \qquad a_{j0}+a_{j1}+\cdots+a_{jm-j} \equiv 0 \pmod{n}, \quad (j=0,\cdots,m),$$

every residue of $n$ is an unitary branch. Otherwise the solutions of the congruence (7) constitute the complete set of branches of period unity. If (7) cannot be satisfied no unitary branch exists.

5. **The functions whose characteristic branches are of period 2.** A necessary and sufficient condition for a binary branch of $F(\xi,\eta)$ is the co-existence of the following congruences for some pair of residues $\alpha,\beta$:

$$(9) \quad F(\beta,\alpha)=\sum_{j=0}^{m}\sum_{k=0}^{m-j} a_{jk}\,\beta^{m-j-k}\,\alpha^{k} \equiv 0, \qquad F(\alpha,\beta)\equiv 0 \pmod{n}.$$

It was proved by Stieltjes* that a necessary and sufficient condition in order that $q$ linear homogeneous congruences, (mod. $n$), in $q$ variables at least one of which is prime to $n$, should be consistent, is that the determinant of the system should vanish, modulo $n$. In accordance with this theory we may eliminate $\alpha$ from (9) by Sylvester's method since one of the quantities eliminated is then unity. The form of this eliminant, for example, when $m=2$, is,

$$C=\begin{vmatrix} a_{02} & a_{01}\beta+a_{11} & a_{00}\beta^2+a_{10}\beta+a_{20} & 0 \\ 0 & a_{02} & a_{01}\beta+a_{11} & a_{00}\beta^2+a_{10}\beta+a_{20} \\ a_{00} & a_{01}\beta+a_{10} & a_{02}\beta^2+a_{11}\beta+a_{20} & 0 \\ 0 & a_{00} & a_{01}\beta+a_{10} & a_{02}\beta^2+a_{11}\beta+a_{20} \end{vmatrix} \equiv 0 \pmod{n}.$$

It is a congruence of order 4, in general of order $m^2$, in the residue $\beta$. Suppose, in the general case, that the resultant, when expanded, is,

$$(10) \qquad C=C_0\beta^{m^2}+C_1\beta^{m^2-1}+\cdots+C_{m^2} \equiv 0 \pmod{n}.$$

THEOREM 2. *A necessary and sufficient condition for a branch of period $w=2$ of $F(\xi,\eta)$ is that $C\equiv 0$ (mod. $n$) for some residue $\beta$ of $n$.*

For, if there exists a $\beta$ for which $C\equiv 0$, then, since $C$ is an eliminant, there exists a pair $(\alpha,\beta)$ to satisfy (9). We recognize the following cases: If $C\equiv 0$ is not satisfied by $C_i\equiv 0 \pmod{n}$, i. e., identically, it is possible to choose an $\alpha$ and then determine a $\beta$ by solving $C\equiv 0 \pmod{n}$, such that $(\alpha,\beta)$ is a branch of period 2. If $C$ is identically zero (mod. $n$) it is possible to choose an $\alpha$ such that every other residue $\beta$ forms, with $\alpha$, a branch $(\alpha,\beta)$ of period 2. If (10) cannot be satisfied there exists no branch of period 2.

* Ann. Fac. Sci. Toulouse, 4 (1890), 1–103. Œuvres Complètes, 317.

An example is,

$$F(\xi, \eta) = 2\xi^3 + 3\xi\eta + \eta^3 + 7\xi + 2\eta + 1 \equiv 0 \pmod{10}. \tag{11}$$

The congruences (7), (10) are as follows:

$$6\alpha^2 + 9\alpha + 1 \equiv 0 \pmod{10},$$
$$6\beta^2 + 4\beta + 6 \equiv 0 \pmod{10}.$$

They have no solutions. Hence there are no branches of $F$ of periods $w = 1$ or $w = 2$.

6. **Functions with characteristic periods** $w \geqq 3$. A necessary and sufficient condition in order that $s$ residues $(\alpha_1, \alpha_2, \cdots, \alpha_s)$, in this order, may form a branch of $F(\xi, \eta)$, of period $s$, is the simultaneity of the following $s$ congruences:

$$F(\alpha_2, \alpha_1) = \sum_{j=0}^{m} \sum_{k=0}^{m-j} a_{jk}\, \alpha_2^{m-j-k}\, \alpha_1^k \equiv 0, \qquad F(\alpha_3, \alpha_2) \equiv 0, \cdots, \tag{12}$$
$$F(\alpha_s, \alpha_{s-1}) \equiv 0, \qquad F(\alpha_1, \alpha_s) \equiv 0 \pmod{n}.$$

For brevity we write $\chi(\alpha_i, \alpha_j) = \chi_{ij}$ and let $R_{hj}$ represent the resultant of two forms $\chi_{ij}$, $\psi_{hi}$, obtained by eliminating $\alpha_i$. We then construct the following series of resultants:

$$R_{s2} = R(F_{1s}, F_{21}), \quad R_{s3} = R(R_{s2}, F_{32}), \cdots, R_{st} = R(R_{st-1}, F_{tt-1}), \cdots, \tag{13}$$
$$R_{ss-1} = R(R_{ss-2}, F_{s-1s-2}), \qquad R_{ss} = R(R_{ss-1}, F_{ss-1}).$$

The last resultant in this chain, $R_{ss}$, is a polynomial, of order $\leqq n$ in the residue $\alpha_s$ alone, whose coefficients are rational and integral in the numbers $a_{jk}$. The form $R_{ss}$ may vanish identically or as a congruence. If no resultant $R_{s2}, \cdots, R_{ss-1}$ vanishes identically (mod. $n$), the vanishing of $R_{ss}$ evidently compels the existence of at least one whole set $(\alpha_1, \alpha_2, \cdots, \alpha_s)$ for which the congruences (12) are consistent. We refer to this as the normal problem. If some of the resultants (13), $t < s$, vanish identically we may still have a period $w = s$ under more complicated conditions. For instance, if $s = 3$ and $R_{32} \equiv 0$ identically, $F_{13}$ and $F_{21}$ are consistent for some residue $\alpha_1$, with $\alpha_2$ and $\alpha_3$ arbitrary. Thus any pair of residues $\alpha_2$, $\alpha_3$ which satisfy $F(\alpha_3, \alpha_2) \equiv 0 \pmod{n}$ form, with the properly chosen $\alpha_1$, a period $w = 3$. This case and the normal problem exhaust all possibilities in connection with the period 3.

THEOREM 3. *In the normal problem a necessary and sufficient condition for a branch of period $w = s$ of $F(\xi, \eta)$ is $R_{ss} \equiv 0 \pmod{n}$, for some residue $\alpha_s$ of $n$. Referring to characteristic periods in the normal problem*

*the exponent $r$ to which the explicit function $\Delta$, obtained from $F(\xi, \eta)$, belongs is the subscript of the first resultant which can vanish in the series, $R_{22}, R_{33}, \cdots, R_{tt}, \cdots$.*

In the case of the example (11) we find,

$$F(\alpha_3, \alpha_2) \equiv \alpha_2^2 + (3\alpha_3 + 2)\alpha_2 + (2\alpha_3^2 + 7\alpha_3 + 1) \pmod{10},$$
$$R_{23} \equiv \alpha_3^4 + (\alpha_2 + 8)\alpha_3^3 + (8\alpha_2^2 + 3)\alpha_3^2 + (4\alpha_2^3 + 4\alpha_2^2 + 3\alpha_2 + 8)\alpha_3 + (6\alpha_2^4 + 8\alpha_2^3 + 6\alpha_2 + 6) \pmod{10},$$
$$R_{33} \equiv 5\alpha_3^7 + 5\alpha_3^6 + 6\alpha_3^5 + 4\alpha_3^4 + 7\alpha_3^3 + 3\alpha_3^2 + 2\alpha_3 + 4 \pmod{10}.$$

The congruence $R_{33} \equiv 0 \pmod{10}$ has no solutions. Hence the function, $\xi \equiv \Delta(\eta)$, defined by,

$$2\xi^2 + 3\xi\eta + \eta^2 + 7\xi + 2\eta + 1 \equiv 0 \pmod{10},$$

has no branch of period 3.

In the case of the linear function,

$$F(\xi, \eta) = a\xi + b\eta + c,$$

the series of resultants which determines the exponent to which $\Delta$ appertains is the following in which the congruence (7) is included under the notation $R_{11}$:

$$(14) \quad R_{tt} = (a^t + (-1)^{t-1} b^t)\alpha_t + \frac{a^t + (-1)^{t-1} b^t}{a + b} c, \qquad (t = 1, 2, \cdots, n)$$

The following conclusion is typical of much more complicated results for the forms $F(\xi, \eta)$ of higher order:

LEMMA. *A set of necessary and sufficient conditions that a linear function $F(\xi, \eta)$ should have a primitive characteristic branch is, $R_{tt} \not\equiv 0 \pmod{n}$ for $t < n$, and that both of the following expressions should be congruent to zero:*

$$S_n = a^n + (-1)^{n-1} b^n, \qquad S_n/(a + b).$$

The condition for a branch of a quadratic $F(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$, of period 2, is found to be,

$$I = \beta^4(a^4 + c^4 + 2ab^2c - 2a^2c^2 - a^2b^2 - b^2c^2) \equiv 0 \pmod{n}.$$

Such expressions have interpretations as invariants. Here $I$ is invariantive under the linear transformations of the quadratic which leave unaltered the property of possessing a binary period.

## Applications in two-dimensional systems of residues.

**7. Plexus of binary forms.** The author has sought applications of the foregoing new theory in a field in which new elementary results might be expected, that concerned with systems of residues (mod. $n$) which are conveniently regarded as two-dimensional.*

Assume any infinite series of integers,

$$S_1(\alpha):\ \alpha_1, \alpha_2, \cdots, \alpha_r, \cdots, \infty,$$

arranged, for convenience, in a vertical column and taken to be the initial numbers in a (triangular) table $Q_1$ in which any element is a sum $\sigma = x\gamma + y\delta$, where $\delta$ is the element immediately preceding $\sigma$ in the row and $\gamma$ is the one next beneath $\delta$ in the column of the latter. The numbers $x, y$ may be arbitrary but are here regarded as integers. It is easy to show that the element in the $v$th row and $u$th column is the binary form,

$$(15)\qquad \begin{aligned} f &= \binom{\alpha_{v-u+1}, \alpha_{v-u+2}, \cdots, \alpha_v}{u, v}(x, y)^{u-1} \\ &= \alpha_{v-u+1}\, x^{u-1} + \binom{u-1}{1}\alpha_{v-u+2}\, x^{u-2}\, y + \cdots + \alpha_v\, y^{u-1}. \end{aligned}$$

The special cases needed for a proof by induction are readily obtained. Assume that the elements of the $v$th row and column number $u-1$, and of the $(v-1)$th row and column number $u-1$, are the following respectively:

$$\delta = \sum_{r=0}^{u-2}\binom{u-2}{r}\alpha_{v-u+2+r}\, x^{u-2-r}\, y^r, \qquad \gamma = \sum_{r=0}^{u-2}\binom{u-2}{r}\alpha_{v-u+1+r}\, x^{u-2-r}\, y^r.$$

Then

$$x\gamma + y\delta = \sum_{s=-1}^{u-2}\left[\binom{u-2}{s+1} + \binom{u-2}{s}\right]\alpha_{v-u+2+s}\, x^{u-2-s}\, y^{s+1} = f,$$

since,

$$\binom{u-2}{s+1} + \binom{u-2}{s} = \binom{u-1}{s+1}.$$

A table $Q$ of residues obtained by reducing the elements of $Q_1$, modulo $n$, contains numerous noteworthy configurations† formed by special arrangements of zeros and other residues. If $S_1(\alpha)$ is periodic (mod. $n$) we shall refer to the strip of residues between two lines which intersect the end elements of a period, and which are parallel to the lower edge of the plexus, as a strip $T$.

* Carmichael, Quart. Journ. Math., 48 (1920), 343; Kempner, Trans. Amer. Math. Soc., 22 (1921), 240; Glenn, Ann. of Math., 25 (1923), 57.

† Cf. example (and figure), Ann. of Math., loc. cit., 67.

THEOREM 4. *Suppose $n = p$ to be an odd prime. Let $Q$ be the plexus which is residual to a $Q_1$ emanating from any sequence $S(\alpha)$ which is periodic (mod. $p$) of period $\mu = p^a$. Then $Q$ is a mosaic of period parallelograms of identical content, each bounded by the boundary lines of a strip $T$ and two consecutive recurrences of $S(\alpha)$ as columns of $Q$. The width of the period parallelogram is $p^a \lambda$ where $\lambda$ is a definite divisor of $p-1$.*

We have

$$\alpha_1 \equiv \alpha_{\mu+1} \equiv \alpha_{2\mu+1} \equiv \cdots \equiv \alpha_{q\mu+1} \equiv \cdots \pmod{p}.$$

It* was shown by E. Lucas that if $p$ is a prime, and

$$M = p m_1 + \eta, \qquad N = p n_1 + \theta, \qquad \eta < p, \qquad \theta < p,$$

then,

$$\binom{M}{N} \equiv \binom{m_1}{n_1}\binom{\eta}{\theta} \pmod{p}.$$

Hence,

$$\binom{m p^a}{j p^a} \equiv \binom{m p^{a-1}}{j p^{a-1}} \equiv \binom{m}{j} \pmod{p}, \qquad (j \leqq m < p),$$

and there results the following formula:

$$(\varrho + \sigma)^{m p^a} \equiv \sum_{j=0}^{m} \binom{m}{j} \varrho^{(m-j)p^a} \sigma^{j p^a} \pmod{p}, \tag{16}$$

where $\varrho$, $\sigma$ are algebraic indeterminates. All numbers,

$$\binom{m p^a}{i}, \qquad (i = 0, \cdots, m p^a),$$

excepting those of the form given by $i = j p^a$ are divisible by $p$. It now follows that the elements of column number $1 + m p^a$ of $Q_1$, which are included between the two boundary lines of a strip $T$ are congruent, respectively, to

$$\alpha_s (x^{p^a} + y^{p^a})^m, \qquad (s = 1, \cdots, p^a), \text{ (cf. (15))}.$$

For all sets $(x, y)$ which satisfy the condition,

$$g = x^{p^a} + y^{p^a} \not\equiv 0 \pmod{p},$$

therefore, it results from Fermat's theorem that $S(\alpha)$ is repeated when $m = \lambda$ where $\lambda$ is the exponent to which $g$ appertains (mod. $p$) and is a divisor of $p-1$. Thus $S(\alpha)$ is repeated as column number,

$$p^a (p-1)/d + 1, \qquad (p - 1 = d\lambda).$$

This proves the theorem.

* Bull. Soc. Math. France, 6 (1877-8), 52.

8. **Periodic functions and the drift-line phenomenon.** Let $F_w$, $(w = 1, \cdots, \nu)$, be a set of $\nu$ functions (3) having respective circular periods $W_w$. If we choose an integer $\alpha_w$ from each period of $W_w$ residues and proceed to develop the $\nu$ periods of the set $F_w$ simultaneously by the method of paragraph 1 it is evident that the period of the set will be,

$$W = L.C.M.(W_1, W_2, \cdots, W_\nu).$$

That is, the $\nu$ numbers $\alpha_1, \cdots, \alpha_\nu$ will be repeated in this order for the first time at the end of a period of $W$ residues. Use is to be made of this principle in explaining certain periodicities in the plexus $Q$.

When tables $Q$ are actually calculated from given periodic initial series $S(\alpha)$ a very persistent property is that they are scored by parallel and equally spaced lines of zero residues. These are finite lines running across the tables oblique to the boundary lines of the strips $T$. In another paper I referred to these as drift-lines.*

DEFINITION. A line drawn athwart the plexus $Q_1$ through $\alpha_i$ and the element number $i-1$ of the third column is said to be a drift-line if all elements of $Q_1$ which it intersects are congruent to zero (mod. $n$).

These elements are,

$$(17)\quad \alpha_{i-s}x^{2s} + \binom{2s}{1}\alpha_{i-s+1}x^{2s-1}y + \cdots + \binom{2s}{2s}\alpha_{i+s}y^{2s}, \quad (s = 0, \cdots, i-1).$$

We shall require that they all vanish (mod. $n$) in a way that is independent of the values of the binomial coefficients. The following congruences are assumed:

$$g(i, s, t) = \alpha_{i-s+t}x^{2(s-t)} + \alpha_{i+s-t}y^{2(s-t)} \equiv 0 \text{ (mod. } n), \qquad (t = 0, \cdots, s).$$

Thus $\alpha_i \equiv 0$. Since $g(i, j+k, t) = g(i, j, t-k)$, the vanishing of any element of a drift-line causes the next element to vanish provided the sum of the two end terms of the latter is congruent to zero. Hence the $i$ elements of the drift-line are, essentially, $\alpha_i \equiv 0$, and, in order, the following simultaneous forms:

$$(18)\qquad \alpha_{i-s}x^{2s} + \alpha_{i+s}y^{2s} \equiv 0 \text{ (mod. } n), \qquad (s = 1, \cdots, i-1).$$

PROPOSITION. *To determine a series $S(\alpha)$ and a pair $(x, y)$ such that the corresponding $Q$ has a drift-line passing through each element $\alpha_{qe}$ $(\equiv 0)$, $(i = qe,\ q = 1, 2, \cdots, \infty)$.*

* Ann. of Math., loc. cit., 61.

The set which is to be simultaneous is,

$$(19)\qquad \begin{aligned} \alpha_{qe-1}^{s}\, x^{2s} + (-1)^{s+1}\, \alpha_{qe+1}^{s}\, y^{2s} &\equiv 0 \,(\mathrm{mod.}\, n), \\ \alpha_{qe-s}\, x^{2s} + \alpha_{qe+s}\, y^{2s} &\equiv 0 \,(\mathrm{mod.}\, n), \qquad (s = 2, \cdots, qe-1). \end{aligned}$$

We assign $\alpha_{qe-1}$ arbitrarily and the same for all values of $q$ determining for $\alpha_{e+1}, \alpha_{2e+1}, \cdots, \alpha_{qe+1}, \cdots$, any periodic sequence for which,

$$\Delta^{(q)} = \begin{vmatrix} (-1)^{e}\, \alpha_{e-1}^{e-1} & \alpha_{(q+1)e+1}^{e-1} \\ \alpha_{qe+1} & \alpha_{e-1} \end{vmatrix} \equiv 0 \,(\mathrm{mod.}\, n).$$

To make the periodicity depend upon simple considerations we may substitute, in $\Delta^{(q)}$,

$$C\alpha_{qe+1} \equiv \alpha_{(q+1)e+1} \,(\mathrm{mod.}\, n),$$

assuming $\alpha_{qe+1}$ prime to $n$ and that the number,

$$-\alpha_{e-1}\, \alpha_{qe+1}^{\varphi(n)-1}$$

appertains to the exponent $e$, $\varphi(n)$ being Euler's function. Then

$$(20)\qquad C^{e-1} \equiv 1 \,(\mathrm{mod.}\, n),$$

$$\alpha_1 \equiv \alpha_1, \quad \alpha_{e+1} = C\alpha_1, \quad \alpha_{2e+1} \equiv C^2\alpha_1, \cdots, \alpha_{qe+1} \equiv C^q\, \alpha_1 \,(\mathrm{mod.}\, n).$$

The periodicity of this sequence is determined by (20).

The determinants of the matrix of (19) are,

$$\Delta_s^{(q)} \equiv \begin{vmatrix} \alpha_{qe-1}^{s} & (-1)^{s+1}\, \alpha_{qe+1}^{s} \\ \alpha_{qe-s} & \alpha_{qe+s} \end{vmatrix} \equiv 0 \ (\mathrm{mod.}\, n), \qquad (s = 1, \cdots, qe-1).$$

The $\alpha_{qe\pm1}$ being known we use $\alpha_{qe+s}$, $\alpha_{qe-s}$ in the rôles of $\xi, \eta$, $(\xi_s, \eta_s)$, as in (3). Since

$$(q+1)e-(e-s) = qe+s,$$

if we change $q$ into $q+1$ and $s$ into $e-s$ in $\Delta_s^{(q)}$ the element $\alpha_{qe+s}$ is automatically transferred from the position it occupies in $\Delta_s^{(q)}$ to the lower left-hand corner of $\Delta_{e-s}^{(q+1)}$. This transfer will be made analogous to the successive interchanges of the rôles of $\xi$ and $\eta$ in the determination of the numbers $\beta$ in paragraph 1. In fact the elements $\alpha_{qe\pm s}$, for which (17) is a drift-line, are determined from periodic branches of the quadratic form,

$$(21)\qquad \Delta_s = \Delta_s^{(q+r)}\,\Delta_{e-s}^{(q+r+1)} = \begin{vmatrix} \lambda^s & -\mu^s \\ \eta_s & \xi_s \end{vmatrix} \begin{vmatrix} \lambda^{e-s} & -\mu^{e-s} \\ \eta_s & \xi_s \end{vmatrix}$$
$$= \lambda_e\,\xi_s^2 + (\lambda^s\,\mu^{e-s} + \lambda^{e-s}\,\mu^s)\,\xi_s\,\eta_s + \mu^e\,\eta_s^2,$$

where,

$$\lambda = \alpha_{qe-1}, \quad -\mu = \alpha_{qe+1}.$$

The period of $\Delta_s$ is developed as follows: Assign $\eta_s$ ($= \alpha_{e-2}$ first in practice), and determine $\xi_s$ from the first factor of $\Delta_s$ made congruent to zero (mod. $n$). Substitute the residue found for a new $\eta_s$. Solve for $\xi_s$ from the second factor of $\Delta_s$. Substitute this for a new $\eta_s$ in the first factor and continue the process. The period found can be used for the series,

$$\alpha_{e+s},\ \alpha_{3e-s},\ \alpha_{3e+s},\ \alpha_{5e-s},\ \cdots,\ \alpha_{(2t-1)e+s},\ \alpha_{(2t+1)e-s},\ \alpha_{(2t+1)e+s},\ \cdots.$$

Let $s$ vary from 2 to $e$—2 in this series and assign the elements $\alpha_1, \cdots, \alpha_{e-2}$, arbitrarily.

The series $S(\alpha)$ is now determined. To find $(x, y)$ we assign $y$ and solve

$$-\alpha_{i-1}^s\,\alpha_{i+s}\,y^{2s} \equiv (-1)^s\,\alpha_{i+1}^s\,\alpha_{i-s}\,y^{2s} \equiv \alpha_{i-1}^s\,\alpha_{i-s}\,x^{2s} \pmod{n},$$

for $x$ ($i = qe$).

A series $S(\alpha)$ determined by these methods is the following, of period 20 ($n = 10$):

$S(\alpha)$: 7, 4, 1, 3, 0, 3, 9, 4, 3, 0, 7, 6, 1, 3, 0, 3, 9, 6, 3, 0, ....

In the figure is exhibited the portion of the corresponding $Q$, which extends from the 35th to the 44th rows as determined from $S(\alpha)$ with $x = 3$, $y = 1$. Parts of two drift-lines are shown.

3659779527362772770988162740223677777709386 1
1386042952125055013301525924071700000138565
4568869293715505740978918142982100007459 73
7740611988880554923677491169383300099786 2
0922506300045035171709540518550900306762
3101529900875142282132325299053701927 32
6378990706362160283437319907568176152 3
9877079129123584257382660796741235897
3301944794774126142588089497997741 76
0976501451099785161104675564060992 3

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# ON SYSTEMS OF TOTAL DIFFERENTIAL EQUATIONS.*

By Joseph Miller Thomas.†

The object of this paper is to coördinate the results of a number of investigators by showing that the existence theorems obtained by them for particular systems of partial differential equations are special cases of a general theorem.‡ This is accomplished in two steps: first, by reducing to a problem of elimination the problem of stating a similar theorem for *any* system which is composed of partial differential equations and a number of finite relations among the unknowns and whose general solution involves no arbitrary functions (§§ 3, 4); and second, by actually indicating in two cases the process of elimination which is to be employed, and which is essentially that used by the original investigators (§§ 5, 6). A further feature of the present treatment of the particular systems is that we shorten the calculations by not concerning ourselves with the explicit forms of the second members, see, for example, equations (5.1). The results so obtained are entirely satisfactory for the general theory of the systems and have the advantage of being applicable to a greater variety of problems.§

The results of the present paper are also presented in such a form that they leave the type of differentiation undetermined to a certain extent; in particular, it is indifferent whether we interpret the process of differentiation as partial or covariant in the application of the theorems to cases where the unknowns are the components of tensors (§ 2). Other types of invariant differentiation are also admissible.

The paper closes with a short discussion of more general systems whose solutions involve arbitrary functions. The projective problem of the homogeneous first integral for the geometry of paths is shown to be of this type.

1. **The general theorem.** By a *system of total differential equations* is meant one which can be solved for all the first (partial) derivatives of all the unknown functions $Z^\alpha$ in terms of $Z^\alpha$ and the independent variables $x^i$. We shall find it convenient to denote such a system by

* Presented to the American Mathematical Society, February 26, 1927; received January 29, 1927.

† National Research Fellow in Mathematics.

‡ O. Veblen and J. M. Thomas, Annals of Math., (2), vol. 27 (1926), p. 290. Professor J. W. Alexander has remarked several times that the theorems in question could probably be obtained from some general theorem.

§ J. A. Schouten, Trans. Amer. Math. Soc., 27 (1925), pp. 441–473.

$$\frac{\partial Z}{\partial x} = f(Z, x),$$

the $f$ denoting that the right hand members depend on $Z^\alpha$ and $x^i$, and upon those variables alone.

A necessary and sufficient condition for the compatibility of such systems is known.* We give the theorem below for convenience of reference. For the moment, we suppose $D_x = \frac{\partial}{\partial x}$, $D_i = \frac{\partial}{\partial x^i}$. It will be seen below that other meanings can be given this symbol without rendering the theorem invalid.

THEOREM A. *Consider the system of total differential equations*

$$D_x Z = f(Z, x). \tag{1.1}$$

*Let the finite relations among the $Z$'s and $x$'s obtained by expressing the relations between the second derivatives $D_i(D_j Z)$ and $D_j(D_i Z)$ and eliminating $D_x Z$ by means of* (1.1) *be denoted by $F_1 = 0$. Let $F_2 = 0$ denote the system obtained by differentiating $F_1 = 0$ with respect to each independent variable in turn and eliminating $D_x Z$ by* (1.1). *Let the sequence*

$$F_\lambda = 0 \qquad (\lambda = 1, 2, \cdots) \tag{1.2}$$

*be obtaining by forming $F_\lambda = 0$ from $F_{\lambda-1} = 0$ in the way that $F_2 = 0$ was formed from $F_1 = 0$. A necessary and sufficient condition for the compatibility of the system* (1.1) *is the existence of a positive integer $N$ such that $F_1 = 0$, $F_2 = 0, \cdots, F_N = 0$ are consistent in the unknowns $Z$ and that any solution of them satisfies $F_{N+1} = 0$.*

2. **Systems of total covariant differential equations.** A particular type of the systems considered above occurs when the unknowns $Z$ are the components of a number of tensors $T$. The form in which such a system is usually given is

$$T_{,i} = f(T, x), \tag{2.1}$$

where the left hand member denotes the covariant derivative of $T$. Since we have identically for any tensor

$$T_{,i} = \frac{\partial T}{\partial x^i} + L(T, x), \tag{2.2}$$

$L$ denoting linear homogeneous dependence on $T$, we see that system (2.1) is of the type (1.1), and we can apply theorem $A$ to the equations obtained by combining (2.1) and (2.2).

* O. Veblen and J. M. Thomas, loc. cit.

It is possible, however, to apply Theorem A directly to equations (2.1). Instead of forming $F_1 = 0$ by means of the identities

$$\frac{\partial^2 T}{\partial x^i \partial x^j} - \frac{\partial^2 T}{\partial x^j \partial x^i} = 0, \tag{2.3}$$

we can use the conditions of integrability for covariant derivatives*

$$T_{i,j} - T_{j,i} = L(T, x); \tag{2.4}$$

and in forming $F_\lambda = 0$ from $F_{\lambda-1} = 0$, we can use covariant differentiation. The sequence $F_\lambda = 0$ obtained in this manner is algebraically equivalent to that obtained by partial differentiation: for the identities (2.4) are proved by adding certain expressions, linear and homogeneous in the components of $T$, to both sides of (2.3); and the identities (2.2) show that for a tensor $U$ the systems

$$U = 0, \qquad \frac{\partial U}{\partial x^i} = 0,$$

and

$$U = 0, \qquad U_{,i} = 0,$$

are algebraically equivalent.

THEOREM B. *If the unknowns in a system of total differential equations are the components of certain tensors, the process of differentiation in Theorem* A *can be regarded as partial or covariant at will.*

3. **Mixed systems.** Consider the system

$$D_x Z = f(Z, x), \qquad f(Z, x) = 0. \tag{3.1}$$

An examination of the proof of Theorem A† shows that the following theorem is capable of proof in the same way.

THEOREM C. *A necessary and sufficient condition for the existence of solutions of the mixed system composed of* (1.1) *and a number of finite relations,*

$$f(Z, x) = 0, \tag{3.2}$$

*is furnished by Theorem* A *provided we include* (3.2) *among* $F_1 = 0$ *before proceeding to the formation of the sequence* $F_\lambda = 0$.

4. **Systems reducible to mixed systems.** Suppose that instead of being given all the derivatives of all the unknown functions, we are only given relations between them insufficient in number to determine them.

* G. Ricci and T. Levi-Cività, Math. Annalen, vol. 54 (1901), pp. 125–201.

† O. Veblen and J. M. Thomas, loc. cit.

It may happen that by differentiating some or all of the given relations a sufficient number of times we can find a system of total differential equations, in a greater number of unknown functions, which is satisfied by any solution of the original system. Those of the original equations which do not figure in the new system are to be looked upon as finite relations among the unknowns of the new system. Theorem C is then to be applied to give the desired condition.

5. **The linear first integral.** The equations of condition are of the type

$$D_i h_j + D_j h_i = f(h, x). \tag{5.1}$$

The number of relations is $n(n+1)/2$, and the number of derivatives $n^2$. Hence we differentiate. In order to keep our system of the first order, we introduce the notation

$$D_i h_j = h_{ji}.$$

We find

$$D_k h_{ij} + D_k h_{ji} = f(h_\alpha, h_{\beta\gamma}, x).$$

Permuting the indices in these last relations cyclically and forming the combination

$$(ijk) - (jki) + (kij),$$

we find $D_k h_{ij}$. Hence theorem C is to be applied to the mixed system

$$\begin{cases} D_j h_i = h_{ij}, \quad D_k h_{ij} = f(h_\alpha, h_{\beta\gamma}, x), \\ \qquad\qquad h_{ij} + h_{ji} = f(h_\alpha, x). \end{cases} \tag{5.2}$$

To find the original theorem of Veblen and T. Y. Thomas,* we put $D_j h_i = h_{i,j}$ = covariant derivative, and remark that from the linear, homogeneous character of the right hand members of (5.2) and (2.4) it follows in this case that the equations of condition are linear and homogeneous in the unknowns $h_i$ and $h_{i,j}$.

6. **The quadratic first integral.** The equations of condition are of the type

$$D_i g_{jk} + D_j g_{ki} + D_k g_{ij} = f(g_{\alpha\beta}, x) \qquad (g_{ij} = g_{ji}). \tag{6.1}$$

Two differentiations give

$$D_p g_{ijkl} + D_p g_{jkil} + D_p g_{kijl} = f(g_{\alpha\beta}, g_{\gamma\delta\varepsilon}, g_{\xi\eta\theta\varkappa}, x). \tag{6.2}$$

It is clear that the left-hand side of (6.2) is transformed into itself by all six permutations of $i, j, k$ and also by the interchange of $l$ and $p$. For

* O. Veblen and T. Y. Thomas, Trans. Amer. Math. Soc., vol. 25 (1923), p. 599.

a given set of indices $i, j, k, l, p$ there are accordingly at most $120 \div 12 = 10$ distinct equations (6.2). By making the substitution $(klp)$ in (6.2), and then permuting cyclically all five indices in the result and in (6.2) itself, we get ten linear equations in the ten unknown derivatives. This system can be solved by evaluating two determinants of the tenth order, most of whose elements are zero. It is also easy to determine multipliers so as to eliminate all but one unknown. A linear combination serving the purpose is

$$\begin{aligned} &2(ijklp) + 2(klpij) + 2(pijkl) + 2(ijlpk) - (jklpi) \\ &\quad - (lpijk) - (jkpil) - (klijp) - (lpjki) - (piklj), \end{aligned}$$

where $(ijklp)$ denotes the left-hand member of (6.2).

Hence our mixed system is

$$(6.3)\quad \begin{cases} D_k g_{ij} = g_{ijk},\ D_l g_{ijk} = g_{ijkl},\ D_p g_{ijkl} = f(g_{\alpha\beta},\ g_{\gamma\delta\epsilon},\ g_{\xi\eta\theta\kappa},\ x), \\ g_{ijk} + g_{jki} + g_{kij} = f(g_{\alpha\beta},\ x), \end{cases}$$

and Theorem C gives the desired result.

To obtain the theorem for the quadratic first integral in tensor form, we put $D_k g_{ij} = g_{ij,k}$, and as before we can see that the equations of condition will be linear and homogeneous in the unknown functions $g_{ij}$, $g_{ij,k}$, $g_{ij,k,l}$.*

**7. Geometries of paths projectively equivalent to a Riemann geometry.** As a further illustration consider conditions of the type

$$(7.1)\qquad D_i g_{jk} + P_{ijk} = f(g_{\alpha\beta},\ x),$$

where the $P$'s are known functions of $g_{\alpha\beta}$, $D_\alpha g_{\beta\gamma}$ and $x$ which satisfy identically the relations

$$P_{ijk} + P_{jki} + P_{kij} = 0.$$

By permuting the indices in (7.1) cyclically and adding we find that $g_{ij}$ must satisfy the same type of system as in § 6. Hence we adjoin equations (7.1) to $F_1 = 0$ as formed in the preceding section and apply Theorem C.

* O. Veblen and T. Y. Thomas, loc. cit., pp. 599–608. The method of elimination employed here is not exactly the same as theirs. It seems to us simpler to employ the covariant derivative, $g_{ij,k,l}$, rather than the second extention $g_{ij,kl}$, because the covariant derivatives of higher order are more easily calculated for the former. If it is desired, however, to obtain their theorem exactly as given, we can parallel their elimination on pp. 599–603, and then apply Theorem C. The two sequences $F_\lambda = 0$ can also be transformed one into the other by means of the identities

$$g_{ij,k,l} = g_{ij,kl} - g_{\alpha j} A^\alpha_{kl} - g_{i\alpha} A^\alpha_{jkl}.$$

The problem of determining whether a given geometry of paths is projectively equivalent to a Riemann geometry requires the discussion of the equations*

$$2(n-1)\,g_{ij,k}+g^{\alpha\beta}(g_{\alpha\beta,i}\,g_{jk}+g_{\alpha\beta,j}\,g_{ik}-2\,g_{\alpha\beta,k}\,g_{ij})=0, \tag{7.2}$$

where $g_{ij}$ is related to the fundamental tensor $g'_{ij}$ of the Riemann space by the formulas

$$g'_{ij}=e^{-4\varphi}\,g_{ij}, \tag{7.3}$$

and where the Christoffel symbols $\Gamma'$ are given by

$$\Gamma'^{i}_{jk}=\Gamma^{i}_{jk}-\delta^{i}_{j}\,\varphi_k-\delta^{i}_{k}\,\varphi_j, \qquad \varphi_j=\frac{\partial\varphi}{\partial x^j}. \tag{7.4}$$

The system (7.2) is a particular case of (7.1). The conditions $F_\lambda=0$ are algebraic† in $g_{ij}$ and $g_{ij,k}$.

If we suppose the $\Gamma$'s are the Christoffel symbols of the second kind for a given Riemann space, then the solution of system (7.2) determines all Riemann spaces capable of geodesic representation on the given one. The above discussion, therefore, shows that *it can be determined by algebraic operations whether a given Riemann space admits others in geodesic correspondence.*

One set of finite relations which any solution of (7.2) must satisfy is easily obtained. Denote the curvature tensors by $B^{i}_{jkl}$ and $B'^{i}_{jkl}$. Then we have

$$B'^{\alpha}_{jkl}=B^{\alpha}_{jkl}+\delta^{\alpha}_{l}\,\Phi_{jk}-\delta^{\alpha}_{k}\,\Phi_{jl}, \tag{7.5}$$

where

$$\Phi_{jk}=\varphi_{j,k}+\varphi_j\,\varphi_k.$$

Multiplying equations (7.5) by

$$g'_{i\alpha}\,g'^{jk}=g_{i\alpha}\,g^{jk},$$

we get

$$g'^{\alpha\beta}\,B'_{k\alpha\beta l}=g^{\alpha\beta}\,B_{k\alpha\beta l}+g_{kl}\,\Phi-\Phi_{kl},$$

where

$$B_{k\alpha\beta l}=g_{k\lambda}\,B^{\lambda}_{\alpha\beta l}, \qquad \Phi=g^{\lambda\mu}\,\Phi_{\lambda\mu}.$$

We therefore have

$$g'^{\alpha\beta}(B'_{k\alpha\beta l}-B'_{l\alpha\beta k})=g^{\alpha\beta}(B_{k\alpha\beta l}-B_{l\alpha\beta k}). \tag{7.6}$$

For the Riemann tensor we have

$$B'_{l\alpha\beta k}=B'_{\beta k l\alpha}=B'_{k\beta\alpha l}.$$

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* Cf. J. M. Thomas, Proc. Nat. Acad. Sci., vol. 12 (1926), p. 123.

† Cf. the solution which was previously given by me, Proceedings, loc. cit. The test for compatibility given there requires the integration of a complete system.

Hence the desired relations are

$$g^{\alpha\beta}(B_{k\alpha\beta l} - B_{l\alpha\beta k}) = 0.$$

The expression on either side of (7.6) is an invariant of the transformation (7.3) combined with (7.4). All the integrability conditions of (7.2) must of course have this property of invariance.

8. **More general systems.** A system of total differential equations is characterized by its having a finite number of arbitrary constants in its general solution. To get an example of a system which does not fall in this category, consider the projective problem of the homogeneous linear first integral for the geometry of paths. The condition is the existence of a pair of vectors $h_i$ and $\psi_i$ which satisfy the relations*

$$h_{i,j} + h_{j,i} + h_i \psi_j + h_j \psi_i = 0. \tag{8.1}$$

Suppose we have a solution $h$, $\psi$ of these equations. Let $\varphi$ be any scalar function. Then the functions

$$h'_i = e^{-2\varphi} h_i, \qquad \psi'_i = \psi_i + 2\frac{\partial \varphi}{\partial x^i}$$

also satisfy (8.1). Hence the methods of the present paper are insufficient for the treatment of this problem.

The above remarks apply with equal force to the projective problem of the homogeneous first integral of any degree.

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* J. M. Thomas, loc. cit., p. 119.

PARIS, FRANCE.

# ON POSITIVE SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS.*

By Lloyd L. Dines.

Among the most elegant chapters of algebra is undoubtedly that one which treats the solution of a system of linear equations

$$\sum_{s=1}^{n} a_{rs} x_s = 0, \qquad (r = 1, 2, \cdots, m) \tag{1}$$

from the point of view of the matrix $\| a_{rs} \|$ of coefficients. This theory furnishes criteria for the existence of solutions (other than the obvious identically zero solution), methods for determining and expressing the degree of arbitrariness of the general solution, and an algorithm for determining the general solution.

One point upon which the classical theory does not touch is the character of the solutions $(x_1, x_2, \cdots, x_n)$ with respect to the *signs* of the components $x_s$. This question, conceivably important in certain types of applications, has found its way into several recent papers.†

The purpose of the present note is to present a simple algorithm for determining whether a given system of equations (1) with real numerical coefficients admits a solution $(x_1, x_2, \cdots, x_n)$ in which every component is *positive*, and for determining such a solution if one exists. Such a solution will be called a *positive solution*.

**1. The case of a single equation.** If we consider a single equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0,$$

in which at least one coefficient is different from zero, we note that a necessary condition for the existence of a positive solution is that some coefficients be positive and some negative.

Assuming then that both positive and negative coefficients are present, let us distinguish the two classes

those which are positive: $a_i$, $\qquad (i = i_1, i_2, \cdots, i_P)$;
those which are negative: $a_j$, $\qquad (j = j_1, j_2, \cdots, j_N)$.

* Received February 11, 1927.

† See papers by Carver, Annals of Math., vol. 23, p. 212; and the author, Annals of Math., vol. 27, p. 57, and vol. 28, p. 41. Reference should also be made to an abstract of a paper presented by Professor Gummer to the Mathematical Association in September 1926. This abstract appears in the Amer. Math. Monthly of December 1926, and came to the author's notice after the present paper was written.

Then the given equation can be written in the form

$$\sum_i a_i x_i = -\sum a_j x_j,$$

and a positive solution* is obtained by taking

$$x_i = -\sum_j a_j, \qquad (i = i_1, i_2, \cdots, i_P);$$

$$x_j = \sum_i a_i, \qquad (j = j_1, j_2, \cdots, j_N).$$

2. **The reduction from $m$ equations to $m-1$ equations.** Let us consider next the system of $m$ equations (1). If any one of the equations contains positive but no negative coefficients or negative but no positive coefficients, a positive solution is obviously impossible. Let us then assume that each equation contains both positive and negative coefficients.

We separate the coefficients of the *first* equation into two classes

those which are positive or zero: $a_{1i}$,

those which are negative: $a_{1j}$,

and write the first equation in the form

$$\sum_i a_{1i} x_i = -\sum_j a_{1j} x_j. \tag{2}$$

Next we write the remaining $m-1$ equations in the form

$$\sum_j a_{rj} x_j = -\sum_i a_{ri} x_i, \qquad (r = 2, 3, \cdots, m), \tag{3}$$

it being understood that the ranges of the summation indices are those determined by the *first* equation, that is $i$ runs over all values for which $a_{1i}$ is positive or zero, and $j$ over those for which $a_{1j}$ is negative.

We now form $m-1$ new equations, by multiplying the two sides of each equation of (3) by the corresponding sides of (2) as follows:

$$\sum_i a_{1i} x_i \cdot \sum_j a_{rj} x_j = \sum_j a_{1j} x_j \cdot \sum_i a_{ri} x_i, \quad (r = 2, 3, \cdots, m). \tag{4}$$

This system may be written in the form

$$\sum_{ij} a^{(1)}_{rij} x_{ij} = 0, \qquad (r = 2, 3, \cdots, m). \tag{5}$$

where

$$a^{(1)}_{rij} = a_{1i} a_{rj} - a_{1j} a_{ri}, \qquad x_{ij} = x_i x_j.$$

* This is of course only one particularly simple positive solution. No attempt is made to give expression to a *general* positive solution.

The double summation index $ij$ runs over all the combinations of the values of $i$ and $j$.

The system (5) is thus seen to be a system of $m-1$ linear homogeneous equations in the new unknowns $x_{ij}$. The significant relation between the systems (1) and (5) is expressed in the following

THEOREM: *To every positive solution of the system* (1) *there corresponds a positive solution of the system* (5), *and conversely.*

The first half of the theorem is almost obvious. For if $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m)$ is a positive solution of (1), then this set of values also satisfies (2) and (3) and (4), and hence the set of values $x_{ij} = \bar{x}_i \bar{x}_j$ satisfies (5).

To prove the converse, suppose that the system (5) admits a positive solution $(\bar{x}_{ij})$. Then upon substitution of the explicit values of the coefficients $a_{rij}^{(1)}$, this fact may be written in the form

$$\sum_{ij} (a_{1i}\, a_{rj} - a_{1j}\, a_{ri})\, \bar{x}_{ij} = 0, \qquad (r = 2, 3, \cdots, m).$$

This in turn may be written in the form

$$\sum_i a_{ri} \Big[-\sum_j a_{1j} \bar{x}_{ij}\Big] + \sum_j a_{rj} \Big[\sum_i a_{1i} \bar{x}_{ij}\Big] = 0, \qquad (r = 2, 3, \cdots, m),$$

and this exactly expresses the fact that the last $m-1$ equations of the given system (1) admit the solution

$$x_i = -\sum_j a_{1j} \bar{x}_{ij}, \qquad x_j = \sum_i a_{1i} \bar{x}_{ij}. \tag{6}$$

It may be verified directly that this solution also satisfies the first equation of (1). Furthermore it is obviously a *positive* solution. The proof of the theorem is thus complete.

**3. The algorithm for a system of $m$ equations.** The developments of the preceding section afford a definite procedure for finding a positive solution of a system of $m$ equations when such a solution exists. The system is first reduced to a similar system of $m-1$ equations which is equivalent in the sense indicated in our theorem. The reduction may then be repeated and an equivalent system of $m-2$ equations obtained. By $m-1$ successive reductions we finally obtain a single equation which can be examined for positive solutions by the method of § 1. From a positive solution of this single equation, a positive solution of the preceding *pair* of equations can be determined by the procedure indicated in § 2, from this a positive solution of the preceding *triple* of equations, and so on backward through the $m-1$ successive reduced systems, until finally a positive solution of the given system of $m$ equations is obtained.

The following two points are to be noted:

(*a*) If at any stage of the reduction any one equation of the system under temporary consideration contains some positive coefficients but no negative coefficients, or *vice versa*, then a positive solution of that system is impossible, and hence the same is true of the original system.

(*b*) That step of the indicated process which consists in obtaining a positive solution of a system of equations from a known positive solution of the system derived from it by reduction, is a matter of a simple formal substitution. For example, a positive solution $(x_s)$ of the system (1) may be obtained from a known positive solution $(\bar{x}_{ij})$ of the reduced system (5) as follows. For each $i$, the component $x_i$ is equal to the right side of (2) with $x_j$ replaced by $\bar{x}_{ij}$. For each $j$, the component $x_j$ is equal to the left side of (2) with $x_i$ replaced by $\bar{x}_{ij}$.

§ 4. **An example.** As an illustration let us seek a positive solution of the system of equations

$$\begin{aligned} &x_1 - x_2 + x_3 + x_4 + x_5 - x_6 - x_7 - x_8 = 0 \\ &x_1 + 2x_2 - x_3 + 2x_4 - x_5 + 2x_6 - x_7 - 2x_8 = 0 \\ &x_1 + 2x_2 + 3x_3 - 4x_4 - 5x_5 - 6x_6 + 7x_7 - 8x_8 = 0. \end{aligned} \tag{7}$$

As explained in § 2, we write the first equation in the form

$$x_1 + x_3 + x_4 + x_5 = x_2 + x_6 + x_7 + x_8. \tag{8}$$

Noting that $i = 1, 3, 4, 5$, and $j = 2, 6, 7, 8$, we must then write the remaining two equations

$$\begin{aligned} 2x_2 + 2x_6 - x_7 - 2x_8 &= -x_1 + x_3 - 2x_4 + x_5 \\ 2x_2 - 6x_6 + 7x_7 - 8x_8 &= -x_1 - 3x_3 + 4x_4 + 5x_5. \end{aligned}$$

We now form the two equations corresponding to (5), and write them with terms arranged according to the $i-j$ classification dictated by the signs in the first of them, as follows:

$$\begin{aligned} &3x_{12} + 3x_{16} + x_{23} + 4x_{24} + x_{25} + x_{36} + 4x_{46} + x_{47} + x_{56} \\ &\quad = x_{18} + 2x_{37} + 3x_{38} + 2x_{57} + 3x_{58}, \\ &-7x_{18} + 10x_{37} - 5x_{38} + 2x_{57} - 13x_{58} \\ &\quad = -3x_{12} + 5x_{16} - 5x_{23} + 2x_{24} + 3x_{25} + 3x_{36} \\ &\qquad - 8_{17} + 12x_{48} + 10x_{46} - 3x_{47} + 11x_{56}. \end{aligned} \tag{6}$$

As regards positive solutions, this pair of equations is equivalent to the given system (7) in the sense of the theorem of § 2. Multiplying the corresponding sides of this pair of equations together, we obtain a *single* equivalent equation. If we denote a product $x_{ij}\,x_{kl}$ by the contracted form $x_{ijkl}$ (or interchangeably by $x_{klij}$), this single equation may be written

$$
\begin{aligned}
&36\,x_{1237}+12\,x_{1257}+20\,x_{1687}+20\,x_{2387}+10\,x_{2338}+12\,x_{2357}+\;2\,x_{2358}\\
&+36\,x_{2437}+\;4\,x_{2457}+\;4\,x_{2587}+\;4\,x_{3687}+20\,x_{3746}+16\,x_{3747}+\;4\,x_{3847}\\
&+\;8\,x_{4757}+\;8\,x_{1718}+16\,x_{1737}+24\,x_{1738}+16\,x_{1757}+24\,x_{1758}=18\,x_{1218}\\
&+\;6\,x_{1238}+30\,x_{1258}+26\,x_{1618}+30\,x_{1638}+\;4\,x_{1657}+54\,x_{1658}+\;2\,x_{1828}\\
&+30\,x_{1824}+26\,x_{2438}+58\,x_{2458}+10\,x_{1825}+14\,x_{2538}+\;4\,x_{2557}+22\,x_{2558}\\
&+10\,x_{1836}+14\,x_{3638}+\;4\,x_{3657}+22\,x_{3658}+38\,x_{1846}+50\,x_{3846}+12\,x_{4657}\\
&+82\,x_{4658}+\;4\,x_{1847}+\;4\,x_{4758}+18\,x_{1856}+12\,x_{3756}+38\,x_{3856}+20\,x_{5657}\\
&+46\,x_{5658}+12\,x_{1848}+24\,x_{3748}+36\,x_{3848}+24\,x_{4857}+36\,x_{4858}.
\end{aligned}
\tag{10}
$$

According to the principle established in § 1, a positive solution of this equation may be obtained by assigning to each unknown in the left member of the equation a value equal to the sum of all the coefficients in the right member, and to each unknown in the right member a value equal to the sum of all the coefficients in the left member. On account of the homogeneity of the equation the magnitude of these values may be reduced by the removal of any common factor. Applying this principle, we find the following solution of equation (10)

each unknown in the left member = 105,
each unknown in the right member = 37.

Now by the principle of § 3(b), a positive solution of the pair (9) may be obtained by formal substitution in the right and left sides of the first of these equations. For example, $x_{12}$ is found by substitution in the right side, and $x_{18}$ by substitution in the left side, as follows

$$
\begin{aligned}
x_{12} &= x_{1218}+2\,x_{1237}+3\,x_{1238}+2\,x_{1257}+3\,x_{1258} = 4\times105+7\times37 = 679,\\
x_{18} &= 3\,x_{1812}+3\,x_{1816}+\;x_{1823}+4\,x_{1824}+\;x_{1825}\\
&\quad+\;x_{1836}+4\,x_{1846}+\;x_{1847}+\;x_{1856} = 19\times37 = 703.
\end{aligned}
$$

A complete positive solution of the system (9) found by this method is

$$
\begin{aligned}
x_{12} &= 679, & x_{16} &= 543, & x_{17} &= 1155, & x_{18} &= 703,\\
x_{23} &= 1087, & x_{24} &= 679, & x_{25} &= 543, & x_{36} &= 543,\\
x_{37} &= 1927, & x_{38} &= 839, & x_{46} &= 543, & x_{47} &= 883,\\
x_{48} &= 407, & x_{56} &= 407, & x_{57} &= 1315, & x_{58} &= 771.
\end{aligned}
$$

Again by the principle of § 3 (*b*), a positive solution of the system (7) may be obtained by formal solution in the two members of equation (8). For example, for the component $x_1$ we have

$$x_1 = x_{12} + x_{16} + x_{17} + x_{18} = 3080.$$

On account of the homogeneity, the magnitude of the components may again be reduced by the removal of a common factor. The resulting positive solution of the given system is

$$x_1 = 770, \quad x_2 = 747, \quad x_3 = 1099, \quad x_4 = 628,$$
$$x_5 = 759, \quad x_6 = 509, \quad x_7 = 1320, \quad x_8 = 680.$$

5. **Systems of non-homogeneous equations.** Non-homogeneous equations can be made homogeneous by the introduction of a new unknown. The system

$$\sum_{s=1}^{n} a_{rs} x_s + b_r = 0 \qquad (r = 1, 2, \cdots, m), \tag{11}$$

is equivalent to the system

$$\sum_{s=1}^{n} a_{rs} x_s + b_r x_{n+1} = 0 \qquad (r = 1, 2, \cdots, m), \tag{12}$$

if $x_{n+1} = 1$.

To every positive solution of (11) corresponds a positive solution of (12), and conversely. The system (12) may be investigated for positive solutions by the method of the preceding sections.

6. **Solutions which are somewhere positive or nowhere negative.** For some purposes solutions $(x_1, x_2, \cdots, x_n)$ of a system (1), in which at least some of the components $x_s$ are positive while none are negative, are desired. Such a solution may be called *M-positive.** Clearly it is possible for a system of equations to admit such a solution while admitting no positive solution.†

A slight alteration of the algorithm above described suffices to determine such solutions when they exist. When seeking *positive* solutions, the procedure is blocked if an equation is encountered in which there are not both positive and negative coefficients. When seeking *M*-positive solutions,

* For the significance of this term, see the author's paper *On sets of functions of a general variable,* Trans. Amer. Math. Soc., vol. 29 (1927), p. 463.

† For example, the system

$$x_1 - x_2 + x_3 = 0, \qquad x_3 = 0,$$

admits no positive solution, but admits the *M*-positive solution $(x_1, x_2, x_3) = (1, 1, 0)$.

if an equation appears which contains coefficients of one sign but none of the opposite sign, all unknowns with non-vanishing coefficients in such equation must be put equal to zero. The solution of the remaining equations for positive values for the remaining unknowns may then be proceeded with as in the algorithm. If the process of reduction, supplemented by this additional feature (of assigning zero values to certain unknowns) can be continued until a single equation is reached, and if this single equation contains both positive and negative coefficients, then it admits a positive solution, and the original system admits an $M$-positive solution which may be found by the reversal process indicated in the algorithm.

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# ON COMPLETELY SIGNED SETS OF FUNCTIONS.*

By Lloyd L. Dines.

Relative to a set $\{f_i\}$ of linearly independent functions

$$f_1(x),\ f_2(x),\ \cdots,\ f_m(x),$$

continuous on an interval $X$

$$a \leqq x \leqq b,$$

I have recently† proved the *equivalence* of two additional properties which the set $\{f_i\}$ may or may not have. These properties are:

($a$) Every linear combination of the functions changes sign on $X$.

($b$) There exists a positive continuous function $\pi(x)$, orthogonal to each function of the set on the interval $X$.

While this equivalence theorem is satisfactory from the point of view of elegance, it may be noted that the two properties ($a$) and ($b$) are both (and apparently equally) difficult to recognize in a given set of functions. The object of the present note is to define a third property ($c$) relative to such sets of functions, which is more easily recognized than either ($a$) or ($b$), and which when present is *sufficient* (even without the assumption of linear independence) to ensure both ($a$) and ($b$). It will be seen also that if the set $\{f_i\}$ has the property ($c$), it is usually a comparatively simple matter to determine the positive function $\pi(x)$ whose existence is asserted in ($b$), a determination which does not follow readily from a knowledge of ($a$) alone.

1. The definition of the new property depends upon the notion of a *completely signed matrix*, which I have defined in a previous note.‡ Let us consider a matrix

$$\| a_{ij} \| \qquad (i = 1, 2, \cdots, m) \quad (j = 1, 2, \cdots, 2^m),$$

of $m$ rows and $2^m$ columns in which all elements are real and different from zero, and let us denote by $\sigma_{ij}$ the sign of the element $a_{ij}$. If the $2^m$ columns of signs

* Received February 11, 1927.

† *A theorem on orthogonal functions with an application to integral inequalities*, Trans. Amer. Math. Soc.

‡ Ann. of Math., vol. 28, p. 42. The roles played by *rows* and *columns* have for convenience been interchanged in the present paper.

$$\sigma_{1j}, \sigma_{2j}, \cdots, \sigma_{mj} \qquad (j = 1, 2, \cdots, 2^m)$$

include all the $2^m$ possible arrangements of sign (+ and —), then the matrix $\| a_{ij} \|$ is said to be a *completely signed matrix.*

The property (c) referred to above may now be defined as follows:

*The set of functions $\{f_i\}$ is said to be a completely signed set on the interval $X$ if there exist $2^m$ points $x_j$ $(j = 1, 2, \cdots, 2^m)$ on the interval $X$, such that the matrix*

$$\| f_i(x_j) \| \qquad (i = 1, 2, \cdots, m) \quad (j = 1, 2, \cdots, 2^m)$$

*is a completely signed matrix.**

2. THEOREM: *If the set $\{f_i\}$ is a completely signed set on the interval $X$, then it has the properties (a) and (b).*

In view of the equivalence of (*a*) and (*b*), if will be sufficient to prove one of them. We will prove (*b*), inasmuch as the proof exhibits a method for actually determining the positive function $\pi(x)$ whose existence is asserted.

By hypothesis there is a set of points $x_j (j = 1, 2, \cdots, 2^m)$ on $X$ for which the matrix (1) is completely signed. And from the continuity of the functions $f_i$ it follows that, corresponding to each $x_j$, a closed interval can be determined which contains $x_j$, and in which no one of the functions changes sign. Let us denote this interval by $X_j$. That is $f_i(x)$ has the sign of $f_i(x_j)$ so long as $x$ is on $X_j$.

Next we consider, relative to each interval $X_j$, a function $\pi_j(x)$ restricted as follows. It is continuous on $X$, and is zero except at the inner points of $X_j$, where it is positive. We define the function $\pi(x)$ as follows:

$$\pi(x) = c_0 + \sum_{j=1}^{2^m} c_j \pi_j(x), \tag{2}$$

the coefficients $c$ being constants to be determined.

The function $\pi(x)$ so defined is continuous on $X$, and is positive if $c_0, c_1, \cdots, c_{2^m}$ are positive. We shall show that these $c$'s can be given positive values such that the conditions

$$\int_a^b \pi(x) f_i(x)\, dx = 0 \qquad (i = 1, 2, \cdots, m) \tag{3}$$

will be satisfied.

Upon substitution of (2) in (3), we obtain a system of $m$ linear homogeneous equations in the $2^m + 1$ unknown $c$'s, which may be written

$$\sum_{j=0}^{2^m} a_{ij} c_j = 0 \qquad (i = 1, 2, \cdots, m), \tag{4}$$

* An example of a completely signed set is: $f_1(x) = x$, $f_2(x) = x^2 - 1$, $-2 \leqq x \leqq 2$.

where

$$a_{i0} = \int_a^b f_i(x)\,dx, \qquad a_{ij} = \int_a^b \pi_j(x) f_i(x)\,dx, \quad \begin{matrix}(i = 1, 2, \cdots, m)\\ (j = 1, 2, \cdots, 2^m).\end{matrix}$$

From the definition of $\pi_j(x)$, it follows that the constant $a_{ij}$ has the same sign as $f_i(x_j)$. Hence the matrix

$$\|a_{ij}\| \qquad (i = 1, 2, \cdots, m)\ (j = 1, 2, \cdots, 2^m)$$

is a completely signed matrix. According to the corollary to Theorem III of the note previously referred to in § 1, the system of equations (4) admits therefore a solution $c_0, c_1, \cdots, c_{2^m}$ in which all the $c$'s are positive.* The function $\pi(x)$ defined by (2), upon the substitution of these values, furnishes the function $\pi(x)$ required for the property $(b)$.

3. *Remark*: The notion of a completely signed set may be extended directly to a set of functions of a general variable† $q$ whose range is an arbitrary class of elements $Q$. By application of Theorem III of the note already cited it may be shown that *every linear combination of a completely signed set of functions on a general range changes sign.*

* An algorithm for determining a positive solution of a system such as (4) is described in the preceding paper.

† Such sets of functions have been considered in the recent paper: *On sets of functions of a general variable,* Trans. Amer. Math. Soc., vol. 29 (1927), p. 463.

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# CONCERNING CONTINUOUS CURVES AND CORRESPONDENCES.*

By William Leake Ayres.

## I. Introduction.

In this paper we shall discuss plane continuous curves† and especially continuous (1, 1) correspondences‡ between continuous curves. Part II contains a new definition of end-points and several theorems concerning continuous curves which are useful in the latter part of the paper. In part III we define and give several results concerning branch points and end-point homogeneity of continuous curves. In part IV we give a very general set of conditions under which a correspondence of subsets of two acyclic§ continuous curves can be extented to a continuous (1, 1) correspondence between the continuous curves themselves. The paper is confined to two dimensions but it is evident that a number of the theorems extend easily to $n$ dimensions.

Finally I wish to thank Professor J. R. Kline, who suggested the problem of this paper and who has assisted materially in its preparation by many suggestions and criticisms.

## II. Theorems on continuous curves.‖

Theorem 1. *If $M$ is a continuous curve and $N$ is a subset of $M$ which is also a continuous curve, then, for any positive number $\epsilon$, $M-N$ contains at most a finite number of maximal connected subsets¶ of diameter greater than $\epsilon$.***

*Proof.* Wilder†† has shown that $M-N$ consists of a set of maximal domains $[d]$, with respect to $M$, whose boundaries with respect to $M$ belong

* Received January 11, 1927.

† For definitions and theorems relating to continuous curves see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bull. Amer. Math. Soc., vol. 29 (1923), pp. 289–302. Hereafter we shall refer to this paper as "*Report*".

‡ *Report*, p. 289.

§ A continuous curve which contains no simple closed curve is said to be *acyclic*.

‖ Presented to the American Mathematical Society October 31, 1925.

¶ A connected set $K$ is said to be a *maximal connected subset* of a set $H$ if $K$ is a subset of $H$ and $H$ contains no connected subset of which $K$ is a proper subset. A subset $N$ of a set $M$ is said to be a *proper subset* if $M-N$ is not vacuous.

** The *diameter of a point-set* is the upper bound of the set of numbers $d(x, y)$, where $x$ and $y$ are points of the set and the symbol $d(x, y)$ denotes the distance from $x$ to $y$.

†† R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377, theorem 9.

to $N$.* Suppose the set of maximal domains $[d]$ of $M-N$ of diameter greater than $\epsilon$ is infinite. If $\bar{d}$ denotes any domain of the set, $\bar{d}$ contains two points $A$ and $B$ such that $d(A, B) > \epsilon$. There exists an arc $AB$ from $A$ to $B$ every point of which lies in $\bar{d}$.† Since $M$ is bounded and no two of the sets $[d]$ have common points, there exist two points $\bar{A}$ and $\bar{B}$ and two sequences $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ such that (1) the points $A_1, A_2, A_3, \cdots$ are distinct points of the set $[A]$ and $\bar{A}$ is the sequential limit point of this sequence, (2) the points $B_1, B_2, B_3, \cdots$ are distinct points of the set $[B]$ and $\bar{B}$ is the sequential limit point of this sequence, (3) for every $i$, the points $A_i$ and $B_i$ belong to the same domain of the set $[d]$.

It follows that $d(\bar{A}, \bar{B}) \geqq \epsilon$. There exists an integer $k$ such that if $i > k$, $A_i$ and $B_i$ lie within circles $C_1$ and $C_2$ of radii $\frac{1}{4}\epsilon$ with centers at $\bar{A}$ and $\bar{B}$ respectively. Each arc $A_iB_i$ $(i > k)$ contains a last point $X_i$ in the order $A_i$ to $B_i$ in common with $C_1$ and the subarc $X_iB_i$ of $A_iB_i$ contains a first point $Y_i$ in common with $C_2$. The sequence of arcs $X_{k+1}Y_{k+1}$, $X_{k+2}Y_{k+2}$, $X_{k+3}Y_{k+3}$, $\cdots$ contains a subsequence $\bar{X}_1\bar{Y}_1$, $\bar{X}_2\bar{Y}_2$, $\bar{X}_3\bar{Y}_3$, $\cdots$ having the following properties: (1) $\bar{X}_i$ is on $C_1$ and $\bar{Y}_i$ is on $C_2$ and every other point of $\bar{X}_i\bar{Y}_i$ lies outside both circles, (2) $C_1$ contains three points $P_1$, $Q_1$ and $X$ such that every point $\bar{X}_i$ lies on the arc $P_1Q_1X$ of $C_1$ and in the order $P_1\,Q_1\,\bar{X}_1\,\bar{X}_2\,\bar{X}_3\cdots X$, and $C_2$ contains three points $P_2, Q_2$ and $Y$ such that every point $\bar{Y}_i$ lies on the arc $P_2Q_2Y$ of $C_2$ and in the order $P_2\,Q_2\,\bar{Y}_1\,\bar{Y}_2\,\bar{Y}_3\cdots Y$, (3) the sequence $\bar{X}_1, \bar{X}_2, \bar{X}_3\cdots$ has the point $X$ as its sequential limit point and the sequence $\bar{Y}_1, \bar{Y}_2, \bar{Y}_3, \cdots$ has as its sequential limit point the point $Y$, (4) each of the arcs $\bar{X}_i\bar{Y}_i$ lies in a maximal domain of $M-N$ of diameter greater than $\epsilon$ and no two lie in the same domain.

Each arc $\bar{X}_i\bar{Y}_i$ contains a point $Z_i$ at a distance greater than or equal to $\frac{1}{2}\epsilon$ from both $\bar{A}$ and $\bar{B}$. The set $[Z_i]$ contains a subset $Z_{n_1}, Z_{n_2}, Z_{n_3}, \cdots$ which has a sequential limit point $Z$. It follows that $Z$ is at a distance greater than or equal to $\frac{1}{2}\epsilon$ from both $\bar{A}$ and $\bar{B}$. Let $C_3$ denote a circle of radius $\epsilon/8$ with center at $Z$. Since $M$ is connected im kleinen there exists within $C_3$ a concentric circle $C_4$ such that every point of $M$ lying within $C_4$ lies together with $Z$ in a connected subset of $M$ which lies

* A connected subset $D$ of a continuous curve $M$ is said to be a *domain* with respect to $M$, or an $S$-domain, if for every point $P$ of $D$ there exists a circle $C$ with center at $P$ such that the set of all points of $M$, which (1) lie interior to $C$ and (2) lie with $P$ in a connected subset of $M$ that lies wholly interior to $C$, belong to $D$. The set of all limit points of $D$ which do not belong to $D$ constitutes the *boundary* of $D$. Cf. R. L. Wilder, loc. cit.

† For proof of this see R. L. Moore, *Concerning continuous curves in the plane*, Math. Zeit., vol. 15 (1922), pp. 254–261. See also R. L. Wilder, *loc. cit.*

entirely within $C_3$. Let $Z_{m_1}$ be the first point of $[Z_{n_i}]$ which lies within $C_4$. The curve $M$ contains a simple continuous arc $\alpha_1$ from $Z_{m_1}$ to $Z$ which lies entirely within $C_3$.* The arc $\alpha_1$ contains a last point $T_1$ in the order $Z_{m_1}$ to $Z$ in common with $\overline{X}_{m_1}\overline{Y}_{m_1}$ and the arc $T_1 Z$ of $\alpha_1$ contains a first point $T_2$ in common with $\overline{X}_{m_1'}\overline{Y}_{m_1'}$.† Since $C_3$ is of radius $\varepsilon/8$ the arc $T_1 T_2$ of $\alpha_1$ lies entirely within the simple closed curve $J_1$ formed by $\overline{X}_{m_1}\overline{Y}_{m_1}$, $\overline{X}_{m_1'}\overline{Y}_{m_1'}$, the arc $\overline{X}_{m_1}\overline{X}_{m_1'}$ of $C_1$ which lies on $P_1 Q_1 X$ and the arc $\overline{Y}_{m_1}\overline{Y}_{m_1'}$ of $C_2$ which lies on $P_2 Q_2 Y$. The arc $T_1 T_2$ contains in its interior a point $V_1$ of $N$, for otherwise the maximal domain of $M-N$ which contains $\overline{X}_{m_1}\overline{Y}_{m_1}$ would contain $\overline{X}_{m_1'}\overline{Y}_{m_1'}$ also which is contrary to the hypothesis that the arcs lie in distinct domains of the set $[d]$ of $M-N$.

Let $C_5$ denote a circle with center at $Z$ lying within $C_4$, containing no point of $\overline{X}_{m_1}\overline{Y}_{m_1}$ and radius less than $\varepsilon/16$. Repeating the above argument with $C_5$ taking the place of $C_3$ we determine a point $V_2$ of $N$ within the simple closed curve $J_2$ formed by $\overline{X}_{m_2}\overline{Y}_{m_2}$, $\overline{X}_{m_2'}\overline{Y}_{m_2'}$, the arc $\overline{X}_{m_2}\overline{X}_{m_2'}$ of $C_1$ which lies on $P_1 Q_1 X$ and the arc $\overline{Y}_{m_2}\overline{Y}_{m_2'}$ of $C_2$ which lies on $P_2 Q_2 Y$, where $Z_{m_2}$ is the first point of the set $[Z_{n_i}]$ which lies in $C_6$ $(m_2 > m_1)$ and where $C_6$ is a circle concentric with $C_5$ such that every point of $M$ within $C_6$ lies with $Z$ in a connected subset of $M$ lying wholly within $C_5$. Continue this process indefinitely each time taking the circle $C_{2i+1}$ of radius less than $\varepsilon/2^{i+2}$ and such that no point of $\overline{X}_{m_{i-1}}\overline{Y}_{m_{i-1}}$ lies within it.

Thus there exists a set of points $V_1, V_2, V_3, \cdots$ such that for every value of $i$ (1) $V_i$ belongs to $N$ and is at a distance from $Z$ less than $\varepsilon/2^{i+2}$, (2) $V_i$ lies interior to the simple closed curve $J_i$ formed by $\overline{X}_{m_i}\overline{Y}_{m_i}$, $\overline{X}_{m_i'}\overline{Y}_{m_i'}$ the arc $\overline{X}_{m_i}\overline{X}_{m_i'}$ of $C_1$ which lies on $P_1 Q_1 X$ and the arc $\overline{Y}_{m_i}\overline{Y}_{m_i'}$ of $C_2$ which lies on $P_2 Q_2 Y$,

By property (1), $Z$ is the sequential limit point of $V_1, V_2, V_3, \cdots$ and since $N$ is closed, $Z$ belongs to $N$. As $N$ is connected im kleinen by hypothesis, there exists within the circle $C_3$ a circle $\overline{C}$ with center at $Z$ such that every point of $N$ within $\overline{C}$ can be joined to $Z$ by an arc of $N$ which lies wholly within $C_3$. By property (1), $N$ contains a point $V_j$ within $\overline{C}$. Let $\overline{\alpha}$ denote the arc of $N$ from $V_j$ to $Z$ lying wholly within $C_3$. The point $V_j$ is in the interior of the simple closed curve $J_j$ and $Z$ is in the exterior, thus $\overline{\alpha}$ must contain a point of $J_j$. Since the radius of $C_3$ is $\varepsilon/8$, $\overline{X}_{m_j}\overline{X}_{m_j'}$ and $\overline{Y}_{m_j}\overline{Y}_{m_j'}$ lie in the exterior of $C_3$ and thus can have no point in common with $\overline{\alpha}$. The arc $\overline{\alpha}$ can have no point in common with either $\overline{X}_{m_j}\overline{Y}_{m_j}$ or $\overline{X}_{m_j'}\overline{Y}_{m_j'}$ since these arcs belong to $M-N$. But these four arcs form $J_j$

* For a justification of the term "arc" here, see J. R. Kline, *Concerning approachability of simple closed and open curves*, Trans. Amer. Math. Soc., vol. 21 (1920), p. 453 and footnote.

† If $m_i$ denotes $n_j$, then $m_i'$ denotes $n_{j+1}$.

and therefore $\bar{\alpha}$ contains no point of $J_j$, which is a contradiction. This completes the proof of the theorem.

We may state the following more general form of this theorem. The proof, except for a few minor changes, is the same as for Theorem 1.

THEOREM 2. *If $M$, $N_1$, $N_2$, $N_3$, $\cdots$, $N_k$ are continuous curves and $N_i (0 < i \leqq k)$ is a subset of $M$, then, for any positive number $\epsilon$, $M-(N_1+N_2+\cdots+N_k)$ contains at most a finite number of maximal connected subsets of diameter greater than $\epsilon$.*

*Definition.* A point $P$ of a continuous curve $M$ is said to be a *W-end-point,** or simply an *end-point,* of $M$ if for every arc $\alpha$ of $M$ joining $P$ to any other point $P'$ of $M$, it is true that the set $M-(\alpha-P)$ contains no connected subset consisting of more than one point which contains $P$.

THEOREM 3. *In order that a point $P$ of a continuous curve $M$ be an end-point of $M$, it is necessary and sufficient that* (1) *$P$ belong to no simple closed curve which is a subset of $M$,* (2) *$P$ be a non-cutpoint† of $M$.*

*Proof.* (a). *The conditions are necessary.* (1) Suppose $P$ is an end-point of $M$ and lies on a simple closed curve $J$ of $M$. Let $P'$ be any other point of $J$. Then $J$ is the sum of two arcs from $P$ to $P'$,

$$J = PXP' + PYP'.$$

Let $PXP' = \alpha$. The set $M-(\alpha-P)$ contains a connected subset $PYP'-P'$ consisting of more than one point and containing $P$. Then $P$ cannot be an end-point of $M$ by definition, which is a contradiction.

(2) Suppose that $P$ is an end-point of $M$ which is also a cutpoint of $M$. Let $P'$ be any other point of $M$ and let $\alpha$ be an arc of $M$ whose end-points are $P$ and $P'$. As $P$ is a cutpoint,

$$M-P = M_1 + M_2,$$

where $M_1$ and $M_2$ are non-vacuous mutually separated sets.‡ Now $\alpha-P$, being connected, must lie wholly in $M_1$ or $M_2$. Suppose it lies in $M_1$. By a theorem due to Wilder,§ $M_2$ is a set of domains with respect to $M$

* This definition was given by R. L. Wilder, *loc. cit.*, p. 358. Cf. also H. M. Gehman, *Concerning end points of continuous curves and other continua.*, Bull. Amer. Math. Soc., vol. 32 (1926), p. 218 (abstract).

† If $M$ is a continuum and $P$ is a point of $M$, then $P$ is said to be a *non-cutpoint* or a *cutpoint* according as $M-P$ is connected or not. Cf. R. L. Moore, *Concerning the cut-points of continuous curves and of other closed and connected point-sets,* Proc. Nat. Acad., vol. 9 (1923), pp. 101–106.

‡ Two sets $H$ and $K$ are said to be *mutually exclusive* if they have no common points. They are said to be *mutually separated* if they are mutually exclusive and neither contains a limit point of the other. The symbol $H \cdot K$ is used to denote the set of common points of the sets $H$ and $K$.

§ *Loc. cit.*, Theorem 9, p. 360.

whose boundary points with respect to $M$ are contained in $M_1+P$. Let $D$ be one of these domains. As $D$ is a subset of $M_2$, no point of $M_1$ can be a limit point of $D$ and thus $P$ must be the entire boundary of $D$. Then $D+P$ is connected, consists of more than one point and contains $P$, which is contrary to the assumption that $P$ is an end-point.

(b). *The conditions are sufficient.* Let $P$ be a point of $M$ which satisfies the two conditions and let us suppose that $P$ is not an end-point of $M$. Then there exists a point $P'$ of $M$ and an arc $\alpha$ of $M$ whose end-points are $P$ and $P'$ such that $M-(\alpha-P)$ contains a connected set $N$ consisting of more than one point and containing $P$. Since $P$ is a non-cutpoint, $M-P$ is connected. Divide $M-P$ into two sets, $\alpha-P$ and $\bar{M}$. Then $\bar{M}$ contains $N-P$. Let $q$ be any point of $N-P$ and let $D_q$ be that maximal domain with respect to $M$ which lies in $\bar{M}$ and contains $q$. As $N$ is connected, one of the sets $N\cdot D_q$ and $N-N\cdot D_q$ must contain a limit point of the other. The set $N\cdot D_q$ cannot contain a limit point of $N-N\cdot D_q$ as no point of the domain $D_q$ is a limit point of a set of points of $M$ lying outside the domain. Since $D_q$ is a maximal domain of $\bar{M}$ and $N-(N\cdot D_q+P)$ lies in $\bar{M}$, no point of $N-(N\cdot D_q+P)$ is a limit point of $N\cdot D_q$. Therefore $P$ is a limit point of $N\cdot D_q$, and consequently of $D_q$. As $D_q$ is a maximal domain of $\bar{M}$, $D_q$ and $\bar{M}-D_q$ are mutually separated; and as $M-P$ is connected, $\alpha-P$ must contain a point $x$ which is a limit point of $D_q$. There exist* arcs $\alpha'$ and $\alpha''$, whose end-points are $x$ and $q$ and $P$ and $q$ respectively such that $D_q$ contains $\alpha'-x$ and $\alpha''-P$.

There exists an arc $\beta$, whose end-points are $x$ and $P$, every point of which belongs to $\alpha'+\alpha''$ and such that $D_q$ contains $\beta-(P+x)$. The arc $\beta$ and the arc $Px$ of $\alpha$ have no points in common except their end-points and thus their sum is a simple closed curve which is a subset of $M$ and contains $P$. But this is contrary to the hypothetis that $P$ lies on no simple closed curve which is a subset of $M$.

THEOREM 4.† *If $x$ and $y$ are two distinct points of a continuous curve $M$, then no point of an arc $xy$ of $M$, except possibly the points $x$ and $y$, is an end-point of $M$.*

*Proof.* Let $P$ denote any point of an arc $xy$ of $M$ distinct from $x$ and from $y$. Let $\alpha$ denote the arc $Px$ which is a subset of the arc $xy$. Then $M-(\alpha-P)$ contains a connected subset, namely the arc $Py$ of $xy$, consisting of more than one point and containing $P$. Thus, by definition, $P$ cannot be an end-point of $M$.

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* Cf. R. L. Wilder, *loc. cit.*, Theorem 1, p. 342.

† R. L. Wilder, *loc. cit.*, Theorem 14, p. 364, gives a proof of this theorem for the acyclic case. Cf. also R. L. Moore, *Concerning the cut-points etc.*, loc. cit., Theorem C.

THEOREM 5. *If $K$ denotes the set of all end-points of a continuous curve $M$ then $K$ contains no connected subset consisting of more than one point.*

*Proof.** Suppose that $K$ contains a connected subset $\bar{K}$ which contains more than one point. Let $P$ be any point of $\bar{K}$. Let $P'$ be any other point of $M$ and $\alpha$ be an arc of $M$ whose end-points are $P$ and $P'$. Let $P''$ denote a point of $\alpha-(P+P')$. Then by Theorem 4, no point of the arc $PP''$ of $\alpha$ except possibly the point $P$ is a point of $K$. Thus the set $M-(PP''-P)$ contains the set $K$ and consequently $\bar{K}$. But this is contrary to the assumption that $P$ is an end-point of $M$.

THEOREM 6. *If $K$ denotes the set of all end-points of a continuous curve $M$, then no subset of $K$ disconnects $M$ even in the weak sense.*†

*Proof.* Suppose $K$ contains a subset $\bar{K}$ such that $M-\bar{K}$ is not connected in the strong sense. Then $M-\bar{K}$ contains two points $A$ and $B$ which lie in no closed and connected subset of $M-\bar{K}$. The curve $M$ contains an arc $AB$ whose end-points are $A$ and $B$.‡ Since $A$ and $B$ lie in no closed and connected subset of $M-\bar{K}$, the arc $AB$ must contain a point of $\bar{K}$, which we will call $P$, and $P$ must be distinct from $A$ and from $B$. The point $P$ belongs to $\bar{K}$ and thus to $K$. But this is impossible by Theorem 4.

## III. On branch points and homogeneity.§

*Definition.* We shall say that a continuous curve $M$ is *homogeneous with respect to $n$ end-points* if for each two sets of $n$ distinct end-points of $M$, $A_1, A_2, A_3, \cdots, A_n$ and $B_1, B_2, B_3, \cdots, B_n$, there exists a continuous (1,1) transformation $T$‖ of $M$ into itself such that $T(A_i) = B_i$ for each

* For a proof of this theorem for the acyclic case, see R. L. Wilder, *loc. cit.*, Theorem 15, p. 365.

† A point-set is said to be *connected in the weak sense* if no matter how it is divided into two mutually exclusive subsets, one of these subsets contains a limit point of the other. The set $M$ is said to be *connected in the strong sense* if every two points of $M$ lie together in some closed subset of $M$ which is connected in the weak sense. A point set $K$ is said to *disconnect a connected point-set $M$ in the weak sense* if $M-M\cdot K$ is not connected in the strong sense. For these definitions, see R. L. Moore, *Concerning the cut-points etc., loc. cit.;* Theorem F of Moore's paper together with Theorem 3 of this paper give a proof of the acyclic case of Theorem 6.

‡ Every two points of a continuous curve $M$ can be joined by a simple continuous arc which is a subset of $M$. Cf. R. L. Moore, *A theorem concerning continuous curves,* Bull. Amer. Math. Soc., vol. 23 (1917), pp. 233–236. See also R. Tietze, *Über stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven,* Math. Zeit., vol. 5 (1919), pp. 284–291; S. Mazurkiewicz, *Sur les lignes de Jordan,* Fund. Math., vol. 1 (1920), pp. 166–209, and earlier papers referred to therein.

§ The results of this section, except Theorems 11 and 12, were presented to the Society January 2, 1926. Theorems 11 and 12 were presented May 1, 1926.

‖ Cf. *Report,* p. 289. We shall use the terms "transformation" and "correspondence" interchangeably.

$0 < i \leqq n$. If a continuous curve is homogeneous with respect to $n$ end-points for every positive integer $n$, we shall say that it is *homogeneous with respect to its end-points* or *end-point homogeneous.**

*Definition.* Suppose $P$ is a point of a continuous curve $M$ which lies on no simple closed curve of $M$. Then $M-P$ is a finite or at most countable† set of trees $[T]$.‡ If the set $[T]$ consists of just one tree, $P$ is an end-point of $M$ by Theorem 3. If the set $[T]$ consists of just two trees, we shall call $P$ an *ordinary cut-point* of $M$. If the set $[T]$ consists of three or more trees, we shall call $P$ a *branch point* of $M$. If the set $[T]$ consists of a finite number $k$ $(k>2)$ of trees we shall say that $P$ is a branch point *of order* $k$. If the set $[T]$ is infinite, either there are only a finite number $n$ of trees each of which contains just one end-point of $M$ or there are an infinite number of such trees. In these two cases we shall say that $P$ is a branch point *of order* $n\infty$ and $\infty\,\infty$ respectively. (Note: $n$ may be zero.)

THEOREM 7. *Suppose* (1) $M$ *is an acyclic continuous curve and* $A$ *and* $B$ *are end-points of* $M$, (2) *there exists a continuous* (1,1) *transformation* $T$ *of* $M$ *into itself such that* $T(A) = B$ *and* $T(B) = A$, *then the arc* $AB$§ *of* $M$ *contains a point which remains invariant under* $T$.‖

The proof of Theorem 7 affords no difficulty and will be omitted.

THEOREM 8. *If all, except possibly three, of the end-points of an acyclic continuous curve* $M$ *remain invariant under a continuous* (1,1) *transformation* $T$ *of* $M$ *into itself, then all of the branch points of* $M$ *remain invariant under* $T$.

*Proof.* Under the hypothesis that not more than three end-points of $M$ are affected by $T$ three cases arise: (1) all of the end-points of $M$ remain invariant under $T$, (2) all except two, $E_1$ and $E_2$, of the end-points of $M$ remain invariant and $T(E_1) = E_2$, $T(E_2) = E_1$, (3) all except three,

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* S. Mazurkiewicz, *Sur les continus homogènes*, Fund, Math., vol. 5 (1924), pp. 137–146, has defined a homogeneous set of points and has shown that a plane bounded continuous curve which is homogeneous is a simple closed curve. The above definition gives a somewhat similar notion which is applicable to acyclic continuous curves.

† Cf. Theorem 1. Cf. also R. L. Wilder, *loc. cit.*, Theorem 9.

‡ If $N$ is a subcontinuum of a continuous curve $M$, a maximal connected subset of $M-N$ which has only one limit point in $N$ is called a *tree* with respect to $N$ or a tree in $M-N$. The limit point in $N$ is called the *foot of the tree.* Cf. H. M. Gehman, *On extending a continuous* (1, 1) *correspondence of two plane continuous curves to a correspondence of their planes,* Trans. Amer. Math. Soc., vol. 28 (1926), pp. 252–265.

§ If $A$ and $B$ are two points of an acyclic continuous curve $M$ then $AB$ denotes the arc of $M$ from $A$ to $B$.

‖ W. Scherrer, *Über ungeschlossene stetige Kurven*, Math. Zeit., vol. 24 (1925), pp. 125–130, proves that if $T$ is *any* continuous (1,1) transformation of $M$ into itself then $M$ contains an invariant point. Then Theorem 7 states that under condition (2) one such point lies on the arc $AB$.

$E_1$, $E_2$ and $E_3$ remain invariant and $T(E_1) = E_2$, $T(E_2) = E_3$, $T(E_3) = E_1$. We shall show that in each of these cases all of the branch points must remain fixed under $T$.

In case (2) the arc $E_1E_2$ contains a point $P$ which is invariant under $T$ by Theorem 7. We shall now show that $E_1E_2$ cannot contain a branch point distinct from $P$ and thus prove that the arc $E_1E_2$ contains not more than one branch point and if there exists one it remains invariant under $T$. Suppose the arc $E_1E_2$ contains a branch point $B$ distinct from $P$. The point $B$ must lie on either $\overset{\frown}{PE_1}$ or $\overset{\frown}{PE_2}$,* suppose $\overset{\frown}{PE_1}$. As $B$ is a branch point there is at least one end-point $E_3$ such that the arc $BE_3$ has only the point $B$ in common with $E_1E_2$. Since $T(PE_1) = PE_2$, $B'$ (i. e. $T(B)$) must lie on $\overset{\frown}{PE_2}$. The point $E_3$ is invariant under $T$ since we are considering the case in which only two end-points are affected by $T$. Then $T(E_3B) = E_3B'$ and, as the arc $E_3B'$ contains $P$, the arc $E_3B$ must contain a point $Q$ such that $T(Q) = P$. Thus $M$ contains two points, $P$ and $Q$ which have $P$ as their correspondent under $T$, which is contradictory to the hypothesis that $T$ is a (1,1) correspondence.

We shall show that in case (3) each of the three arcs, $E_1E_2$, $E_2E_3$, $E_1E_3$, contains but one branch point and this branch point is common to the three arcs and remains invariant under $T$. Let $P$ be the first point of the arc $E_1E_2$ in the order from $E_1$ to $E_2$ which is a point of $E_2E_3$. The point $P$ is distinct from $E_1$, $E_2$ and $E_3$ by Theorem 4. Then the subarc $PE_3$ of $E_2E_3$ has no point except $P$ in common with the arc $E_1E_2$, $P$ is the only point which belongs to all three of the arcs $E_1E_2$, $E_2E_3$, $E_1E_3$ and $P$ is a branch point of $M$. We have now

$$T(E_1E_2) = E_2E_3, \qquad T(E_2E_3) = E_3E_1, \qquad T(E_1E_3) = E_2E_1.$$

Now $P$ lies on each of the three arcs $E_1E_2$, $E_2E_3$, $E_1E_3$, and thus $P'$ must lie on each of the three transformed arcs. But as $P$ is the only point which lies on each of the three arcs $E_2E_3$, $E_3E_1$ $E_2E_1$, $P' = T(P) = P$. Thus $P$ is invariant under $T$. Now suppose one of the three arcs from $P$, say $PE_1$, contains a branch point $B$ distinct from $P$. The argument continues exactly as in case (2) above and we obtain a point $Q$ such that $Q \neq P$ and $T(Q) = P$, which contradicts the hypothesis that $T$ is a (1,1) transformation.

Now, in any one of the three cases, let us suppose that $M$ contains a branch point $B$ such that $T(B) = B' \neq B$. Let $E_1E_2$ be a maximal arc of $M$ containing $B$ and $B'$;† then $E_1$ and $E_2$ are end-points of $M$.

* If $AB$ is an arc, $\overset{\frown}{AB}$ denotes $AB-(A+B)$.

† S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fund. Math., vol. 2 (1921), pp. 119–130.

We suppose the order $E_1BB'E_2$. We see easily that, as $B$ is a branch point of $M$, $B'$ is also and thus there exist two end-points $E_3$ and $E_4$ of $M$ such that $B$ and $B'$ are the only points which the arcs $E_3B$ and $E_4B'$ have in common with the arc $E_1E_2$. Not more than one of the four points $E_i$ can be affected by $T$, for if two fail to remain fixed under $T$ the arc joining these two must contain one of the two points $B$ or $B'$, neither of which remains invariant under $T$. But we proved above that if two end-points fail to remain fixed under $T$ then the arc of $M$ joining them can contain at most one branch point and this branch point must remain fixed under $T$. It follows then that not more than one of the four points $E_i$ can be affected by $T$. And if one of the points $E_i$ fails to remain fixed under $T$, $E_i$ and $E_i'$ lie in the same tree of $M-BB'$.

Then at least one of the two points $E_2$ and $E_4$, say $E_2$, and at least one of the points $E_1$ and $E_3$, say $E_1$, remain invariant under $T$. Since $T(E_1B) = E_1B'$ and $\underset{\frown}{E_1B'}$ contains $B$, $\underset{\frown}{E_1B}$ contains a point $P_1$ such that $T(P_1) = B$. Similarly $T(E_3B) = E_3'B'$ and $\underset{\frown}{E_3'B'}$ contains $B$ since $E_3$ and $E_3'$, whether identical or not, lie in the same tree of $M-BB'$. Therefore $\underset{\frown}{E_3B}$ contains a point $P_2$ such that $T(P_2) = B$. But $P_1 \neq P_2$ and $B$ is the correspondent of each point under $T$, which is contrary to the hypothesis that $T$ is a (1,1) correspondence.

THEOREM 9. *In order that an acyclic continuous curve $M$ be homogeneous with respect to its end-points it is necessary and sufficient that $M$ contain a point $P$ such that $M$ consists of a set of arcs from $P$ to the end-points of $M$, no two of which have any point in common except $P$.*

*Proof.* The condition is obviously sufficient; we shall show that it is also necessary. Let $E$ denote the set of all end-points of $M$ and let $e_1$ and $e_2$ be points of $E$. The arc $e_1e_2$ of $M$ must contain a branch point $b_1$ for otherwise $M$ consists of merely the arc $e_1e_2$ and the theorem is true. The set $M-b_1$ consists of at least three trees, one $M_{e_1}$ containing $e_1$, one $M_{e_2}$ containing $e_2$, and at least one other $M_{e_3}$ containing a point $e_3$ of $E$. We shall prove that $M_{e_1} = e_1b_1 - b_1$.

Suppose that $M_{e_1}$ contains a point $p$ which is not on $e_1b_1$. Let $e_4b_1$ be a maximal arc of $M_{e_1} + b_1$ which contains $p$ and $b_1$ and let $b_2$ be the first point which the arc $e_4b_1$ has in common with $e_1b_1$ in the order $e_4$ to $b_1$. Then $e_4$ is a point of $E$ and $b_2$ is a branch point of $M$. By hypothesis there exists a continuous (1,1) transformation $T$ of $M$ into itself such that $T(e_1) = e_2$, $T(e_2) = e_1$, $T(e_3) = e_3$, $T(e_4) = e_4$. By Theorem 7 the arc $e_1e_2$ contains a point $q$ which corresponds to itself under $T$. Five cases may arise according as (1) $q$ lies between $e_1$ and $b_2$, (2) $q = b_2$, (3) $q$ lies between $b_2$ and $b_1$, (4) $q = b_1$, (5) $q$ lies between $b_1$ and $e_2$.

Let us consider the first case. Since $e_1' = e_2$ and $e_2' = e_1$, $T(e_1 q) = e_2 q$ and $T(e_2 q) = e_1 q$. Then $b_1'$ and $b_2'$ lie on $e_1 q$. The set $e_3 b_1 + e_4 b_2$ is not connected while $T(e_3 b_1 + e_4 b_2) = e_3 b_1' + e_4 b_2'$ is connected since each component is an arc and they have the point $q$ in common. But this is a contradiction since the connectivity of a set is invariant under a continuous transformation. In each of the other four cases we arrive at a contradiction in exactly the same way by showing that $e_3 b_1 + e_4 b_2$ is not connected while $e_3 b_1' + e_4 b_2'$ is connected. Thus $M_{e_1} = e_1 b_1 - b_1$.

Let $M_e$ be any other tree of $M - b_1$, and let $e$ be a point of $E$ in $M_e$. Suppose $M_e$ contains a point which is not a point of $eb_1$. Then $M_e$ contains an end-point $\bar{e}$ distinct from $e$. We may arrive at a contradiction exactly as in the case of $M_{e_1}$ above by considering the continuous (1,1) transformation $\bar{T}$ of $M$ into itself such that

$$\bar{T}(e) = e_1, \qquad \bar{T}(e_1) = e, \qquad \bar{T}(e_4) = e_4, \qquad \bar{T}(\bar{e}) = \bar{e}.$$

Thus $M_e = eb_1 - b_1$. Therefore every tree of $M - b_1$ is a set consisting of an arc of $M$ from $b_1$ to an end-point of $M$ minus the point $b_1$, which proves the necessity of the condition.

As $M$ is connected im kleinen we see that the set of arcs of the preceding theorem is finite or at most countably infinite and, if infinite, $b_1$ is the only limit point of $E$, the set of all end-points of $M$. By noticing that in the proof of the preceding theorem we make use of four end-points in each step we obtain the following more general result.

THEOREM 10. *A necessary and sufficient condition that an acyclic continuous curve $M$ be homogeneous with respect to $n$ end-points $(n > 3)$ is that $M$ contain a point $P$ such that $M$ consists of a set of arcs from $P$ to the end-points of $M$, no two of which have any point in common except $P$.*

The following example shows that the condition of Theorem 10 is *not necessary* unless $n > 3$. In this example any three end-points may be carried into any three end-points by a continuous (1,1) transformation of $M$ into itself and yet $M$ fails to satisfy the condition of Theorem 10.

Consider the ordinary Euclidean plane with a rectangular coördinate system. Let $M_{11}$ denote the interval from (1,0) to (0,0) and let $M_{12}$ be vacuous. Let $p_i$ $(i = 1, 2, \cdots)$ be the point of $M_{11}$ whose distance from the point (0,0) is $\frac{1}{3} \cdot 2^{1-i}$ and let $q_i$ $(i = 2, 3, 4, \cdots)$ be the point of $M_{11}$ whose distance from (1,0) is $\frac{1}{3}(1 + 2^{1-i})$. At each point $p_i$ and $q_i$ erect perpendiculars to $M_{11}$ of length $\frac{1}{15} \cdot 2^{1-i}$ with $p_i$ or $q_i$, as the case may be, as midpoints. Let $M_{21}$ be the set of points consisting of all such perpendiculars and let $M_{22}$ be vacuous. Let $M_2$ denote $M_{11} + M_{21}$. Suppose that $AB$ is any interval of $M_2$ such that $A$ is an end-point of $M_2$, $B$ is a branch point of $M_2$ and $\overset{\frown}{AB}$ contains no branch point of $M_2$. Let

$d$ denote the length of $AB$. Let $u_i$ $(i = 1, 2, 3, \cdots)$ be the point of $AB$ whose distance from $B$ is $\frac{1}{3}\cdot 2^{1-i}\cdot d$ and let $v_i$ $(i = 2, 3, 4, \cdots)$ be the point of $AB$ whose distance from $A$ is $\frac{1}{3}(1+2^{1-i})d$. At each point $u_i$ and $v_i$ erect perpendiculars to $AB$ of length $\frac{1}{15}\cdot 2^{1-i}\cdot d$ with $u_i$ and $v_i$, as the case may be, as midpoints. Let $M_{31}$ be the set of points consisting of all such perpendiculars for all such intervals $AB$ of $M_2$. Let $CD$ be any interval of $M_2$ such that $C$ and $D$ are branch points of $M_2$ and $\overset{\smile}{CD}$ contains no branch points of $M_2$; and let $k$ be the length of $CD$. Let $a_i$ $(i = 1, 2, 3, \cdots)$ be the point of $CD$ whose distance from $C$ is $2^{-i}\cdot k$ and $b_i$ $(i = 2, 3, 4, \cdots)$ be the point of $CD$ whose distance from $D$ is $2^{-i}\cdot k$. At each point $a_i$ and $b_i$ erect perpendiculars to $CD$ of length $\frac{1}{3}\cdot 2^{-i}\cdot k$ with $a_i$ or $b_i$, as the case may be, as midpoints. Let $M_{32}$ be the set of points consisting of all such perpendiculars for all such intervals $CD$ of $M_2$. Let $M_3 = \sum_{i=1}^{3}\sum_{j=1}^{2} M_{ij}$. By erecting similar perpendiculars to all intervals $AB$ and $CD$ of $M_3$ we may define $M_{41}$ and $M_{42}$, and so on indefinitely. Let $K_1 = \sum_{i=1}^{\infty}\sum_{j=1}^{2} M_{ij}$ and let $K_2$ be the set obtained by reflecting $K_1$ in the line $x = 0$, $K_3$ be the set obtained by reflecting $K_1 + K_2$ in the line $x = y$. Let $M = K_1 + K_2 + K_3$.

By examination we see that $M$ is an acyclic continuous curve and that any three end-points of $M$ may be carried into any three end-points of $M$ by a continuous (1,1) transformation of $M$ into itself. But $M$ does not satisfy the condition of Theorem 10.

THEOREM 11. *The set of branch points of an acyclic continuous curve is countable.**

LEMMA A. If $T$ is a tree of a continuous curve and $P$ its root then $T+P$ is a continuous curve.

*Proof.* Let $M$ be the continuous curve. From the definition of a tree it follows that $T$ and $M-(T+P)$ are mutually separated sets. Thus $T+P$ is bounded, closed and connected and we need only to show that $T+P$ is connected im kleinen at each of its points. As $M-T$ is closed, the connectivity im kleinen of $T+P$ at a point $Q$ of $T$ follows at once from the fact that the set $M$ is connected im kleinen at $Q$. The set $T+P$ is also connected im kleinen at $P$ for if $C_1$ is any circle with center at $P$, there exists a concentric circle $C_2$ such that every point of $M$ lying within

* This theorem was presented to the Society May 1, 1926. I later found a proof by Menger and, as mine is very similar, it is omitted here the reader being referred to Menger's paper (*Über reguläre Baumkurven*, Math. Ann., vol. 97 (1926), pp. 572–582). Since this paper was in print, I have found a still earlier proof by Wazewski, Ann. Soc. Polonaise de Math., vol. 2 (1923), pp. 49–170.

$C_2$ lies with $P$ in a connected subset of $M$ lying wholly within $C_1$. Let $x$ be any point of $T$ within $C_2$ and let $M_x$ be an arc of $M$ from $x$ to $P$ and lying wholly within $C_1$. Since $T$ and $M-(T+P)$ are mutually separated, the arc $M_x$ must lie wholly in $T+P$ and thus $T+P$ is connected im kleinen at $P$. Since $T+P$ is bounded, closed, connected and connected im kleinen it is continuous curve.

*Definition.* Suppose $M$ is a continuous curve which contains just one simple closed curve $C$. The set $M-C$ consists of a countable number of trees, $d_1, d_2, d_3, \cdots$, by Theorem 1. Each tree $d_i$ has just one limit point $P_i$ in $C$ which is the root of the tree by definition. We shall call the points $P_i$ the *cyclic branch points* of $M$.

THEOREM 12. *The set of all branch points (ordinary and cyclic) of a continuous curve containing just one simple closed curve is countable.*

*Proof.* Let $M$ be the continuous curve and $C$ be the simple closed curve in $M$. Any maximal connected subset of $M-C$ is a tree and the number of such trees is countable by Theorem 1. Let these trees be denoted by $T_1, T_2, T_3, \cdots$ and the corresponding limit points in $C$ by $P_1, P_2, P_3 \cdots$. Every set $T_i+P_i$ is a continuous curve by Lemma $A$. Also it is evident that no continuous curve $T_i+P_i$ contains a simple closed curve. Then if $B_i$ denotes the set of all the branch points of $T_i+P_i$, $B_i$ is countable by Theorem 11. Then if $B$ denotes the set of points consisting of all the points of the sets $B_i$ $(i = 1, 2, 3, \cdots)$ together with all the points $P_i$, $B$ is a countable set being formed of a countable number of countable sets. The set $B$ contains all the branch points of $M$ and therefore the set of all the branch points of $M$ is countable.

## IV. On continuous (1,1) correspondences between continuous curves.*

In this section we give a very general set of conditions under which two continuous curves, which are acyclic or each of which contains just one simple closed curve, can be put into continuous (1,1) correspondence. We also give a set of conditions under which a correspondence between subsets of two acyclic continuous curves can be extended to a continuous (1,1) correspondence between the continuous curves themselves.† From this we can formulate the corresponding theorem for the case of the continuous curves containing just one simple closed curve but the conditions are necessarily so numerous as to make the theorem of little interest.

* Presented to the Society May 1, 1926.

† We say that a correspondence $C$ between two subsets, $N_1$ and $N_2$, of two sets, $M_1$ and $M_2$, respectively *can be extended* to the sets themselves provided there exists a correspondence $C'$ between $M_1$ and $M_2$ such if $P$ is a point of $N_1$, $C(P) = C'(P)$.

The conditions given are conditions on the subsets, whose correspondence we desire to extend, and the branch points of the continuous curves. Thus we see from Theorems 11 and 12 that the conditions restrict only a countable set of points in addition to the subsets themselves.

THEOREM 13. *Let $M_1$ and $M_2$ be acyclic continuous curves and $C_1$ be a* (1,1) *correspondence between $N_1$ and $N_2$, subsets of $M_1$ and $M_2$ respectively. Then this correspondence can be extended to a continuous* (1,1) *correspondence $T$ between the continuous curves provided there exists a* (1,1) *correspondence $C_2$ between $B_1$ and $B_2$, the branch points of $M_1$ and $M_2$, and $C_1$ and $C_2$ satisfy the following conditions:*

(1) *if $P$ is a point of $N_1$ and $B_1$ then $C_1(P) = C_2(P)$ and if $P'$ is a point of $N_2$ and $B_2$ then $C_1^{-1}(P') = C_2^{-1}(P')$,*

(2) *if $P$ is a point of $B_1$, $P$ and $C_2(P)$ are branch points of the same order,*

(3) *if $P$ is a point of $N_1$, $P$ is an ordinary cut-point if, and only if, $C_1(P)$ is an ordinary cut-point,*

(4) *if $P_1$, $P_2$, $P_3$ are points of $K_1(K_i = N_i + B_i)$ then $P_1$, $P_2$, $P_3$ lie on an arc of $M_1$ and $P_2$ is between $P_1$ and $P_3$ if, and only if, $\bar{C}(P_1)$, $\bar{C}(P_2)$, $\bar{C}(P_3)$ ($\bar{C}$ is the correspondence between $K_1$ and $K_2$ which is the same as $C_1$ for points of $N_i$ and the same as $C_2$ for points of $B_i$) lie on an arc of $M_2$ and $\bar{C}(P_2)$ is between $\bar{C}(P_1)$ and $\bar{C}(P_3)$,*

(5) *a sequence of points of $K_1$, $P_1$, $P_2$, $P_3$, $\cdots$, has a sequential limit point $Q$ if, and only if, $\bar{C}(P_1)$, $\bar{C}(P_2)$, $\bar{C}(P_3)$, $\cdots$ has a sequential limit point $R$,*

(6) *if either $Q$ or $R$ is an ordinary cut-point then the other is also,*

(7) *the point $Q$ belongs to $N_1$ if, and only if, $R$ belongs to $N_2$.*

It will be noticed that $M_1$ and $M_2$ enter into the hypothesis of the theorem in a completely symmetrical manner. Thus any statement which is proved concerning $M_1$ or any of its subsets gives a corresponding statement concerning $M_2$ or corresponding subsets of $M_2$. This being the case we will usually omit the statement concerning $M_2$. Before proceeding to the proof of the theorem we will state several lemmas which follow out of the hypothesis of the theorem.

LEMMA B. *Under the hypothesis of Theorem* 13, *if $Q$ is a point of $K_1$ then $R$ is a point of $K_2$ and $\bar{C}(Q) = R$.*

*Proof.* We will show that the assumption $\bar{C}(Q) = Q' \neq R$ leads to a contradiction, thus proving the lemma, for $Q' = R$ implies $R$ is a point of $K_2$. If $Q$ belongs to $N_1$, $R$ belongs to $N_2$ by condition (7). If $Q$ belongs to $B_1$, by condition (6) $R$ belongs to $B_2$ or is an end-point of $M_2$. Thus $R$ either belongs to $K_2$ or is an end-point of $M_2$. We may suppose that the points $Q$, $P_1$, $P_2$, $P_3$, $\cdots$ are distinct points as the other cases may be easily settled or reduced to this. Then the points $R$, $P_1'$, $P_2'$, $P_3'$, $\cdots$,

($P_i' = \bar{C}(P_i)$) are distinct. For every $i$ let $b_i'$ be the first point which the arc of $M_2$ from $P_i'$ to $R$ has in common with the arc $RQ'$ of $M_2$. If $R$ is an end-point no $b_i'$ can be $R$.* If $R$ is a point of $K_2$, then there exists a point $\bar{R}$ in $M_1$ such that $\bar{C}(\bar{R}) = R$ and $\bar{R}$ is distinct from $Q$, If for infinitely many values of $i$, $b_i' = R$, then for every such $i$ we have the order $P_i'RQ'$ in $M_2$ and consequently (Condition (4)), $P_i\bar{R}Q$ in $M_1$ and as $M_1$ is connected im kleinen $Q$ could not be the sequential limit point of $[P_i]$. Thus in either case not more than a finite number of the points $b_i'$ are $R$. It follows that there exists a sequence of positive integers, $n_1, n_2, n_3, \cdots (n_i < n_{i+1})$, such that every point $b_{n_i}'$ lies on $\overset{\frown}{RQ'}$, $b_{n_i}' \neq b_{n_j}'$ unless $i = j$ and we have the order $Q'b_{n_1}'b_{n_2}'b_{n_3}' \cdots R$. For every $i$, $b_{n_1}'$ lies between $Q'$ and $P_{n_i}'$. The point $b_{n_1}'$ is a point of $B_2$ and thus a point $b_{n_1}$, such that $\bar{C}(b_{n_1}) = b_{n_1}'$, exists and lies between $Q$ and $P_{n_i}$ for every $i$ (Condition (4)). As $M_1$ is connected im kleinen $Q$ cannot be the sequential limit point of the sequence $[P_{n_i}]$. Thus the assumption that $Q' \neq R$ has led to a contradiction.

LEMMA C. *Under the hypothesis of Theorem* 13, *$Q$ is a branch point, an end-point or an ordinary cut-point if, and only if, $R$ is also.*

*Proof.* By condition (6) if either $Q$ or $R$ is an ordinary cut-point, the other is also. Then the only possible case contradictory to the lemma is that either $Q$ or $R$, say $Q$, may be a branch while the other is an end-point. But in this case, by Lemma B, $R$ is a point of $K_2$ and $Q' = R$. As $Q'$ is a branch point and $Q' = R$, the point $R$ must be a branch point also.

LEMMA D. *Under the hypothesis of Theorem* 13, *if $P$ is a point of $N_1$ then $P$ and $C_1(P)$ are either both end-points, both branch points or both ordinary cut-points.*

The lemma follows immediately from conditions (1) and (3) of the hypothesis of the theorem.

*Proof of Theorem* 13. If $P$ is any point of $K_1$ we denote $\bar{C}(P)$ by $P'$. We will define $T$ as follows:

If $P$ is any point of $K_1$, then $T(P) = P'$. If $P$ is not a point of $K_1$ but is a limit point of $K_1$, then $K_1$ contains a sequence, $P_1, P_2, P_3, \cdots$, of points which has $P$ as its sequential limit point. In this case we define $T(P)$ to be the sequential limit point of $P_1', P_2', P_3', \cdots$ which exists by condition (5) and, by Lemma B, $T(P)$ is not a point of $K_2$.

The transformation $T$ has been defined for all points and limit points of $K_1$, a closed set $U$. Let $P$ be any other point of $M$ and $E_{1P}E_{2P}$ be a maximal arc of $M_1$ containing $P$. Let $p_1$ and $p_2$ be the first points of

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* Cf. Theorem 4. Cf. also R. L. Wilder, loc. cit., Theorem 14, and R. L. Moore, *Concerning the cut-points etc.*, loc. cit., Theorem C.

the sets $U+E_{1P}$ and $U+E_{2P}$ on the arcs from $P$ to $E_{1P}$ and from $P$ to $E_{2P}$ respectively. Neither $p_1$ nor $p_2$ is $P$ unless $P$ is itself an end-point of $M_1$. The arc $p_1p_2$ may be one of three types: (1) $p_1 = E_{1P}$ and $p_2 = E_{2P}$, (2) $p_1$ and $p_2$ both belong to $U$, (3) either $p_1$ or $p_2$ is an end-point of $M_1$ and the other is a point of $U$.

In case (1) $B_1$ and $B_2$ are vacuous and $M_1$ and $M_2$ reduce to simple continuous arcs. Here all seven conditions of the hypothesis are satisfied and Theorem 13 is obviously true since any two arcs may be put into continuous (1,1) correspondence.

In case (2) we will show that there is no point of $T(U)$ on the arc of $M_2$ between $T(p_1)$ and $T(p_2)$. Suppose $T(U)$ has a point $Q'$ between $T(p_1)$ and $T(p_2)$. If $Q'$ is not a point of $K_2$ it is a limit point of $K_2$. Let $\varepsilon>0$ be a number smaller than both $d(T(p_1),Q')$ and $d(T(p_2),Q')$. As $M_2$ is connected im kleinen there exists a number $\delta_\varepsilon>0$ such that every point of $M_2$ within a distance $\delta_\varepsilon$ of $Q'$ can be joined to $Q'$ by an arc of $M_2$ every point of which is a distance from $Q'$ less than $\varepsilon$. As $Q'$ is a limit point of $K_2$, $K_2$ contains a point $u$ such that $d(u,Q')<\delta_\varepsilon$. The first point which the arc from $u$ to $Q'$ has in common with the arc from $T(p_1)$ to $T(p_2)$ must be a point of $K_2$ and lie between $T(p_1)$ and $T(p_2)$. Thus we may assume that $Q'$ is a point of $K_2$ for, if not, we may replace it by this new point which belongs to $K_2$.

Let $Q$ denote the point of $M_1$ such that $\bar{C}(Q) = Q'$; $Q$ exists since $Q'$ is a point of $K_2$. The first point the arc from $Q$ to $p_1$ has in common with the arc $p_1p_2$ is either $p_1$ or $p_2$, say $p_2$. If $p_2$ is not a point of $K_1$ then we may replace it in the argument by a point $\bar{p}_2$ of $K_1$ so that the original order is preserved, as follows:

Let $\eta$ be a positive number such that the points $Q$ and $p_1$ lie in the exterior of the circle of radius $\eta$ with center at $p_2$ and such that the entire arc of $M_2$ from $Q'$ to $T(p_1)$ lies in the exterior of the circle of radius $\eta$ with center at $T(p_2)$. Since $M_1$ and $M_2$ are connected im kleinen there exists a positive number $\delta_\eta$ such that every point of $\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$ within a distance $\delta_\eta$ of $\begin{Bmatrix} p_2 \\ T(p_2) \end{Bmatrix}$ can be joined to $\begin{Bmatrix} p_2 \\ T(p_2) \end{Bmatrix}$ by an arc of $\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$ every point of which is within a distance $\eta$ of $\begin{Bmatrix} p_2 \\ T(p_2) \end{Bmatrix}$. There exists a point $q$ of $K_1$ such that $d(q,p_2)<\delta_\eta$ and $d(T(q),T(p_2))<\delta_\eta$. Let $\bar{p}_2$ be the first point the arc of $M_1$ from $q$ to $p_2$ has in common with the arc $Qp_1$. Then $\bar{p}_2$ is a point of $K_1$ and lies between $Q$ and $p_1$. It follows that $T(\bar{p}_2)$ is a point of $K_2$ and $Q'$ is between $T(p_1)$ and $T(\bar{p}_2)$. Similarly if $p_1$ is not a point of $K_1$ we may determine a point $\bar{p}_1$ of $K_1$ preserving

order relations. If $p_1$ or $p_2$ are points of $K_1$ let $p_1 = \overline{p}_1$ and $p_2 = \overline{p}_2$; and then in any case we have three points $\overline{p}_1, Q, \overline{p}_2$ of $K_1$ so that $\overline{p}_2$ is between $Q$ and $\overline{p}_1$ and three corresponding points of $K_2$ such that $Q'$ is between $\overline{p}_1'$ and $\overline{p}_2'$. But this is contrary to condition (4) and thus the assumption that $T(U)$ contains a point between $T(p_1)$ and $T(p_2)$ has led to a contradiction. Then we may set up any continuous (1,1) correspondence between the arcs $p_1p_2$ and $T(p_1)T(p_2)$, such that $p_1$ and $T(p_1)$ and $p_2$ and $T(p_2)$ correspond, and call this correspondence $T$ for the points of these arcs.

In case (3) we will suppose that $p_1$ is the end-point of $M_1$ and $p_2$ is the point of $U$. We will consider three possibilities according as $p_2$ is (a) an end-point, (b) an ordinary cut-point, or (c) a branch point.

In (a), $M_1$ and $M_2$ reduce to simple continuous arcs and $N_1$ and $N_2$ to a single end-point each. Theorem 13 is here obvious.

In (b), $M_2 - T(p_2)$ is the sum of two trees $D_1$ and $D_2$. If neither $D_1$ nor $D_2$ is an arc minus one of its end-points, each contains a branch point of $M_1$, $b_1'$ and $b_2'$ respectively, and $T(p_2)$ is between $b_1'$ and $b_2'$. Let $b_1$ and $b_2$ be the points of $M_1$ such that $C_2(b_1) = b_1'$ and $C_2(b_2) = b_2'$. Then $p_2$ is not between $b_1$ and $b_2$. If $p_2$ is a point of $K_1$, then by condition (4) we have a contradiction immediately, and if not, then as we have done in case (2) above we may replace $p_2$ by a point of $K_1$ without disturbing the order arrangement and then obtain a contradiction by condition (4). Therefore at least one of the two trees of $M_2 - T(p_2)$ is a simple arc. If both are simple arcs then $M_1$ and $M_2$ reduce to simple arcs and $N_1$ and $N_2$ each to a single cut-point and Theorem 13 is evidently true. If just one of the two trees of $M_2 - T(p_2)$ is a simple arc minus one of its end-points, let the end-point which the tree contains be $T(p_1)$ and we may set up any continuous (1,1) correspondence between the two arcs $p_1p_2$ and $T(p_1)T(p_2)$, such that $p_1$ and $T(p_1)$ and $p_2$ and $T(p_2)$ correspond, and call this correspondence $T$ for these two arcs.

In (c), we proceed as follows: The branch points $p_2$ and $p_2'$ are of the same order by condition (2). We shall show that $M_1 - p_2$ and $M_2 - p_2'$ have the same number (if it is finite, and if one set is infinite then the other is also) of trees that are simple arcs minus one end-point and contain no point of $N_1$ or $N_2$. First, let us suppose that $p_2$ is of order $n$. Every tree of $M_1 - p_2$ which is not of the above type contains at least one point of $K_1$. Let $T_1, T_2, \cdots, T_k$ $(0 \leqq k \leqq n)$ be the trees of $M_1 - p_2$ which are not of this type. We shall show that the number of trees $M_2 - p_2'$ not of this type is greater than or equal to $k$. For $k$ equal 0 or 1 this is evident. Consider $k > 1$. For each $i$ $(0 < i \leqq k)$, $T_i$ contains a point $q_i$ of $K_1$. For every $i$ and $j$ $(i \neq j)$, $p_2$ lies between $q_i$ and $q_j$. Then $p_2'$ lies between $q_i'$ and $q_j'$ by condition (4) and thus $q_i'$ and $q_j'$ lie in different trees of

$M_2-p_2'$. But as a tree of $M_2-p_2'$ which contains a point $q_i'$ cannot be a tree of the above mentioned type, the number of trees of $M_2-p_2'$ not of this type is greater than or equal to $k$. Conversely in exactly the same manner we may show that $k$ is greater than or equal to the number of trees of $M_2-p_2'$ which are not of the above type. Therefore $M_1-p_2$ and $M_2-p_2'$ each contain exactly $n-k$ trees that are simple arcs minus one end-point and contain no point of $N_1$ or $N_2$. The cases where the order of $p_2$ is of the form $n\omega$ or $\omega\omega$ may be treated in a similar manner. Thus in any case the number of trees of $M_1-p_2$ and $M_2-p_2'$ that are simple arcs minus one end-point and contain no point of $N_1$ or $N_2$, as the case may be, is the same. We may arbitrarily set up a (1,1) correspondence between the trees of $p_2$ which are of this type and those of $p_2'$ which are of this type and we may select any continuous (1,1) correspondence between corresponding trees (arcs), such that $p_2$ and $p_2'$ correspond, and call this $T$ for these arcs.

We have now defined the correspondence $T$ for every point of $M_1$. We have yet to show that $T$ is a (1,1) correspondence between $M_1$ and $M_2$ and that $T$ is continuous.

To every point of $M_1$ there corresponds under $T$ just one point of $M_2$. It is easy to see that $M_2$ cannot contain a point which corresponds to no point of $M_1$. Now suppose two points $P_1$ and $P_2$, of $M_1$ correspond to a single point $P$ of $M_2$. If $P$ belongs to $K_2$, then $P_1$ and $P_2$ belong to $K_1$ and we have a contradiction for $\overline{C}$ is a (1,1) correspondence and for points of $K_1$, $T=\overline{C}$. If $P$ is not a point of the set $K_2$ but is a limit point of it, then $P_1$ and $P_2$ do not belong to the set $K_2$ but are limit points of it. Let $P_{11}, P_{12}, P_{13}, \cdots$ and $P_{21}, P_{22}, P_{23}, \cdots$ be sequences of points of $K_1$ which have $P_1$ and $P_2$ respectively as sequential limit points. By definition of $T$, $P$ is the sequential limit point of both $P_{11}', P_{12}', P_{13}', \cdots$ and $P_{21}', P_{22}', P_{23}', \cdots$ and hence of $P_{11}', P_{21}', P_{12}', P_{22}', \cdots$. But as the sequence $P_{11}, P_{21}, P_{12}, P_{22}, \cdots$ has no sequential limit point this is a contradiction of condition (5). If $P$ is not a point of $T(U)$, then neither $P_1$ nor $P_2$ are points of $U$. As above we define the points $p_1$ and $p_2$ such that $P$, but no point of $T(U)$, is between $p_1$ and $p_2$ and $p_1$ and $p_2$ are points of $T(U)$ or end-points of $M_2$. Correspondingly we obtain points $p_{11}, p_{12}, p_{21}, p_{22}$ which are points of $U$ or end-points of $M_1$ such that $P_1$, but no point of $U$, is between $p_{11}$ and $p_{12}$, and $P_2$, but no point of $U$, is between $p_{21}$ and $p_{22}$, $T(p_{11}) = T(p_{21}) = p_1$, $T(p_{12}) = T(p_{22}) = p_2$. Then either $p_{11} = p_{21}$ and $p_{12} = p_{22}$ and thus there are two arcs of $M_1$ from $p_{11}$ to $p_{12}$ which is impossible since $M_1$ is acyclic, or two points of $U$ correspond to one point of $T(U)$ which we have already shown impossible. Therefore $T$ is a (1, 1) correspondence between $M_1$ and $M_2$.

We shall show that if $P$ is a limit point of $L$, a subset of $M_1$, then $T(P)$ is a limit point of $T(L)$. The converse follows from the symmetry. Let $P_1, P_2, P_3, \cdots$ be a sequence of distinct points of $L$ which has $P$ as its sequential limit point. If infinitely many of the points of the sequence, $P_{k_1}, P_{k_2}, P_{k_3}, \cdots$, belong to $K_1$ then $P''$* is the sequential limit point of the sequence, $P''_{k_1}, P''_{k_2}, P''_{k_3}, \cdots$, by Lemma B or definition of $T$. In this case $P''$ is a limit point of $L''$. Suppose infinitely many, $P_{n_1}, P_{n_2}, P_{n_3}, \cdots$, of the points of the sequence are points of $U$ but not of $K_1$. For each $i$, there exists a point $Q_{n_i}$ of $K_1$ such that $d(Q_{n_i}, P_{n_i}) < 2^{-i}$ and $d(Q''_{n_i}, P''_{n_i}) < 2^{-i}$. The point $P$ is the sequential limit point of $[P_{n_i}]$ and consequently of $[Q_{n_i}]$. By condition (5), the sequence $[Q''_{n_i}]$ has a sequential limit point and, by definition of $T$ and Lemma B, this point is $P''$. Since $d(Q''_{n_i}, P''_{n_i}) < 2^{-i}$, $P''$ is the sequential limit point of the sequence $[P''_{n_i}]$. In this case then $P''$ is a limit point of the set $L''$.

If neither of the above two conditions exist, infinitely many, $P_{m_1}, P_{m_2}, P_{m_3}, \cdots$, of the points of the sequence $[P_i]$ must belong to $M_1 - U$. Each such point lies on an arc $p_{m_i 1} P_{m_i} p_{m_i 2}$ and each point $P''_{m_i}$ on an arc $p''_{m_i 1} P''_{m_i} p''_{m_i 2}$ as these arcs are defined above in the definition of $T$. One of the two points, $p_{m_i 1}, p_{m_i 2}$, is either a point or a limit point of $K_1$ and therefore for every $i$ there exists in $K_1$ a point $X_{m_i}$ such that

$$d(X_{m_i}, P_{m_i}) < d_i,$$

where $d_i = 2^{-i} + d(\text{arc } p_{m_i 1} p_{m_i 2})$, and such that for the corresponding point $X''_{m_i}$ it is true that

$$d(X''_{m_i}, P''_{m_i}) < d'_i,$$

where $d'_i = 2^{-i} + d(\text{arc } p''_{m_i 1} p''_{m_i 2})$. But

$$\lim_{i \to \infty} d_i = 0 \text{ and } \lim_{i \to \infty} d'_i = 0,\dagger$$

therefore as $P$ is the sequential limit point of the sequence $[P_{m_i}]$ it is the sequential limit point of the sequence $[X_{m_i}]$. By condition (5), the sequence $[X''_{m_i}]$ has a sequential limit point which is $P''$ by definition of $T$ and Lemma B. Then, since $\lim_{i \to \infty} d'_i = 0$, $P''$ is the sequential limit point of the sequence $[P''_{m_i}]$. Thus in any case if $P$ is a limit point of $L$, a subset of $M_1$, $P''$ is a limit point of $L''$.

We have defined a correspondence $T$ between $M_1$ and $M_2$ such that for any point $P$ of $N_1$, $T(P) = C_1(P)$, and have shown that $T$ is a (1, 1)

* If $P$ is a point or set of points of $M_1$, $P'$ denotes the point or set of points $T(P)$.

† Cf. my paper, Bull. Amer. Math. Soc., vol. 33 (1927), p. 201-208, Theorem 4. Cf. also R. L. Wilder, loc. cit., Theorem 8.

correspondence and is continuous, which completes the proof of the theorem.

*Independence examples.* Condition (1) of Theorem 13 is simply that the two correspondences $C_1$ and $C_2$ be consistent. The other conditions are independently necessary as shown by the following examples. In each example every condition except the one mentioned is satisfied and it is obvious that no continuous (1, 1) correspondence between $M_1$ and $M_2$ exists. In each example except (3) and (7), the sets $N_1$ and $N_2$ are considered vacuous. We take the continuous curve $M_1$ in a Euclidean $xy$-plane and $M_2$ in a Euclidean $x'y'$-plane.

Condition (2). The set $M_1$ consists of the points of the interval from $(1, 0)$ to $(-1, 0)$ together with those of the interval from $(0, 1)$ to $(0, 0)$. Let $M_2$ be the interval (in the $x'y'$-plane) from $(1, 0)$ to $(-1, 0)$ together with the interval from $(0,1)$ to $(0, -1)$. Also to show the necessity of distinguishing various types of branch points of infinite order we give the example: Let $L_1$ consist of the intervals from $(0, 0)$ to $(-2, 0)$ and from $(-1, \frac{1}{2})$ to $(-1, 0)$. Let $L_{i+1}$ be the set obtained from $L_i$ by shrinking $L_i$ to one-half its size about the point $(0, 0) = P$ and then rotating it about $P$ through an angle of $2^{-i}\frac{\pi}{2}$ in a clockwise direction. Let $M_1 = \sum_{i=1}^{\infty} L_i$, and let $M_2$ be the same set of points in the $x'y'$-plane as $M_1$ in the $xy$-plane with the addition of the interval from $(1, 0)$ to $(0, 0)$. The point $P$ is a branch point of order $0\,\infty$ while the point $P'(= (0, 0)$ in the $x'y'$-plane) is of order $1\,\infty$.

Condition (3). Let $M_1$ be the interval from $(1, 0)$ to $(-1, 0)$ and $N_1$ be the point $(0, 0)$. Let $M_2$ be the interval from $(1, 0)$ to $(-1, 0)$ in the $x'y'$-plane and $N_2$ be the point $(1, 0)$.

Condition (4). Let $M_1$ consist of the intervals $(2, 0)$ to $(-2, 0)$, $(1, 1)$ to $(1, -1)$, $(-1, 1)$ to $(-1, -1)$ and $(0, 1)$ to $(0, 0)$. Let $M_2$ consist of the intervals in the $x'y'$-plane from $(2, 0)$ to $(-2, 0)$, $(1, 1)$ to $(1, 0)$, $(0, 1)$ to $(0, -1)$ and $(-1, 1)$ to $(-1, -1)$.

Condition (5). Let $M_1$ consist of the intervals $(2, 0)$ to $(-2, 0)$ and, for every positive integer $i$, from $(1/i, 1/i)$ to $(1/i, 0)$ and from $(-1/i, 1/i)$ to $(-1/i, 0)$. Let $M_2$ consist of the intervals in its plane from $(3, 0)$ to $(-3, 0)$ and, for every positive integer $i$, from $(1+1/i, 1/i)$ to $(1+1/i, 0)$ and from $(-1-1/i, 1/i)$ to $(-1-1/i, 0)$.

Condition (6). (a) Let $M_1$ be the sum of the intervals from $(0, 0)$ to $(2, 0)$ and, for every positive $i$, from $(1/i, 1/i)$ to $(1/i, 0)$. In this case $Q(= (0, 0))$ is an end-point. Let $M_2$ be the same as $M_1$ in its plane with the addition of the interval from $(0, 0)$ to $(-1, 0)$. In $M_2$ the point $R(= (0, 0))$ is an ordinary cut-point.

(b) Let $M_1$ be the same set of points in its plane as $M_2$ in (a) except with the addition of the interval from (0, 1) to (0, 0). In $M_1$ the point $Q = (0, 0)$ is a branch point. Let $M_2$ be the same as $M_2$ in (a) except for the addition of the interval from $(-\frac{1}{2}, 1)$ to $(-\frac{1}{2}, 0)$. In $M_2$ the point $R = (0, 0)$ is an ordinary cut-point.

Condition (7). Let $M_1$ be the same set of points in its plane as $M_2$ in 6(a). Let $N_1$ be the point (0, 0). Let $M_2$ be the same as the $M_2$ of 6(a). Let $N_2$ be the point $(-\frac{1}{4}, 0)$.

As a corollary of Theorem 13, we have, in case $N_1$ and $N_2$ are vacuous:

THEOREM 14. *There exists a continuous* (1, 1) *correspondence between two acyclic continuous curves $M_1$ and $M_2$ providing there exists a* (1, 1) *correspondence $C$ between $B_1$ and $B_2$, the branch points of $M_1$ and $M_2$, satisfying the following conditions:*

(1) *$P$ and $C(P)$ are branch points of the same order,*

(2) *three points of $B_1$, $P_1$, $P_2$, $P_3$, lie on an arc of $M_1$ and $P_2$ is between $P_1$ and $P_3$ if, and only if, $C(P_1)$, $C(P_2)$, $C(P_3)$ lie on an arc of $M_2$ and $C(P_2)$ is between $C(P_1)$ and $C(P_3)$,*

(3) *a sequence of points of $B_1$, $P_1$, $P_2$, $P_3$, $\cdots$, has a sequential limit point $Q$ if, and only if, the sequence, $C(P_1)$, $C(P_2)$, $C(P_3)$, $\cdots$, has a sequential limit point $R$,*

(4) *if either $Q$ or $R$ is an ordinary cut-point then the other is also.*

*Definition.* Suppose $P$ and $P'$ are cyclic branch points of two continuous curves $M$ and $M'$ each of which contains just one simple closed curve, $C$ and $C'$. Let $\sum D_P$ and $\sum D_{P'}$ be the sets of trees of $M - C$ and $M' - C'$ which have $P$ and $P'$ as roots. Then the points $P$ and $P'$ will be said to be *similar* providing there exists a continuous (1, 1) correspondence $T$ between $P + \sum D_P$ and $P' + \sum D_{P'}$ such that $T(P) = P'$.

The sets $P + \sum D_P$ and $P' + \sum D_{P'}$ are acyclic continuous curves* and thus the conditions under which two cyclic branch points are similar are given in Theorem 13 in which the sets $N_1$ and $N_2$ are the single points $P$ and $P'$.

THEOREM 15. *If $M_1$ and $M_2$ are continuous curves each of which contains just one simple closed curve, $C_1$ and $C_2$, then there exists a continuous* (1, 1) *correspondence $T$ between $M_1$ and $M_2$ provided there exists a* (1, 1) *correspondence $T'$ between $B_1$ and $B_2$, the cyclic branch points of $M_1$ and $M_2$, satisfying the following conditions:*

(1) *$P$ and $T'(P)$ are similar,*

(2) *the points $P_1$ and $P_2$ separate $P_3$ and $P_4$ on $C_1$ if, and only if, the points $T'(P_1)$ and $T'(P_2)$ separate $T'(P_3)$ and $T'(P_4)$ on $C_2$,*

* Every closed and connected subset of a continuous curve containing only a finite number of simple closed curves is a continuous curve. Cf. abstract of my paper, *Continuous curves of a certain type*, Bull. Amer. Math. Soc., vol. 32 (1926), pp. 307–8.

(3) *the sequence* $P_1, P_2, P_3, \cdots$ *of points of* $B_1$ *has a sequential limit point* $Q$ *if, and only if, the sequence* $T'(P_1), T'(P_2), T'(P_3), \cdots$ *has a sequential limit point* $R$,

(4) *the point* $Q$ *is a point of* $B_1$ *if, and only if,* $R$ *is a point of* $B_2$.

*Proof.* We shall define $T$ for points of $M_1$ as follows:

The points of $B_1$ are countable and may be ordered in a sequence $p_1, p_2, p_3, \cdots$ and let $p_i' = T'(p_i)$. Let $E_i = p_i + \sum D_{p_i}$ and $E_i' = p_i' + \sum D_{p_i'}$ where $\sum D_{p_i}$ and $\sum D_{p_i'}$ denote the sets of trees of $M_1 - C_1$ and $M_2 - C_2$ which have $p_i$ and $p_i'$ as roots. By condition (1) there exists a continuous (1, 1) correspondence $T_i$ between $E_i$ and $E_i'$ such that $T_i(p_i) = p_i'$. For all points of $E_i$, let $T = T_i$.

Let $P$ be any point of $C_1 - B_1$ which is a limit point of $B_1$. Then $P$ is the sequential limit point of a sequence of points $b_1, b_2, b_3, \cdots$ of $B_1$. By condition (3), the sequence $b_1', b_2', b_3', \cdots$ $(b_i' = T'(b_i))$ has a sequential limit point $Q$. We define $T(P) = Q$. By condition (4), $Q$ is a point of $C_2 - B_2$. Let $L$ denote the closed set of points for which $T$ has been defined and let $L'' = T(L)$.

Let $P$ be any point of $M_1 - L$. Since $L$ is closed there exist, unless $B_1$ is vacuous, two points $N_{1P}$ and $N_{2P}$ of $L$ such that one, $C_P$, of the two arcs of $C_1$ from $N_{1P}$ to $N_{2P}$ contains $P$ but no point of $L$ except $N_{1P}$ and $N_{2P}$. In case $B_1$ is vacuous, the curves $M_1$ and $M_2$ are simple closed curves and our theorem is evident. If $N_{1P}$ and $N_{2P}$ are not distinct, $B_1$ consists of a single point and then our theorem is easily proved. Using methods similar to those of Theorem 13 we may prove that at least one of the two arcs from $N_{1P}''$ to $N_{2P}''$* contains no point of $L''$ except $N_{1P}''$ and $N_{2P}''$. If both of the two arcs have this property then $B_1$ and $B_2$ each consist of just two points and we may easily complete the proof for this case. Excepting this simple case, let $\bar{C}_P$ be the arc of $C_2$ from $N_{1P}''$ to $N_{2P}''$ which contains no point of $L''$ with the exception of these two points. For points of the arc $C_P$ we define $T$ to be any continuous (1, 1) correspondence between $C_P$ and $\bar{C}_P$ such that $N_{1P}$ and $N_{2P}$ correspond to $N_{1P}''$ and $N_{2P}''$ respectively.

We have now defined $T$ for every point of $M_1$. It is evident from the definition that no two distinct points of $M_1$ can correspond to the same point of $M_2$. And using methods similar to those of Theorem 13 we may show that to each point of $M_2$ there corresponds at least one point of $M_1$. Thus $T$ is a (1, 1) correspondence between $M_1$ and $M_2$.

It remains to prove that $T$ is continuous. Suppose $P$ is a limit point of a set of points $N$, a subset of $M_1$. We shall prove that $P''$ is a limit

* If $K$ is a point or set of points of $M_1$, then $K'$ denotes $T'(K)$ and $K''$ denotes $T(K)$.

point of $N''$. The set $N$ contains a sequence of distinct points which has $P$ as its sequential limit point and by selecting a subsequence of this we obtain a sequence of points, $p_1, p_2, p_3, \cdots$, of $N$ which has $P$ as its sequential limit point and such that the sequence $p_1'', p_2'', p_3'', \cdots$ of points of $N''$ has a sequential limit point $Q$. If $P'' = Q$, our statement is proved.

If $P$ is a point of $M_1 - C_1$, then all but a finite number of the points of the sequence $[p_i]$ must lie in the same tree of $M_1 - C_1$ as $P$ and, since $T$ is continuous for any single tree by definition, $P'' = Q$. If $P$ is a point of $C_1 - L$, all but a finite number of the points of the sequence $[p_i]$ must lie on the interval $N_{1P}N_{2P}$ and, since $T$ is continuous on any such interval, $P'' = Q$. The remaining case is that $P$ is either a point or a limit point of $B_1$.

If infinitely many of the points $p_i$ lie in trees which have the same root point $r$, then $r = P$ for $r + \sum D_r$ is closed and has only the point $r$ in common with $C_1$. Since $T$ is continuous in a single set $r + \sum D_r$, $P'' = Q$. If infinitely many points $p_i$ lie in $M_1 - C_1$ but no single set $r + \sum D_r$ contains more than a finite number of the points, then there exists a sequence of integers $n_1, n_2, n_3, \cdots$ and a sequence of distinct cyclic branch points of $M_1, r_1, r_2, r_3, \cdots$, such that, for every $i$, $r_i + \sum D_{r_i}$ contains the point $p_{n_i}$. We see easily that $P$ is the sequential limit point of the sequence $[r_i]$, and that $Q$ is the sequential limit point of the sequence $[r_i']$. Then we may replace the sequences $[p_i]$ and $[p_i'']$ by $[r_i]$ and $[r_i']$ in this case in showing that $P'' \neq Q$ is impossible. Then we need only to consider the case where $P$ is a point or a limit point of $B_1$ and every point of the sequence $[p_i]$ belongs to $C_1$. Let $\bar{Q}$ denote the point of $M_1$ such that $T(\bar{Q}) = Q$, and consider the following cases:

*Case I. Infinitely many of the points $p_i$ are points of $B_1$.* Then we may select a subsequence $[p_{k_i}]$ all of whose points belong to $B_1$. If $P$ is a limit point of $B_1$ but does not belong to $B_1$, then $P''$ was defined as the sequential limit point of a sequence $b_1', b_2', b_3', \cdots$ of $B_2$ and $P'' \neq Q$ violates condition (3) since the sequence $p_{k_1}, b_1, p_{k_2}, b_2, \cdots$ has the sequential limit point $P$ while the corresponding sequence of $B_2$ has no sequential limit point. Now suppose $P$ belongs to $B_1$. By condition (4), $Q$ belongs to $B_2$ and let $\bar{r}$ be any point of $B_1$ distinct from $P$ and $\bar{Q}$. Since $P$ is the sequential limit point of the sequence $[p_{k_i}]$ and $Q$ is the sequential limit point of $[p_{k_i}'']$, there exists an integer $t_1$ such that if $i > t_1$, $p_{k_i}$ lies on $\underset{\frown}{P\bar{Q}}$ or $\underset{\frown}{P\bar{r}}$ and $p_{k_i}''$ lies on $\underset{\frown}{P'Q}$ or $\underset{\frown}{Q\bar{r}'}$.* If for any value of $i > t_1$, $p_{k_i}$ lies on $\underset{\frown}{P\bar{r}}$ then $P$ and $\bar{r}$ separate $\bar{Q}$ and $p_{k_i}$ on $C_1$ while $P'$ and $\bar{r}'$

* If $A, B, C$ are points of a simple closed curve $\gamma$, $AB$ denotes the arc of $\gamma$ from $A$ to $B$ which does not contain $C$. Arcs $AC$ and $BC$ have similar meanings.

do not separate $Q$ and $p'_{k_i}$ on $C_2$, which contradicts condition (2). The same contradiction arises if for any value of $i > t_1$, $p'_{k_i}$ lies on $\overset{\frown}{Q\bar{r}'}$. Then for every $i > t_1$, $p_{k_i}$ lies on $\overset{\frown}{P\bar{Q}}$ and $p'_{k_i}$ lies on $\overset{\frown}{QP'}$. Let $s$ be an integer greater than $t_1$. There exists an integer $t_2 > t_1$ such that if $i > t_2$, $p_{k_i}$ lies on the segment of the arc $Pp_{k_s}$ of $C_1$ which does not contain $\bar{Q}$ or $\bar{r}$. From condition (2), it follows that $p'_{k_i}$ for $i > t_2$ must lie on the segment of the arc $P'p'_{k_i}$ which does not contain $Q$ or $\bar{r}'$. But this arc does not contain $Q$ and so $Q$ cannot be the sequential limit point of the sequence $[p'_{k_i}]$. Thus under the assumption that $P'' \neq Q$, in case infinitely many of the points $p_i$ belong to $B_1$, we are led to a contradiction.

*Case* II. *Infinitely many of the points $p_i$ are limit points of $B_1$ but do not belong to $B_1$.* Then there exists a subsequence $[p_{k_i}]$ all of whose points are limit points of $B_1$ but do not belong to $B_1$. We have just shown that if $X$ is the sequential limit point of a sequence of points of $B_1$ then $X''$ is a limit point of the corresponding sequence in $B_2$. Each point of $[p_{k_i}]$ is a limit point of points of $B_1$; to each point $p_{k_i}$ we may associate a point $b_i$ of $B_1$ such that

$$d(p_{k_i}, b_i) < 1/i \text{ and } d(p''_{k_i}, b'_i) < 1/i.$$

As $[p_{k_i}]$ has the sequential limit point $P$, the point $P$ is the sequential limit point of $[b_i]$. And from the preceding case we have that $P''$ is a limit point of $[b'_i]$. Since $d(p''_{k_i}, b'_i) < 1/i$, the point $P''$ is a limit point of $[p''_{k_i}]$. But $Q$ is the only limit point of $[p''_{k_i}]$ and therefore $P'' = Q$.

If neither of the above two cases hold, then all except a finite number of the points $p_i$ belong to $C_1$ but are neither points nor limit points of $B_1$. In this case we may show that the assumption $P'' \neq Q$ leads to a contradiction in an analogous manner to that of the similar case of Theorem 13.

Then in every case we have shown that if $P$ is a limit point of a set of points $N$ belonging to $M_1$, the point $P''$ is a limit point of $N''$. The converse follows from the symmetry. Therefore $T$ is a continuous (1, 1) correspondence between $M_1$ and $M_2$.

We may set up examples similar to those of Theorem 13 to show that the conditions of this theorem are independently necessary.

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# GENERALIZED DERIVATIVES.*

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**Introduction.** The point of departure in this paper is that, if $f(e)$ is a completely additive function of normal sets, then, for any normal measurable set $E$, $f(E) = \int_E Df\,dP$ or $f(E) = f(E \cdot \bar{E}) + \int_E Df\,dP$ according as $f(e)$ is absolutely continuous or not. The set $\bar{E}$ is of measure zero. The function $f(E \cdot \bar{E})$ has a zero derivative nearly everywhere and $Df$ is a derivative of $f$ calculated on any regular family. These results are valid for any finite number of dimensions.‡

This suggests a decomposition of a completely additive function of point sets $f(e)$ in terms of a Stieltjes integral, instead of a Lebesgue integral, in which the derivative of $f(e)$ with respect to $g(e)$ occurs. It is evident that $g(e)$ is the analogue of the measure function in the foregoing theory.

Several papers§ have appeared concerning this more general decomposition and in each a different method of treatment is employed. Moreover a different definition of the derivative is used by each author.

It is the purpose of this paper to obtain and extend these results merely by making use of the results stated at the beginning of this paper and a transformation considered by Radon.‖ Such a method of treatment is desirable because not only does it show the very close connection between the results stated at the beginning of this paper and those of the authors referred to, but it also serves to unify and extend these results.

**A Transformation.** The transformation used by Radon is of fundamental importance in obtaining the results of this paper. It seems advisable, therefore, to consider it in some detail. We shall confine the detailed discussion to the one-dimensional case and merely state the results for two or more dimensions.

Let $g(e)$ be a completely additive, non-negative function, defined for all

* Presented to the American Mathematical Society under the title Stieltjes Derivatives, May 1, 1926; received June 10, 1926, in revised form April 30, 1927.

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‡ de la Vallée-Poussin, Intégrales de Lebesgue, 1916, p. 97.

§ W. H. Young, Proc. Lond. Math. Soc., vol. 15 (1916), p. 35; P. J. Daniell, Trans. Amer. Math. Soc., vol. 19 (1918), p. 353; P. J. Daniell, Bull. Amer. Math. Soc., vol. 41 (1920), p. 444; E. C. Francis, Proc. Cambr. Phil. Soc., vol. 22 (1925), p. 924.

‖ Radon, Wiener Sitzsber., vol. 122 (1913), p. 1342.

normal sets* in the interval $J$ $(0 \leqq x < 1)$. Now divide $J$ successively into 2, $2^2$, $2^3$, $\cdots$ congruent intervals,† which are denoted by $(J_1, J_2)$, $(J_{11}, J_{12}, J_{21}, J_{22})$, $\cdots$ respectively. It is clear that each point of $J$ belongs to one and only one interval for a given division.

Consider the interval $K(0 \leqq y < g(J))$. In the $n$th division of $J$ we have $2^n$ intervals $(J_{\alpha_1 \cdots \alpha_n})$. To each $J_{\alpha_1 \cdots \alpha_n}$ for which $g(J_{\alpha_1 \cdots \alpha_n}) > 0$ let there correspond an interval $K_{\alpha_1 \cdots \alpha_n}$ whose measure is $g(J_{\alpha_1 \cdots \alpha_n})$. If $g(J_{\alpha_1 \cdots \alpha_n}) = 0$ there is no $K_{\alpha_1 \cdots \alpha_n}$ corresponding to it. We lay off these $(K_{\alpha_1 \cdots \alpha_n})$ intervals on $K$ end to end and in such a way that they have the same order as the $(J_{\alpha_1 \cdots \alpha_n})$ intervals, to which they correspond, have on $J$. For the $(n+1)$st division of $J$ we divide each $J_{\alpha_1 \cdots \alpha_n}$ into two intervals $J_{\alpha_1 \cdots \alpha_n, 1}$ and $J_{\alpha_1 \cdots \alpha_n, 2}$. The intervals $K_{\alpha_1 \cdots \alpha_n, 1}$ $K_{\alpha_1 \cdots \alpha_n, 2}$, corresponding to them exactly cover the interval $K_{\alpha_1 \cdots \alpha_n}$ which corresponds to $J_{\alpha_1 \cdots \alpha_n}$. We have assumed $g(J_{\alpha_1 \cdots \alpha_n}) > 0$. If $g(J_{\alpha_1 \cdots \alpha_n}) = 0$ then any further subdivision of that particular interval may be left out of consideration. It is clear that

$$\sum_{1}^{2}{}_{\alpha_1} \cdots \cdots \sum_{1}^{2}{}_{\alpha_n} K_{\alpha_1 \cdots \alpha_n} = g(J)$$

for any $n$.

We shall now define a correspondence between the points of $J$ and $K$. Each point $P$ of $J$ is a limit set of a definite set of intervals $J_{\alpha_1}, J_{\alpha_1 \alpha_2}, \cdots$. The corresponding intervals in $K$ are finite or infinite in number. If they are merely finite in number then $P$ will be said to have no corresponding point in $K$. The set $E_0'$ of points of $J$ of this type is evidently obtained by taking the logical sum of the $J$ intervals for each of which $g = 0$. Clearly $g(E_0') = 0$.

If the corresponding intervals of $K$ are infinite in number, then they have a limit set which consists of a simple point $Q$ or of an interval $A_i$ (open or closed on the right) or the limit set is empty. In the former cases $Q$ or $A_i$ is said to correspond to $P$. The latter case can occur if and only if all of the corresponding $K$ intervals, from some definite interval on, have the same right hand end point $Q$, which obviously belongs to none of them, while their left hand end points converge toward $Q$. If such is the case $P$ will have no corresponding point in $K$. It is clear

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*A set $E$ is said to be normal relative to a completely additive function $f$ if, when $\varepsilon > 0$ is arbitrarily preassigned, we can find an open set $O \supset E$ and a closed set $F \subset E$ such that

$$|f(E' - E'')| < \varepsilon$$

whenever $O \supset E' \supset E'' \supset F$. The sets $E'$, $E''$ are such that $f(E')$, $f(E'')$ exist.

†We shall use the term interval to denote the set of points for which $a \leqq x < b$, unless otherwise stated.

that the set* of points $E_0''$ of $J$ which is of this type is at most countable and is such that $g(E_0'') = 0$. The former statement is obvious. The latter follows from the fact that $g(P) = \lim_{n\to\infty} g(J_{\alpha_1\cdots\alpha_n}) = \lim_{n\to\infty} m(K_{\alpha_1\cdots\alpha_n}) = 0$.

By definition the correspondents, when they exist, of two distinct points of $J$ are distinct. Therefore the $A_i$ do not overlap and hence are at most countable. We thus obtain the

THEOREM 1. *With the exception of a set of points $E_0 = E_0' + E_0''$ such that $g(E_0) = 0$, each point $P$ of $J$ has as a correspondent a point $Q$ or an interval $A_i$ of $K$. The intervals $(A_i)$ are distinct.*

COROLLARY. *If $g(\alpha) > 0$ for any interval $\alpha$ in $J$ then $E_0$ is empty.*

Each point $Q$ of $K$ is a limit set of a unique set of intervals $K_{\alpha_1}, K_{\alpha_1\alpha_2}, \cdots$, or is contained in an $A_i$ which is a limit set of a unique set of intervals. Consider a point $Q$ not in any $A_i$. To each interval in the set of intervals which determines $Q$ there corresponds an interval of $J$. The intervals of $J$ thus defined cannot all have the same right hand end point from some definite one on as is seen by the definition of the correspondence between the $K$ and $J$ intervals. They, therefore, have a limit set which contains one and only one point $P$. This point $P$ is said to correspond to $Q$. It is clear that $Q$ also corresponds to $P$. If $Q$ is any point of some $A_i$ then, to each interval in the set of intervals determining $A_i$ there corresponds an interval in $J$ which has as an interior or left hand end point the point $P$ to which $A_i$ corresponds. Hence the set of intervals thus defined has $P$ as a limit set. The point $P$ is said to correspond to $Q$. We thus have the

THEOREM 2. *To each point $Q$ of $K$ there corresponds one and only one point $P$ of $J$.*

Consider a set $E$ in $J$ which is normal relative to $g$. Then $E$ may be contained in an open set $O_1$ and $CE$ may be contained in an open set $O_2$ so that

$$g(O_1) = \sum g(\bar{\alpha}_i) < g(E) + \frac{\varepsilon}{2}, \quad g(O_2) = \sum g(\bar{\beta}_j) < g(CE) + \frac{\varepsilon}{2},$$

where $\varepsilon > 0$ is arbitrarily preassigned† and $O_1 = \sum \bar{\alpha}_i$, $O_2 = \sum \bar{\beta}_j$. The intervals $(\bar{\alpha})$ and $(\bar{\beta})$ are those intervals arising from the successive subdivisions of $J$.‡

* The set $E_0''$ is not necessarily empty. In fact the function $g(e)$ which is defined as $m(e)$ for any measurable set in the interval $0 \leqq x < a$, 0 in the interval $a \leqq x < c$ and $m(e)$ in the interval $c \leqq x < 1$ where $0 < a < c < 1$ and $a$ is irrational is such that the point $P$, which has the coördinate $a$, has no correspondent in $K$. Clearly $P$ does not belong to $E_0'$.

† de la Vallée Poussin, Intégrales de Lebesgue, p. 87.

‡ de la Vallée Poussin, Intégrales de Lebesgue, p. 13.

Let $E_1 = E - E \cdot E_0$ and $E_2 = CE - CE \cdot E_0$, where $E_0$ is the set of points of $J$ mentioned in Theorem 1, and let $E'$ be the set in $K$ which corresponds to $E_1$. Evidently $CE'$ corresponds to $E_2$.

Since

$$\sum g(\bar{\alpha}_i) + \sum g(\bar{\beta}_j) < g(J) + \varepsilon = m(K) + \varepsilon$$

we obtain

$$\bar{m}(E') + \bar{m}(CE') < m(K) + \varepsilon,$$

where $\bar{m}(E')$ denotes the exterior measure of $E'$. This is obvious since the intervals of $K$ corresponding to $(\bar{\alpha}_i)$ include $E'$ and $g(\bar{\alpha}_i)$ is equal to the measure of the corresponding $K$ interval. Since $\varepsilon > 0$ is arbitrary we have

$$\bar{m}(E') + \bar{m}(CE') \leqq m(K).$$

On the other hand we have

$$\bar{m}(E') + \bar{m}(CE') \geqq m(K)$$

by the definition of exterior measure. Hence $\bar{m}(E') + \bar{m}(CE') = m(K)$ or $\bar{m}(E') = m(K) - \bar{m}(CE') = \underline{m}(E')$, where $\underline{m}(E')$ is the interior measure of $E'$. Therefore $E'$ is measurable.

Given $\varepsilon_1 > 0$ we can find a set of $\bar{\alpha}$ intervals such that

$$g(E) + \varepsilon_1 > \sum g(\bar{\alpha}_i) \geqq m(E').$$

On the other hand, in a similar way, we have

$$g(E) \leqq \sum g(\bar{\alpha}_i) < m(E') + \varepsilon_2.$$

Since $\varepsilon_1, \varepsilon_2 > 0$ are arbitrary we see that $m(E') = g(E)$. We may thus state the

THEOREM 3. *Given a normal set $E$ in $J$. There corresponds to $E - E E_0$ a set $E'$ in $K$ such that $m(E') = g(E)$.*

In an analogous manner we may prove the following

THEOREM 4. *Let $E'$ be any measurable set in $K$. Then if $E$ of $J$ corresponds to $E' - \sum E' A_i$* we have $g(E) = m(E' - \sum E' A_i)$.

A further property of the transformation is given by the following

THEOREM 5. *If $\alpha$ is any interval (which may contain either end point, both, or none) of $J$, and such that $g(\alpha) > 0$, then the set of points in $K$ which correspond to $\alpha - \alpha \cdot E_0$ is an interval.*

We shall prove the theorem for a closed interval. The other cases may be treated in a similar manner. Let $\alpha$ be a closed interval in $J$ and such that $g(\alpha) > 0$. It is easy to see that $\alpha$ is a limit set of intervals $\alpha_1 \supset \alpha_2 \cdots$, each of which is a sum of $\bar{\alpha}$ intervals. Now let $\alpha_n = \sum \bar{\alpha}_i$ where the

$\bar{\alpha}_i$ are $\bar{\alpha}$ intervals. Consider the intervals $\bar{\beta}_i$ on $K$ which correspond to $\bar{\alpha}_i$, then, if $\beta_n = \sum \bar{\beta}_i$, it is clear that $\beta_n$ is an interval and $\beta_1 \supset \beta_2 \cdots$. Hence $\lim_{n\to\infty} \beta_n = \beta$ exists. Moreover $\beta$ satisfies the conditions of the theorem. In fact, a point $P$ in $\alpha - \alpha \cdot E_0$ is contained in all of the $\alpha_n$'s and, therefore, its correspondent $Q$ is contained in all of the $\beta_n$'s and hence belongs to $\beta$. Similarly the correspondent $P$ in $J$ of any $Q$ in $\beta$ belongs to $\alpha$. It is obvious that $\beta$ is an *interval*. Hence the theorem is proved. The interval $\beta$ may be open at the left even if $\alpha$ is closed at that end, and may be closed at that end, even if $\alpha$ is open there.

**Absolute continuity.** A completely additive function $f(e)$ of normal sets (not necessarily non-negative) is said to be absolutely continuous with respect to $g(e)$ (a non-negative function of normal sets) if whenever $E$ is a set for which $f(E)$ is defined and $g(E) = 0$, then $f(E) = 0$.*

**Definition of derived numbers and derivatives.**† The derived numbers of $f$ with respect to $g$ at the point $P$ are defined as

$$\left.\begin{aligned} D^+ f &= \overline{\lim_{m\alpha\to 0}} \\ D_+ f &= \underline{\lim_{m\alpha\to 0}} \\ D^- f &= \overline{\lim_{m\alpha\to 0}} \\ D_- f &= \underline{\lim_{m\alpha\to 0}} \end{aligned}\right\} \frac{f(\alpha)}{g(\alpha)}$$

where in the first two cases $\alpha$ is a closed interval having $P$ as left hand end point and in the remaining cases $\alpha$ is a closed interval having $P$ as right hand end point. If $g(\alpha) = 0$ then we define $f(\alpha)/g(\alpha) = 0$. If $D^+f = D_+f$ we say that $f$ has a right hand derivative with respect to $g$ at $P$. Similarly we define the left hand derivative. If $D^+f = D_+f = D^-f = D_-f$ then $f$ is said to have a derivative with respect to $g$ at $P$. We denote this derivative by $Df$.‡

THEOREM 6. *$Df$ exists and is finite nearly everywhere $(g)$ in $J$, i. e., except for a set $E$ such that $g(E) = 0$. If $f$ is not absolutely continuous with respect to $g$ we shall assume that $g(\alpha) > 0$ for any interval $\alpha$ in $J$.*

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* A consequence of this definition is that any set which is normal relative to $g$ is a set which is normal relative to $f$. Radon, loc. cit., p. 1318.

† In this and the discussion following it is to be understood that $f$ and $g$ are completely additive functions of normal sets in $J$, $g$ non-negative, and $g(e) = f(e) = 0$ if $e \cdot J = 0$, unless otherwise stated. When we say that $E$ is a set for which $g$ or for which $f$ exists, we shall understand $E$ to be normal relative to $g$ or to $f$.

‡ The upper and lower derived numbers of $f$ with respect to $g$ at $P$ may be defined $\overline{\lim_{m\alpha\to 0}} f(\alpha)/g(\alpha)$ and $\underline{\lim_{m\alpha\to 0}} f(\alpha)/g(\alpha)$ where $\alpha$ is *any* interval containing $P$.

Let $f(e)$ and $g(e)$ be the functions stated in the theorem and let $f$ be absolutely continuous with respect to $g$. Then if $E'$ is any measurable set in $K$ such that $E$ in $J$ corresponds to $E'-\sum E'A_i$ we define

$$F(E') = f(E)+\sum \frac{m(E'A_i)}{m(A_i)} f(P_i).$$

The $P_i$'s are the points of $J$ which have the intervals $A_i$ of $K$ corresponding to them.

The $F$ thus defined is a completely additive function of normal sets and, hence, $DF$ calculated in the usual manner on any regular family in $K$ exists and is finite nearly everywhere in $K$.*

Consider any set $E$ in $J$ for which $g(E)$ exists. Then to $E-E\cdot E_0$ ($E_0$ is the set given in Theorem 1) there corresponds $E'$ in $K$. Therefore to $E'-\sum E'A_i$ there corresponds $E-(E\cdot E_0+\sum E\cdot P_i)$. Hence

$$F(E') = f(E)-f(E\cdot E_0)-\sum f(E\cdot P_i)+\sum f(E\cdot P_i) = f(E),$$

since $f(E\cdot E_0) = 0$.

Consider the right hand derivative of $f$ with respect to $g$. Let $E$ be the set of points of $J$ such that each point $P$ of $E$ has one and only one point $Q$ of $K$ as a correspondent. *Any* interval $\alpha$ which has $P$ as left hand end point is such that $g(\alpha)>0$, for otherwise $P$ would have no corresponding point $Q$ in $K$. If $\beta$ is the interval in $K$ corresponding to the closed interval $\alpha$ in $J$ we have

$$\frac{f(\alpha)}{g(\alpha)} = \frac{F(\beta)}{m(\beta)}.$$

Since the intervals $(\beta)$ form a regular family and since $DF$ calculated on any regular family exists and is finite nearly everywhere in $K$, we have $D^+f = D_+f$ and is finite nearly everywhere $(g)$ in $E$. If $P$ is a point such that $g(P)>0$ then for each such point $D^+f = D_+f = f(P)/g(P)$. Hence $D^+f = D_+f$ and is finite nearly everywhere $(g)$ in $J$.

Consider the left hand derivative. Let $E$ be the set of points of $J$ such that each point $P$ of $E$ has one and only one corresponding point $Q$ of $K$. The set of points $\bar{E}$ of $E$ such that for each one there is at least one interval $\alpha$ having it as right hand end point and such that $g(\alpha) = 0$, is at most countable. In fact, if we associate with each point of $\bar{E}$ an interval $\alpha$ such that $g(\alpha) = 0$, then the associated intervals are distinct, since *each* point of $E$ has a corresponding point in $K$. Just as before we can show that $D^-f = D_-f$ nearly everywhere $(g)$ in $J$.

* de la Vallée Poussin, loc. cit., p. 97.

Since $DF = D^+f = D_+f$ nearly everywhere $(g)$ in $J$ and $DF = D^-f = D_-f$, it is clear that $f$ has a derivative with respect to $g$ nearly everywhere $(g)$ in $J$.

If $f$ is not absolutely continuous with respect to $g$ we proceed as follows: $F$ is obviously a function of normal sets. Since $E_0$ is empty [corollary to Theorem 1], $f(\alpha) = F(\beta)$, where $\beta$ is the interval in $K$ corresponding to $\alpha$ in $J$. The remainder of the proof is evident.

COROLLARY. *The derived numbers of $f$ with respect to $g$ defined in the footnote p. 423 are equal to $Df$ nearly everywhere $(g)$ in $J$.*

THEOREM 7. *If $f$ is absolutely continuous with respect to $g$ and $E$ is any set in $J$ for which $g$ is defined, then*

$$f(E) = \int_E Df\, dg.$$

*The integral is a generalized Stieltjes integral.* $Df$ may be either the derivative defined on p. 423 or in the footnote p. 423 when the upper and lower derived numbers are equal.*

The function $F$ defined in the proof of Theorem 6 is absolutely continuous. In fact if $m(E') = 0$ then $m(E'A_i) = 0$ and $m(E' - \sum E'A_i) = 0$. But $m(E' - \sum E'A_i) = g(E)$ Theorem 4. Hence $F(E') = 0$. Hence there is no singular set $E'$ for $F(E')$, and the statement is proved.†

We may, therefore, put

$$F(E') = \int_{E'} DF\, dP$$

where $E'$ is any measurable set in $K$ and $DF$ is a derivative of $F$ calculated on any regular family.‡

Consider

$$\cdots, Y_{-n}, Y_{-n-1}, \cdots, Y_{-1}, Y_0, Y_1, \cdots, Y_n, \cdots$$

where the $Y_{-i}$ are negative and $Y_i$ positive numbers, and the absolute value of the difference between any two consecutive terms does not exceed $\delta$. Let the set of points of $E'$ for which $Y_n \leqq DF < Y_{n+1}$ be $E'_n$ and let the set of points of $E$ for which $Y_n \leqq Df < Y_{n+1}$ be $E_n$.

It is clear that $g(E_n) = m(E'_n)$. Hence

$$\sum_{-\infty}^{+\infty} Y_i\, m(E'_i) = \sum_{-\infty}^{+\infty} Y_i\, g(E_i).$$

It follows that§

$$F(E') = \int_E Df\, dg.$$

* Radon, loc. cit., p. 1322 ff.

† de la Vallée Poussin, loc. cit. p. 97.

‡ de la Vallée Poussin, loc. cit., p. 73.

§ Radon, ibid., p. 1328.

But since $F(E') = f(E)$, as is seen by referring to the proof of Theorem 6, we have $f(E) = \int_E Df\, dg$. This completes the proof of the theorem.

COROLLARY. *If $f$ is not absolutely continuous with respect to $g$ and $E$ is a set for which $f$ and $g$ are defined then $\int_E Df\, dg = \int_{E'} DF\, dP$. $E'$ is the set in $K$ which corresponds to $E$ in $J$.*

THEOREM 8. *If $f$ is not absolutely continuous with respect to $g$ and $g(\alpha) > 0$ for any interval $\alpha$ in $J$, then*

$$f(E) = f(E \cdot \bar{E}) + \int_E Df\, dg.$$

*$E$ is any set in $J$ for with both $f$ and $g$ are defined. $Df(E \cdot \bar{E}) = 0$ nearly everywhere $(g)$ in $J$ and $\bar{E}$ is such that $g(\bar{E}) = 0$.*

Let $E'$ be a measurable set in $K$ such that $f$ is defined for the $E$ in $J$ which corresponds to $E' - \sum E'A_i$. Define

$$F(E') = f(E) + \sum \frac{m(E'A_i)}{m(A_i)} f(P_i)$$

which is evidently a completely additive function of normal measurable sets in $K$.

We may, therefore, put

$$F(E') = F(E'\bar{E}') + \int_{E'} DF\, dP \tag{1}$$

where $DF(E'\bar{E}') = 0$ nearly everywhere in $K$ and $m(\bar{E}') = 0$.*

Now if $E$ is any set in $J$ for which $f$ and $g$ are defined and $E'$ in $K$ corresponds to it, then,

$$F(E') = f(E)$$

since the set $E_0$ is empty [corollary to Theorem 1]. Let $E$ in $J$ correspond to $E'$ in $K$, then $f(E)$ and $g(E)$ exist. In fact, $\bar{E}'$ is measurable Borel† and since the transformation from $K$ to $J$ carries intervals into intervals or points, the corresponding set $\bar{E}$ in $J$ is measurable Borel. But $f$ and $g$ are defined for all sets measurable Borel.‡ Hence $f(\bar{E})$ and $g(\bar{E})$ exist. Since $m(\bar{E}') = 0$ then $F(\sum \bar{E}'A_i) = 0$. It follows from this that $\bar{E}'$ may be assumed to contain no points of $\sum A_i$. Hence

$$F(E'\bar{E}') = f(E \cdot \bar{E}).$$

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* de la Vallée Poussin, loc. cit., p. 97.

† de la Vallée Poussin, ibid., p. 63.

‡ de la Vallée Poussin, ibid., p. 87.

By means of (1) we conclude that

$$f(E) = f(E \cdot \bar{E}) + \int_{E'} DF \, dP.$$

Hence by the corollary to Theorem 7 we have, for any set $E$ for which both $f$ and $g$ exist,

$$f(E) = f(E \cdot \bar{E}) + \int_{E} Df \, dg.$$

Since $m(\bar{E}') = 0$ then $g(\bar{E}) = 0$, Theorem 4. Since $DF(E'\bar{E}') = 0$ nearly everywhere in $K$ then $Df(E \cdot \bar{E}) = 0$ nearly everywhere $(g)$ in $J$ [corollary to Theorem 6]. The theorem is thus proved.

THEOREM 9. *Let $f$ and $g$ be given as in the previous theorem but with the added restriction that $f$ has no point values* where $g$ has none. Then $\bar{E}$ is the set of points in $J$ where $f$ has a unique infinite derivative $(+\infty$ or $-\infty)$ with respect to $g$.*

It follows from the assumption that $f$ has no point values where $g$ has none that the function $F$ defined in the proof of the previous theorem, has no point values. But on $E'$, the set of points of $K$ corresponding to $E$, $F(E')$ has a unique infinite derivative $(+\infty$ or $-\infty)$.† Therefore, by Theorem 6, $f$ has a unique infinite derivative on $E$.

**Derivatives of point functions.** Let $f(x)$ be a function of bounded variation in $J(0 \leqq x < 1)$ and $g(x)$ be a non-decreasing function in the same interval. Denote by $\bar{f}(e)$ and $\bar{g}(e)$ the unique completely additive functions of normal sets defined by $f(x)$ and $g(x)$ respectively.‡ We shall call $\bar{f}$ and $\bar{g}$ the variation functions of $f$ and $g$.

Definition: The function $f(x)$ is said to be absolutely continuous with respect to $g(x)$ if $\bar{f}(e)$ is absolutely continuous with respect to $\bar{g}(e)$.

Definition. The upper and lower derivatives of $f$ with respect to $g$ at $P$ are defined as

$$\overline{\lim_{m\alpha \to 0}} \frac{f(\alpha)}{g(\alpha)} = \bar{D}f, \qquad \underline{\lim_{m\alpha \to 0}} \frac{f(\alpha)}{g(\alpha)} = \underline{D}f.$$

We define $\alpha$ as the interval $x - h_1 \leqq t \leqq x + h_2$ where $x$ is the coördinate of $P$. $f(\alpha) = f(x + h_2) - f(x - h_1)$ and similarly for $g(\alpha)$. When $P$ belongs to $\sum P_i$, the set of points of discontinuity of $g(x)$, we shall choose

* $f$ is said to have a point value at the point $P$ if $f(P) \neq 0$.

† de la Vallée Poussin, loc. cit., p. 94.

‡ G. C. Evans, The Rice Institute Pamphlet, vol. 7 (1920), p. 271. The facts here are as follows: For any intervals $\alpha\,(x_1 \leqq t \leqq x_2)$ in $J$ the relation $f(x_2 + 0) - f(x_1 - 0) = \bar{f}(\alpha)$ holds.

$\alpha$ so that $P$ is an interior point of it. If $g(\alpha) = 0$ for some $\alpha$, then $f(\alpha) = 0$ and we define the meaningless ratio $0/0 = 0$. If at a point $P$ $\bar{D}f = \underline{D}f = Df$ then $f$ is said to have a derivative with respect to $g$ at $P$. In order to define $Df$ at 0 we define $f(x) = f(0)$, $g(x) = g(0)$, $x < 0$.

LEMMA. *Let $g(x)$ be a non-decreasing function of $x$ in $J$. Let $\overline{g}(e)$ be its variation function. Then $\lim_{m\alpha \to 0} g(\alpha)/\overline{g}(\alpha) = 1$ nearly everywhere $(\overline{g})$ in $J$. $\alpha$ has the same meaning as in foregoing.*

In fact

$$\frac{g(\alpha)}{\overline{g}(\alpha)} = \frac{g_1(\alpha)}{\overline{g}(\alpha)} + \frac{S(\alpha)}{\overline{g}(\alpha)}$$

where $g_1$ is a continuous function of $x$ and $S$ is a saltus function.* Clearly $\lim_{m\alpha \to 0} g_1(\alpha)/\overline{g}(\alpha)$ exists nearly everywhere $(\overline{g})$ in $J$, corollary to Theorem 6. In fact, the variation function $\overline{g}_1$ of $g_1$ is such that $\overline{g}_1(\alpha) = g_1(\alpha)$ for any $\alpha$. If $P_i$ is a point of $J$ such that $\overline{g}(P_i) \neq 0$ then $\lim_{m\alpha \to 0} g_1(\alpha)/\overline{g}(\alpha) = 0$ for every point of $\sum P_i$. We wish to show that $\lim_{m\alpha \to 0} g_1(\alpha)/\overline{g}(\alpha) = 1$ almost everywhere $(\overline{g})$ in $J - \sum P_i$. We have

$$\frac{g_1(\alpha)}{\overline{g}(\alpha)} = \frac{\overline{g}_1(\alpha)}{\overline{g}(\alpha)}.$$

Let $\overline{G}_1(E')$ be the function on $K$ corresponding to $\overline{g}_1(E)$. Then $\overline{G}_1(E') = \overline{g}_1(E)$ by definition in Theorem 6, and

$$G_1(E') = m(E' - \sum A_i E')$$

and

$$\frac{\overline{g}_1(\alpha)}{\overline{g}(\alpha)} = \frac{\overline{G}_1(\alpha')}{m\alpha'} = \frac{m(\alpha' - \sum A_i \alpha')}{m\alpha'}.$$

But the limit of this expression as $m\alpha' \to 0$ is the density of the set $K - \sum A_i$, which is unity nearly everywhere in $K - \sum A_i$.†

If $P$ is not in $\sum P_i$, and has a correspondent $P'$ in $K$, then $P'$ is in $K - \sum A_i$ and $m\alpha' \to 0$ with $m\alpha$. Hence in $J - \sum P_i$, $\lim_{m\alpha \to 0} g_1(\alpha)/\overline{g}(\alpha) = 1$ nearly everywhere $(\overline{g})$.‡

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* Lebesgue, Leçons sur l'Integration, p. 58.

† de la Vallée Poussin, Intégrales de Lebesgue, p. 71.

‡ I am indebited to Professor G. C. Evans for the proof that $\lim_{m\alpha \to 0} g_1(\alpha)/\overline{g}(\alpha) = 1$ nearly everywhere $(\overline{g})$ in $J - \sum P_i$.

We assert that $\lim_{m\alpha\to 0} S(\alpha)/\overline{g}(\alpha) = 0$ nearly everywhere $(\overline{g})$ in $J-\sum P_i$ and is 1 at every point of $\sum P_i$. In fact, the variation function $\overline{S}$ of $S$ is such that $\overline{S}(\alpha) \geqq S(\alpha)$* for any $\alpha$ in $J$. Now define $\overline{S}_1$ on $K$ exactly as $F$ was defined in the proof of Theorem 6. We see that $D\overline{S}_1 = 0$ nearly everywhere in $K-\sum A_i$. Hence $\lim_{m\alpha\to 0} \overline{S}(\alpha)/\overline{g}(\alpha) = 0$ nearly everywhere $(\overline{g})$ in $J-\sum P_i$. Therefore $\lim_{m\alpha\to 0} S(\alpha)/\overline{g}(\alpha) = 0$ nearly everywhere $(\overline{g})$ in $J-\sum P_i$. It is obvious that $\lim_{m\alpha\to 0} S(\alpha)/\overline{g}(\alpha) = 1$ at every point $P$ of $\sum P_i$. This completes the demonstration of the lemma.

THEOREM 10. *$\overline{D}f = \underline{D}f = Df$ nearly everywhere $(\overline{g})$ in $J$. If $f$ is not absolutely continuous with respect to $g$ we shall assume that $g(\alpha) > 0$ for any interval $\alpha$ in $J$.*

Since $f(x)$ is of bounded variation it can be written as the difference of two non-decreasing functions of $x$. It will be sufficient, therefore, to prove the theorem for a non-decreasing function.

We write

$$f(x) = f_1(x) + S(x)$$

where $f(x)$ is non-decreasing, $f_1(x)$ is continuous and $S(x)$ is a saltus function.† Let $\overline{f}, \overline{f_1}, \overline{S}$ be the variation functions defined by $f, f_1, S$ respectively. We then have

$$\overline{f}(e) = \overline{f_1}(e) + \overline{S}(e)$$

for any normal set in $J$.

Since

$$\frac{\overline{S}(\alpha)}{\overline{g}(\alpha)}\,\frac{\overline{g}(\alpha)}{g(\alpha)} \geqq \frac{S(\alpha)}{g(\alpha)}$$

we see, by the above lemma, that $DS = 0$ nearly everywhere $(\overline{g})$ in $J-\sum P_i$. Now since $f_1(\alpha) = \overline{f_1}(\alpha)$ for any $\alpha$ we have

$$\frac{f_1(\alpha)}{g(\alpha)} = \frac{\overline{f_1}(\alpha)}{\overline{g}(\alpha)}\,\frac{\overline{g}(\alpha)}{g(\alpha)}.$$

Hence, by the above lemma, $Df_1 = D\overline{f_1}$ nearly everywhere $(\overline{g})$ in $J-\sum P_i$. But $D\overline{f} = D\overline{f_1}$ nearly everywhere $(\overline{g})$ in $J-\sum P_i$, since $D\overline{S}$ is obviously zero nearly everywhere $(\overline{g})$ in $J-\sum P_i$. Therefore $Df = D\overline{f}$ nearly everywhere $(\overline{g})$ in $J-\sum P_i$. If $P_i$ is a point of $\sum P_i$ then clearly $Df = D\overline{f}$. This completes the proof of the theorem.

* G. C. Evans, loc. cit., p. 270.

† Lebesgue, loc. cit., p. 58.

The two following theorems are obvious.

THEOREM 11. *If $f$ is absolutely continuous with respect to $g$, then, for any set $E$ in $J$ for which $\bar{g}$ exists,*

$$\bar{f}(E) = \int_E Df\, d\bar{g}.$$

THEOREM 12. *If $f$ is not absolutely continuous with respect to $g$ and $g(\alpha) > 0$ for any interval $\alpha$ in $J$, then for any set $E$ for which $f$ and $g$ exist,*

$$\bar{f}(E) = \bar{f}(E \cdot \bar{E}) + \int_E Df\, d\bar{g}.$$

*The function $\bar{f}(E\bar{E})$ has a derivative with respect to $g$ which is zero nearly everywhere $(\bar{g})$ in $J$. The set $\bar{E}$ is such that $\bar{g}(\bar{E}) = 0$.**

**Note.** The statement made by Francis, loc. cit., p. 928, that $Df$ and $D\bar{f}$ exist together and are equal is not necessarily true.

Define $f(x)$ and $g(x)$ in $J$ as follows:

$$\begin{aligned}
f(x) &= x, \quad 0 \leqq x \leqq 1/2, \\
f(x) &= 2/2^i + x, \quad x_i < x < x_{i-1}, \quad x_i = 1/2 + 1/2^i, \\
f(x_i) &= f(x_i - 0) = 2/2^{i+1} + x_i, \\
g(x) &= x, \quad 0 \leqq x \leqq 1/2, \\
g(x) &= 2/2^i + x, \quad x_i < x < x_{i-1}, \quad x_i = 1/2 + 1/2^i, \\
g(x_i) &= g(x_i + 0) = 2/2^i + x_i.
\end{aligned}$$

Then at $x = 1/2$, $D\bar{f}$ exists and is equal to 1 while $\bar{D}f = 1$ and $\underline{D}f = 2/3$. It is true, however, that $Df$ and $D\bar{f}$ exist together and are equal nearly everywhere $(\bar{g})$ in $J$, Theorem 10.

## Additive functions of point sets in two or more dimensions.

**A transformation.**† We shall now consider, briefly, the two dimensional case since the treatment of the higher dimensional cases is exactly analogous. Let there be a square whose upper right and lower left hand vertices have the coördinates $(M, M)$ $(-M, -M)$ respectively. We denote by $J$ the interval $(-M \leqq x < M,\ -M \leqq y < M)$. If $(a, b)$, $(c, d)$ are the coördinates of the upper right and lower left hand vertices of a rectangle whose sides are parallel to the coördinate axis in $J$ we shall understand

* Theorem 11 includes the result of Daniell, Trans. Amer. Math. Soc., loc. cit., p. 353. Theorems 10 and 11 include the results of Francis, loc. cit., p. 935 ff. concerning the derivatives of functions of bounded variation. Theorems 10, 11, 12 include the extensions of Lebesgue's theorems given by W. H. Young loc. cit., p. 58 ff.

† Radon, loc. cit., p. 1342.

by an interval the set of points whose coördinates satisfy the relations $c \leqq x < a$, $d \leqq y < b$.

Let $g(e)$ be a completely additive non-negative function of normal sets in $J$. In the following we shall restrict $g$ to be of this type.

Divide $J$ successively into 4, $4^2$, $4^3$, $\cdots$ congruent sub-intervals. Number the intervals arising from the first division $(J_1, J_2, J_3, J_4)$, those arising from $J_i$ in the second division $(J_{i1}, J_{i2}, J_{i3}, J_{i4})$ etc. Each point of $J$ belongs to one and only one interval for a given division. On the linear interval $K(0 \leqq t < g(J))$ lay off end to end the intervals $K_1, K_2, K_3, K_4$, where $m(K_i) = g(J_i)$ if $g(J_i) = 0$, there is no $K_i$. On $K_i$ lay off end to end the intervals $K_{i1}, K_{i2}, K_{i3}, K_{i4}$, where $m(K_{ij}) = g(J_{ij})$ if $g(J_{ij}) > 0$. If $g(J_{ij}) = 0$, there is no $K_{ij}$. The method of continuing the process is evident.

Each point $P$ of $J$ is a limit set of a definite set of intervals $(J_\alpha, J_{\alpha_1\alpha_2}, \cdots)$. The limit set of the corresponding $K$ intervals will correspond to $P$. Conversely, each point $Q$ of $K$ is a limit set of a definite set of $K$ intervals or is contained in an interval (open or closed on the right) $A_i$ which is a limit set of $K$ intervals. The limit set of the corresponding $J$ intervals is said to correspond to $Q$.

The properties of this transformation may be derived, *mutatis mutandis*, as in the one-dimensional case. We, therefore, merely state the results.

THEOREM. *With the exception of a set of points $E_0$ for which $g(E_0) = 0$ each point of $J$ has as a correspondent a point or an interval $A_i$ of $K$. Conversely, with the exception of a set $\bar{E}_0$ for which $m(\bar{E}_0) = 0$ each point of $K$ has as a correspondent one and only one point $P$ of $K$. If $g(\alpha) = 0$ for every interval $\alpha$ in $J$ then $E_0$ is at most countable.*

THEOREM. *A set $E$ of $J$ for which $g$ exists has a measurable set $E'$ in $K$ corresponding to $E - E \cdot E_0$. Furthermore, $g(E) = m(E')$.*

THEOREM. *A measurable set $E'$ in $K$ such that $E' \cdot \sum A_i = 0$ has corresponding to $E' - E'\bar{E}_0$ a set $E$ in $J$ and such that $m(E') = g(E)$.*

*Definition of the derivatives.* Let $f(e)$ be a completely additive function of normal sets in $J$. We define the upper and lower derivatives of $f$ with respect to $g$ at a point $P$ as

$$\overline{\lim_{n\to\infty}} \frac{f(J_{\alpha_1\cdots\alpha_n})}{g(J_{\alpha_1\cdots\alpha_n})} = \bar{D}f,$$

$$\underline{\lim_{n\to\infty}} \frac{f(J_{\alpha_1\cdots\alpha_n})}{g(J_{\alpha_1\cdots\alpha_n})} = \underline{D}f,$$

where $(J_{\alpha_1}, J_{\alpha_1\alpha_2}, \cdots)$ is the set of intervals having $P$ as a limit set. If $\bar{D}f = \underline{D}f = Df$ then $f$ is said to have a derivative with respect to $g$

at $P$. If $g(J_{\alpha_1 \cdots \alpha_n}) = 0$ for any $J_{\alpha_1 \cdots \alpha_n}$ we assume that $f(J_{\alpha_1 \cdots \alpha_n}) = 0$ and define the meaningless ratio $0/0 = 0$.

**Absolute continuity.** A completely additive function $f(e)$ of normal sets is said to be absolutely continuous with respect to $g(e)$ if whenever $E$ is a set for which $f(E)$ exists and $g(E) = 0$, then $f(E) = 0$.

If $f$ is not absolutely continuous with respect to $g$ it will be assumed that $f$ has no point values, where $g$ has none and $g(\alpha) > 0$ for any interval $\alpha$ in $J$.

By means of arguments similar to those used in the one dimensional case we can prove the following theorems.

THEOREM. *If $f$ is absolutely continuous with respect to $g$, then $Df$ exists nearly everywhere $(g)$ in $J$. For any set $E$ such that $g(E)$ exists*

$$f(E) = \int_E Df\, dg.^*$$

THEOREM. *If $f$ is not absolutely continuous with respect to $g$, then $Df$ exists nearly everywhere $(g)$ in $J$. For any set $E$ for which $f$ and $g$ are defined*

$$f(E) = f(E \cdot \bar{E}) + \int_E Df\, dg.$$

*The set $\bar{E}$ is such that $g(\bar{E}) = 0$. $Df(E \cdot \bar{E}) = 0$ nearly everywhere $(g)$ in $J$.*

* This theorem includes the result of Daniell, Bull. Amer. Math. Soc., loc. cit., p. 444.

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# A CREMONA GROUP ISOMORPHIC WITH THE GROUP OF THE TWENTY-SEVEN LINES ON A CUBIC SURFACE.*

By H. L. Black.

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## I. Introduction.

In 1896 E. H. Moore† introduced the now familiar "cross ratio group", a $G_{n!}$ in an $S_{n-3}$. Soon after this H. E. Slaught‡ developed the geometric representation and the complete form system of invariants for the special case $n = 5$. Later E. Kasner§ displayed a relation between the $G_{6!}$ and a double-six on the general cubic surface. In 1915 A. B. Coble‖ generalized the problem by considering the absolute linear invariants of a set $P_n^k$ of

* Received December 27, 1926.

† E. H. Moore, The cross-ratio group of $n!$ Cremona transformations of order $n-3$ in flat space of $n-3$ dimensions. Amer. Journ. of Math., vol. 22, pp. 279–291.

‡ H. E. Slaught, Amer. Journ. of Math., vol. 22, pp. 343–388, vol. 23, pp. 99.

§ E. Kasner, The double-six configuration connected with the cubic surface, and a related group of Cremona transformations. Amer. Journ. of Math., vol. 25 (1903), pp. 107–122.

‖ A. B. Coble, Point sets and Cremona groups. Trans. Amer. Math. Soc., vol. 16, pp. 155–198, vol. 17, pp. 345–385, vol. 18, pp. 331–372. These will be referred to as C I, C II, and C III, respectively.

$n$ points in $S_k$. With such a set there is associated a Cremona group $G_{n!}$ of more general type than the Moore group (cf. § II). By adding to a canonical form of the generators of the $G_{6!}$ in $S_4$ associated with a $P_6^2$ the involutory transformation which inverts the variables there is obtained a group $G_{6,2}$ isomorphic with that of the lines on a cubic surface.

The first object of the present paper is to develop (§ III) a simple method of finding the order of the transformations of such a $G_{n!}$, those for the special cases $n = 6$, $n = 7$ being found in the process.

Section IV is given over to the discussion of certain particular cases of a mapping problem (cf. § II, 3) which arise when special conditions are imposed upon the six points; these are equivalent to requiring the associated cubic surface to have a node. Among the loci connected with these nodal conditions are the fundamental spaces of the transformations of the Cremona Groups $G_{6!}$ and $G_{6,2}$.

In section V it is shown that there is but one linear system of order 5 which is invariant under the $G_{6,2}$. The geometric properties of this system, which is determined by the complex invariants (cf. § II), are there displayed. The following section deals with the relations between these invariants and the discriminant loci already developed; and finally (§ VII) the coördinates of the canonical map (cf. § II, 3) are obtained in terms of the forty complex invariants of the cubic surface.

## II. Preliminary survey.*

In a space $\Sigma_k$ any $k+2$ points are linearly independent; $n$ points are therefore connected by $n-k-1$ linear relations. If, then, we take as coördinates the coefficients of corresponding terms in a set of such equations, we have defined a second set of $n$ points, these lying in a $\Sigma_{n-k-2}$. These two sets of points are denoted by $P_n^k$ and $Q_n^{n-k-2}$, and are called Associated Sets.

Two sets $P_n^2$ and $Q_n^2$ in a plane are said to be congruent under a Cremona transformation $C_m$ with $\varrho (\leqq n)$ $F$-points if $\varrho$ of the pairs $p_i, q_i (i = 1, 2, \cdots, n)$ are corresponding $F$-points of $C_m$ (i. e., of the same multiplicity) and if the remaining $n-\varrho$ pairs are corresponding ordinary points.

A set $P_n^k$ is said to be in the canonical form when the first $k+2$ points fall upon the reference points and the unit point, and the coördinates of the remaining points are affected by such a factor of proportionality that the last coördinate of each is $u$. Then the other $k(n-k-2)$ coördinates, taken with $u$, define the canonical map $P$ in $\Sigma_{k(n-k-2)}$ of the set $P_n^k$.

* The definitions and results here given were developed by Professor Coble in the articles previously mentioned. They are repeated here for convenience in following the discussion.

We define an integral invariant of the point set as a rational integral function of the $\binom{n}{k+1}$ determinants $P(i_1, i_2, \cdots, i_{k+1})$ which is homogeneous and of the same degree in the coördinates of each of the $n$ points.

If in the functions of Joubert for the $P_6^1$

$$A = (25)(13)(46)+(51)(42)(36)+(14)(35)(26)+(43)(21)(56)+(32)(54)(16)$$
$$B = (53)(12)(46)+(14)(23)(56)+(25)(34)(16)+(31)(45)(26)+(42)(51)(36)$$
$$C = (53)(41)(26)+(34)(25)(16)+(42)(13)(56)+(21)(54)(36)+(15)(32)(46)$$
$$D = (45)(31)(26)+(53)(24)(16)+(41)(25)(36)+(32)(15)(46)+(21)(43)(56)$$
$$E = (31)(24)(56)+(12)(53)(46)+(25)(41)(36)+(54)(32)(16)+(43)(15)(26)$$
$$F = (42)(35)(16)+(23)(14)(56)+(31)(52)(46)+(15)(43)(26)+(54)(21)(36)$$

each binary determinant $(ik)$ be replaced by the ternary determinant $(ikx)$ there arise six cubic curves $a=0, \cdots, f=0$ on a $P_6^2$ —$p_1, p_2, \cdots, p_6$— of the plane; these are connected by the identities $\sum a = 0$, $a+b = 4(51x)(42x)(36x)$, etc. Since six cubics on six points are connected by two linear relations, of which we have one, we may take as the second $\sum \bar{a}\, a = 0$. The coefficients $\bar{a}, \bar{b}, \cdots, \bar{f}$, fixed to within a factor of proportionality by requiring also that $\sum \bar{a} = 0$, are given by replacing in $A$ the product $(ij)(kl)(mn)$ by the ternary determinant $(ij, kl, mn)$; and similarly for $\bar{b}, \bar{c}, \cdots, \bar{f}$.

Now the cubic curves on $P_6^2$ map the plane on a general cubic surface $C^{(3)}$ given by $\sum a^3 = 0$, where $\sum a = 0$ and $\sum \bar{a}\, a = 0$. The tritangent planes of $C^{(3)}$ comprise 15 of the type $a+d=0$ and thirty of the type

$$(1) \qquad (\overline{be} - d_2)(b+e) - (\overline{cf} + d_2)(c+f) = 0,$$

where

$$\overline{cf} = a_2 + 2(\bar{c}^2 + \bar{f}^2 + \bar{c}\,\bar{f}),$$
$$(2) \qquad a_2 = \sum(\bar{a}\,\bar{b}),$$
$$d_2 = \begin{vmatrix} (341)(561) & (531)(461) \\ (342)(562) & (532)(462) \end{vmatrix}.$$

We shall further make use of the quantities defined by $a_3 = \sum \bar{a}\,\bar{b}\,\bar{c}$ and the set

$$(3)\qquad \begin{aligned} \bar{\alpha} &= a_3 - a_2\bar{a} - 2\bar{a}^3, \\ &\cdots\cdots\cdots \\ \bar{\zeta} &= a_3 - a_2\bar{f} - 2\bar{f}^3. \end{aligned}$$

Among the double sixes of the $C^{(3)}$ those having four lines in common are said to be azygetic. Two such double sixes determine a third azygetic to both; there are 120 such azygetic triads. The double sixes syzygetic (having 6 lines in common) with the members of an azygetic triad form a symmetric set of three azygetic triads called a complex. These are of two types. Associated with those of the first are ten complex invariants of the form

$$(4)\qquad \Gamma_{123,456} = d_2(123)(456).$$

The thirty of the second kind lead to invariants like

$$(5)\qquad \begin{aligned} 2\Gamma_{12,34,56} &= \bar{\alpha} + \bar{\delta} - d_2(\bar{a} - \bar{d}), \\ 2\Gamma_{12,56,34} &= \bar{\alpha} + \bar{\delta} + d_2(\bar{a} - \bar{d}). \end{aligned}$$

In the plane there are 6! possible permutations of the six points. These form a group which may be generated by the five interchanges (12), (23), (34), (45), (56). We associate with this group the $G_{61}$ of Cremona transformations in four dimensions by the following method. The set of points is taken in the canonical form and transformed by some permutation of the group. For example (12) would result in the following change:

$$(6)\qquad \begin{aligned} 100 &\cdots 010 \\ 010 &\cdots 100 \\ 001 &\cdots 001 \\ 111 &\cdots 111 \\ xyu &\cdots xyu \\ ztu &\cdots ztu \end{aligned}$$

The reference basis is then restored by a linear transformation—in this case, by

$$y_1 = y_2', \quad y_2 = y_1', \quad y_3 = y_3'.$$

Now in this process the last two points have undergone a transformation. We introduce a proportionality factor which again makes the last coördinates equal. The resulting replacement of the $x, y, z, t, u$ is the Cremona transformation corresponding to the given permutation—in this case a collineation interchanging $x$ and $y$, $z$ and $t$. The generators of this Cremona $G_{61}$ are

(7)

| | (12) | (23) | (34) | (45) | (56) |
|---|---|---|---|---|---|
| $x'$ | $y$ | $xt$ | $u-x$ | $uy$ | $z$ |
| $y'$ | $x$ | $ut$ | $u-y$ | $ux$ | $t$ |
| $z'$ | $t$ | $yz$ | $u-z$ | $zy$ | $x$ |
| $t'$ | $z$ | $uy$ | $u-t$ | $xt$ | $y$ |
| $u'$ | $u$ | $yt$ | $u$ | $xy$ | $u$ |

The $G_{6,2}$ is obtained from the $G_{61}$ by adding the transformation which inverts the variables—evidently a quartic transformation. It amounts to passing from the set $P_n^2$ to the set $Q_n^2$, congruent under the quadratic transformation $A_{123}$, with base points at the first three points.

For any set $P_n^2$ the $G_{n!}$ contains a collineation subgroup of order $4(n-4)!$ which is the direct product of the group corresponding to the permutation of the rows of the matrix $\|y_j^{(i)}\|$ and the group generated by the interchange of the columns of the matrix and by the involutory collineation which interchanges the origin and unit point of $\Sigma$ leaving the other reference points unchanged.

## III. The determinant representation.

A. **The $G_{61}$.** A more direct method of obtaining the transformations of the $G_{61}$ from the corresponding permutations is very desirable. That this is possible is indicated a priori by the fact that the properties of the $G_{61}$ are evidently inherent in the canonical form of the $P_6^2$. We proceed to develop an expression for the coördinates $x, y, z, t, u$ as irrational invariants of the set of points.

The values of the three-rowed determinants of the canonical matrix are as follows.

$$
\begin{array}{llllll}
\overline{123} = \overline{124} = \overline{143} = \overline{423} = 1, & \overline{125} = & & \overline{126} = u, \\
\overline{153} = y, & \overline{523} = x, & \overline{163} = t, & \overline{623} = z, \\
\overline{524} = x-u, & \overline{154} = y-u, & \overline{624} = z-u, & \overline{164} = t-u, \\
\overline{543} = x-y, & \overline{643} = z-t, & \overline{156} = u(y-t), & \overline{526} = u(x-z), \\
\overline{563} = xt-yz, & \overline{564} = u(x-y-z+t)+yz-xt. &&
\end{array} \tag{8}
$$

A few trials will show that in general the transforms of $\overline{523}$, $\overline{153}$, etc., under a permutation do not represent the transforms of the corresponding variables under the parallel Cremona transformation. We note, however, that this is true for the collineation $G_8$ generated by (12), (34), and (56). Under (23) the determinants go into others representing linear functions of the variables; to obtain the quadratic functions into which $x, y, z, t,$

and $u$ actually transform the determinants must be multiplied, before permutation, by $\overline{125}$ or $\overline{126}$, according as the point 6 or the point 5 occurs in them. Similarly to satisfy (45) one of the factors $\overline{143}$ or $\overline{423}$ must be introduced. All four of these last factors occur in the expression for $u$, while to give symmetry and at the same time to correct for changes in sign a factor $\overline{124}$ is included in the products representing the first four variables. Thus, as the determinant representation of the variables, we obtain

$$
\begin{aligned}
xu &= \overline{523}\cdot\overline{126}\cdot\overline{143}\cdot\overline{124},\\
yu &= \overline{153}\cdot\overline{126}\cdot\overline{423}\cdot\overline{124},\\
zu &= \overline{623}\cdot\overline{125}\cdot\overline{143}\cdot\overline{124},\\
tu &= \overline{163}\cdot\overline{125}\cdot\overline{423}\cdot\overline{124},\\
uu &= \overline{125}\cdot\overline{126}\cdot\overline{143}\cdot\overline{423}.
\end{aligned}
\tag{9}
$$

If each of these products be multiplied by the factor $\overline{356}\cdot\overline{456}$, the coördinates are expressed as irrational integral invariants of the $P_6^2$.

B. **The $G_{71}$.** A study of the $G_{71}$ of Cremona transformations in a space of six dimensions possesses intrinsic interest and also helps in the following study of the $G_{n1}$. The generating substitutions are

(10)

| | (12) | (23) | (34) | (45) | (56) | (67) |
|---|---|---|---|---|---|---|
| $x'$ | $y$ | $xts$ | $u-x$ | $uy$ | $z$ | $x$ |
| $y'$ | $x$ | $uts$ | $u-y$ | $ux$ | $t$ | $y$ |
| $z'$ | $t$ | $zys$ | $u-z$ | $zy$ | $x$ | $r$ |
| $t'$ | $z$ | $uys$ | $u-t$ | $xt$ | $y$ | $s$ |
| $r'$ | $s$ | $ryt$ | $u-r$ | $ry$ | $r$ | $z$ |
| $s'$ | $r$ | $uyt$ | $u-s$ | $sx$ | $s$ | $t$ |
| $u'$ | $u$ | $yts$ | $u$ | $xy$ | $u$ | $u$ |

The determinant representation in this case is

$$
\begin{aligned}
x &= \overline{523}\cdot\overline{126}\cdot\overline{127}\cdot\overline{143}\cdot\overline{124},\\
y &= \overline{153}\cdot\overline{126}\cdot\overline{127}\cdot\overline{423}\cdot\overline{124},\\
z &= \overline{623}\cdot\overline{125}\cdot\overline{127}\cdot\overline{143}\cdot\overline{124},\\
t &= \overline{163}\cdot\overline{125}\cdot\overline{127}\cdot\overline{423}\cdot\overline{124},\\
r &= \overline{723}\cdot\overline{125}\cdot\overline{126}\cdot\overline{143}\cdot\overline{124},\\
s &= \overline{173}\cdot\overline{125}\cdot\overline{126}\cdot\overline{423}\cdot\overline{124},\\
u &= \overline{125}\cdot\overline{126}\cdot\overline{127}\cdot\overline{143}\cdot\overline{423}.
\end{aligned}
\tag{11}
$$

Multiplication by any one of many products, e. g.,

$$\overline{123}\cdot\overline{145}\cdot\overline{267}\cdot\overline{356}\cdot\overline{357}\cdot\overline{367}\cdot\overline{456}\cdot\overline{457}\cdot\overline{467},$$

expresses the coördinates as irrational invariants of the $P_7^2$.

C. **The $G_{n!}$.** The groups $G_{6!}$ and $G_{7!}$ are special cases of the group $G_{n!}$ in $\Sigma_{2(n-4)}$ associated with a $P_n^2$. The generating transformations of this general group are developed in C I pp. 183–4. It is sufficient to point out here that the interchange (34) leads to a transformation of order $n-4$ of the form

$$y_1'^{(j)} = \frac{y_1^{(j)}\prod_{m=1}^{n-4} y_2^{(m)}}{y_2^{(j)}},$$

$$(12)\qquad y_2'^{(j)} = \frac{u\prod_{m=1}^{n-4} y_2^{(m)}}{y_2^{(j)}}, \qquad (j = 5, 6, \cdots, n),$$

$$u' = \prod_{m=1}^{n-4} y_2^{(m)},$$

where the subscripts indicate columns and the superscripts rows in the canonical form. The transposition (45) results in

$$(13)\qquad y_1'^{(j)} = y_1^{(j)}\,y_2^{(5)}, \quad y_2'^{(j)} = y_2^{(j)}\,y_1^{(5)}, \quad (j = 6, 7, \cdots, n),$$
$$u' = y_1^{(5)}\,y_2^{(5)}.$$

The additional transpositions of the kind $(l, l+1)$ give collineations which replace the first two coördinates of the point $l$ by those of the point $l+1$, and vice versa, leaving the other coördinates invariant. An examination of the following determinant products will show that they satisfy the generating transformations

$$\begin{aligned}
y_1^{(5)} &= \overline{523}\cdot\overline{143}\cdot\overline{124}\cdot\overline{126}\cdot\overline{127}\cdots\overline{12n},\\
y_2^{(5)} &= \overline{153}\cdot\overline{423}\cdot\overline{124}\cdot\overline{126}\cdot\overline{127}\cdots\overline{12n},\\
y_1^{(6)} &= \overline{623}\cdot\overline{143}\cdot\overline{124}\cdot\overline{125}\cdot\overline{127}\cdots\overline{12n},\\
y_2^{(6)} &= \overline{163}\cdot\overline{423}\cdot\overline{124}\cdot\overline{125}\cdot\overline{127}\cdots\overline{12n},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_1^{(n)} &= \overline{n23}\cdot\overline{143}\cdot\overline{124}\cdot\overline{125}\cdot\overline{126}\cdots\overline{12(n-1)},\\
y_2^{(n)} &= \overline{1n3}\cdot\overline{423}\cdot\overline{124}\cdot\overline{125}\cdot\overline{126}\cdots\overline{12(n-1)},\\
u &= \overline{143}\cdot\overline{423}\cdot\overline{125}\cdot\overline{126}\ \cdots\ \cdots\ \overline{12n}.
\end{aligned}\qquad (14)$$

These may be converted into irrational invariants of the $P_n^2$ by multiplication by any of several determinant products; these invariants are of degree $\geqq n-3$, the degree of the coördinates in 1 and 2.

D. **Order of the transformations.** From the above representation we may study the order of the transformations of $G_{n!}$. Under a general permutation the points $1, 2, \cdots, n$ are replaced by others of the set; we indicate these by $1', 2', \cdots, n'$. Now the coördinates $y$ will be replaced by others,

$$
\begin{aligned}
y_1'^{(5)} &= \overline{5'2'3'}\ \overline{1'4'3'}\ \overline{1'2'4'}\ \overline{1'2'6'} \cdots \overline{1'2'n'},\\
y_2'^{(5)} &= \overline{1'5'3'}\ \overline{4'2'3'}\ \overline{1'2'4'}\ \overline{1'2'6'} \cdots \overline{1'2'n'},\\
y_1'^{(6)} &= \overline{6'2'3'}\ \overline{1'4'3'}\ \overline{1'2'4'}\ \overline{1'2'5'} \cdots \overline{1'2'n'},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u' &= \overline{1'4'3'}\ \overline{4'2'3'}\ \overline{1'2'5'}\ \overline{1'2'6'} \cdots \overline{1'2'n'}.
\end{aligned}
\tag{14a}
$$

When the common factors have been removed, these products will give the corresponding transformation. The order of this will be the number of times the points $5, 6, \cdots, n$ occur, minus the number of factors $u$ common to all; the points 1, 2, 3, 4 contributing only numerical factors. Since transformations of the points $5, 6, \cdots, n$ lead simply to collineations, it is evident that the order of the transformations depends only on the way in which the points 1, 2, 3, 4 are replaced. Of these the first two play the same role, as do the last two. The points are therefore designated as $a$, $b$, or $c$, depending on the divisions to which they belong. For example, $bbcc$ indicates that 1 and 2 have been replaced by 3 and 4, while the latter have been replaced by two of the points $5, 6, \cdots, n$. This may be taken as an illustrative case. Assume the transformation $135 \cdot 246$ which is of this type. Our coördinates become

$$
\begin{aligned}
y_1'^{(5)} &= \overline{145}\ \overline{365}\ \overline{346}\ \overline{342}\ \overline{347}\ \overline{348} \cdots \overline{34n},\\
y_2'^{(5)} &= \overline{315}\ \overline{645}\ \overline{346}\ \overline{342}\ \overline{347}\ \overline{348} \cdots \overline{34n},\\
y_1'^{(6)} &= \overline{245}\ \overline{365}\ \overline{346}\ \overline{341}\ \overline{347}\ \overline{348} \cdots \overline{34n},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_1'^{(j)} &= \overline{745}\ \overline{365}\ \overline{346}\ \overline{341}\ \overline{342}\ \overline{348} \cdots \overline{34n},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u' &= \overline{365}\ \overline{645}\ \overline{342}\ \overline{341}\ \overline{347}\ \overline{348} \cdots \overline{34n}.
\end{aligned}
\tag{15}
$$

The total order is $n-2$; and since there are no common factors $u$, that is the order of the transformation. Table (16) below shows the orders of the possible types of transformation.

(16)

| 1 | 2 | $n-4$ | $n-3$ | $n-2$ | $2n-8$ | $2n-7$ | $2n-6$ |
|---|---|---|---|---|---|---|---|
| $aabb$ | $aabc$ | $abab$ | $abbc$ | $abcc$ | $bcaa$ | $bcbc$ | $bccc$ |
| | $aacc$ | $bbaa$ | $bbac$ | $bbcc$ | $bcab$ | $bcac$ | $ccbc$ |
| | | | $abac$ | $accc$ | $ccaa$ | $ccac$ | $cccc$ |
| | | | $acab$ | $accb$ | | $ccab$ | |
| | | | $acac$ | | | $ccbb$ | |
| | | | $acbb$ | | | | |

Since in the $P_6^2$ there are only two points of type $c$, no transformations of the last type appear. We have then the theorem

(17) *The maximum order of a transformation of the $G_{61}$ is* 5.

In general we have the theorem

(18) *The maximum order of a transformation of the $G_{n1}$ is $2n-6$; such transformations always arise for $n \geqq 7$.*

If we restrict ourselves to five points (one of type $c$) many of the cases given above drop out, and the maximum order is seen to be two, as Slaught has shown.

The table shows that the $G_{61}$ consists of transformations of the following orders: 8 collineations, 80 quadratic, 288 cubic, 208 quartic, and 136 quintic. Due to the collineation $G_8$ these occur in sets of eight. The group of permutations contains 15 single, 45 double, and 15 triple interchanges; 40 like 123, and 40 like 123 456; 90 of type 1234, 90 like 1234 56; 144 like 12345, 120 of the kind 123 45, and 120 of the type 123456, in addition to the identity. Hence the transformations comprise the identity, 75 of period 2, 80 of period 3, 180 of period 4, 144 of period 5, and 240 of period 6.

## IV. The Discriminant loci.

A. **Method of Determination.** Up to this point of the discussion we have treated the generic case in which no two of the points of $P_6^2$ coincide, no three are collinear, and the six do not lie on a conic. When one of these situations arises, the point $P$ of the canonical map will trace out a special locus in $\Sigma_4$. Since the above conditions are those under which the cubic surface $C^{(3)}$ has a node, they are known as discriminant conditions, and the corresponding loci are called discriminant loci. We indicate the types of conditions by $\delta_{ij} = 0$, $\delta_{lmn} = 0$, and $d_2 = 0$, respectively. Now as pointed out in C III, the vanishing of some of these factors can readily be expressed by explicit conditions on the coördinates of $P_6^2$—in general a single equation defining a manifold in $\Sigma_4$. This, however, is not the case for those of the type $\delta_{ij} = 0$, nor when the set is taken in the canonical form, for $\delta_{123} = 0$, $\delta_{134} = 0$, $\delta_{234} = 0$, or $\delta_{12n} = 0$.

This difficulty arises from the fact that the points 1, 2, 3, 4 are taken as the reference points and the unit point, and as such two can not be made to coincide or 3 to lie on a line, without invalidating the mapping process.

This difficulty is, however, readily surmountable by use of the determinant representation. Then the coördinates of the map are expressed in terms of the irrational invariants of the $P_6^2$, which are independent of the coördinate system. The common factors $\overline{356}\ \overline{456}$ in no way affect the conditions, so again we study the forms

$$x = \overline{523}\cdot\overline{126}\cdot\overline{143}\cdot\overline{124},\quad y = \overline{153}\cdot\overline{126}\cdot\overline{423}\cdot\overline{124},$$
$$z = \overline{623}\cdot\overline{125}\cdot\overline{143}\cdot\overline{124},\quad t = \overline{163}\cdot\overline{125}\cdot\overline{423}\cdot\overline{124},$$
$$u = \overline{125}\cdot\overline{126}\cdot\overline{143}\cdot\overline{423}.$$

Since collinearity of three points requires the vanishing of their determinant, the discriminant loci are readily obtainable. For example, $\overline{143} = 0$ at once leads to $x = z = u = 0$. At a glance, then, we obtain the loci for $\delta_{126}$, $\delta_{125}$, $\delta_{143}$, $\delta_{423}$, and $\delta_{124}$. The one corresponding to $\delta_{123} = 0$ is obtained from that of $\delta_{124} = 0$ by the collineation (34).

Now for the type $\delta_{ij} = 0$ we proceed as in the following illustration for $\delta_{12} = 0$. Let 2 approach 1 in any direction; it may thus be replaced by $1 + \varepsilon a$. Then the coördinates of $P$ become

$$\begin{array}{rcccc}
x = & (\overline{513} + \varepsilon\,\overline{5a3}) & (\varepsilon\,\overline{1a6}) & (143) & (\varepsilon\,\overline{1a4}), \\
y = & (\overline{153}) & (\varepsilon\,\overline{1a6}) & (\overline{413} + \varepsilon\,\overline{4a3}) & (\varepsilon\,\overline{1a4}), \\
z = & (\overline{613} + \varepsilon\,\overline{6a3}) & (\varepsilon\,\overline{1a5}) & (\overline{143}) & (\varepsilon\,\overline{1a4}), \\
t = & (\overline{163}) & (\varepsilon\,\overline{1a5}) & (\overline{413} + \varepsilon\,\overline{4a3}) & (\varepsilon\,\overline{1a4}), \\
u = & (\varepsilon\,\overline{1a5}) & (\varepsilon\,\overline{1a6}) & (\overline{143}) & (\overline{413} + \varepsilon\,\overline{4a3}).
\end{array} \tag{19}$$

Disregarding infinitesimals of the third order, we see that the locus of $P$ is the plane

$$x = y,\quad z = t.$$

On these special loci the correspondence is no longer 1—1; but this may be restored, as in Cremona transformations in the plane, by considering not only points, but the directions around them.

Table (20) gives the discriminant loci for the types $\delta_{ij} = 0$, $\delta_{lmn} = 0$. Here $A_u$ stands for the unit point, $A_0$ for the origin, $A_1$ for (1, 0, 0, 0, 0), etc.

(20)

| | | | | | |
|---|---|---|---|---|---|
| $\delta_{256}$ | Cone: $xt-yz=0$ | $\delta_{12}$ | $S_3$: $x=z=u$ | $\delta_{236}$ | $S_9$: $z=0$ |
| $\delta_{456}$ | $u(x-y-z+t)+yz-xt=0$ | $\delta_{23}$ | $y=t=u$ | $\delta_{136}$ | $t=0$ |
| $\delta_{123}$ | $S_0$: $A_u$ | $\delta_{45}$ | $x=y=u$ | $\delta_{24}$ | $u=0$ |
| $\delta_{124}$ | $A_0$ | $\delta_{46}$ | $z=t=u$ | $\delta_{145}$ | $u=y$ |
| $\delta_{15}$ | $A_1$ | $\delta_{14}$ | $x=z=0$ | $\delta_{146}$ | $u=t$ |
| $\delta_{25}$ | $A_2$ | $\delta_{24}$ | $y=t=0$ | $\delta_{156}$ | $y=t$ |
| $\delta_{16}$ | $A_3$ | $\delta_{25}$ | $x=y=0$ | $\delta_{256}$ | $z=x$ |
| $\delta_{26}$ | $A_4$ | $\delta_{26}$ | $z=t=0$ | $\delta_{245}$ | $u=x$ |
| $\delta_{125}$ | $S_1$: $u=z=t=0$ | $\delta_{12}$ | $x=y,\ z=t$ | $\delta_{246}$ | $u=z$ |
| $\delta_{126}$ | $u=x=y=0$ | $\delta_{56}$ | $x=z,\ y=t$ | $\delta_{245}$ | $y=x$ |
| $\delta_{124}$ | $u=x=z=0$ | $\delta_{235}$ | $S_8$: $x=0$ | $\delta_{246}$ | $t=z$ |
| $\delta_{234}$ | $u=y=t=0$ | $\delta_{135}$ | $y=0$ | | |

Under the $G_{61}$ there is clearly no way to pass from the conditions of one type to those of another; but under the $G_{6,2}$ they form a conjugate set. This may be seen as follows: Under $A_{123}$ the condition $d_2 = 0$ (all six points a conic) transforms into $\delta_{456} = 0$. All others of the type $\delta_{lmn}$ may be obtained from this by permutation. Those of type $\delta_{ij}$ may be gotten by operating with $A_{123}$ on $\delta_{124}$, giving $\delta_{34}$, and permuting as before.

In section VI we shall exhibit an algebraic method of obtaining the discriminant loci and thereby verify the fact that there are no exceptional loci other than the ones given above.

Table (21) gives the discriminant loci for the $G_{71}$ as found by means of equations (11).

(21)

| | | | | | |
|---|---|---|---|---|---|
| $\delta_{357}$ | Spread: $xs=yr$ | $\delta_{12}$ | $S_3$: $x=y,\ z=t,\ r=s$ | $\delta_{34}$ | $S_5$: $u=0$ |
| $\delta_{356}$ | $xt=yz$ | $\delta_{13}$ | $x=z=r=u$ | $\delta_{145}$ | $u=y$ |
| $\delta_{367}$ | $zs=tr$ | $\delta_{23}$ | $y=t=s=u$ | $\delta_{425}$ | $u=x$ |
| $\delta_{456}$ | $u(x-y-z+t)+yz-xt=0$ | $\delta_{14}$ | $x=z=r=0$ | $\delta_{146}$ | $u=t$ |
| $\delta_{457}$ | $u(x-y-r+s)+yr-xs=0$ | $\delta_{24}$ | $y=t=s=0$ | $\delta_{426}$ | $u=z$ |
| $\delta_{467}$ | $u(z-t-r+s)+tr-zs=0$ | $\delta_{35}$ | $S_4$: $x=y=0$ | $\delta_{147}$ | $u=s$ |
| $\delta_{567}$ | $xt+yr+zs-rt-sx-yz=0$ | $\delta_{36}$ | $z=t=0$ | $\delta_{427}$ | $u=r$ |
| $\delta_{123}$ | $S_0$: $A_u$ | $\delta_{37}$ | $r=s=0$ | $\delta_{156}$ | $y=t$ |
| $\delta_{124}$ | $A_0$ | $\delta_{45}$ | $x=y=u$ | $\delta_{256}$ | $x=z$ |
| $\delta_{15}$ | $A_1$ | $\delta_{46}$ | $z=t=u$ | $\delta_{157}$ | $y=s$ |
| $\delta_{16}$ | $A_3$ | $\delta_{47}$ | $r=s=u$ | $\delta_{257}$ | $x=r$ |
| $\delta_{17}$ | $A_5$ | $\delta_{56}$ | $x=z,\ y=t$ | $\delta_{167}$ | $t=s$ |
| $\delta_{25}$ | $A_2$ | $\delta_{67}$ | $z=r,\ t=s$ | $\delta_{267}$ | $z=r$ |
| $\delta_{26}$ | $A_4$ | $\delta_{57}$ | $x=r,\ y=z$ | $\delta_{345}$ | $x=y$ |
| $\delta_{27}$ | $A_6$ | $\delta_{523}$ | $S_2$ $x=0$ | $\delta_{346}$ | $z=t$ |
| $\delta_{125}$ | $S_1$: $z=t=r=s=u=0$ | $\delta_{153}$ | $y=0$ | $\delta_{347}$ | $r=s$ |
| $\delta_{126}$ | $x=y=r=s=u=0$ | $\delta_{623}$ | $z=0$ | | |
| $\delta_{127}$ | $x=y=z=t=u=0$ | $\delta_{163}$ | $t=0$ | | |
| $\delta_{143}$ | $S_2$: $x=z=r=u=0$ | $\delta_{723}$ | $r=0$ | | |
| $\delta_{423}$ | $y=t=s=u=0$ | $\delta_{173}$ | $s=0$ | | |

**B. The geometric configuration.** In the $G_{61}$ we have then the following configuration of discriminant loci:

The $\Sigma_4$ contains an isolated $\Sigma_3$, $u = 0$ outside of which lie the unit point and the origin (0, 0, 0, 0, 1). In the $\Sigma_3$ there is a skew quadrilateral consisting of the lines $u = z = t = 0$, $u = x = z = 0$, $u = x = y = 0$, and $u = y = t = 0$; the vertices are the reference points $A_1$, $A_2$, $A_3$, $A_4$. These four lines in $u = 0$ are joined to the origin by four planes

$$z = t = 0, \quad x = z = 0, \quad x = y = 0, \quad y = t = 0,$$

and to the unit point by the four

$$u = z = t, \quad u = x = z, \quad u = x = y, \quad u = y = t.$$

Moreover the configuration includes two "Diagonal" planes given by $x = y$, $z = t$ and $x = z$, $y = t$. These pass through both the origin and the unit point, and meet $u = 0$ in the lines of intersection of the planes $x = y$ and $z = t$, $x = z$ and $y = t$ lying in the $\Sigma_3$. Each of the edges of the skew quadrilateral with the two points outside $u = 0$, determine a three-way. The face planes of the quadrilateral, taken with the origin, give four more three-ways; the same is true of the face planes and the unit point. Finally, $\delta_{356}$ and $\delta_{456}$ lead to two cones $xt - yz = 0$ and $xt - yz + u(-x + y + z - t) = 0$, respectively. The first has its vertex at the origin and on it lie the diagonal planes and the four determined

(22)

| Condition | No. of such Loci | Space | Equations |
|---|---|---|---|
| $\delta_{34}$ | 1 | $\Sigma_{2n-7}$ | $u = 0$ |
| $\delta_{123}$ | 1 | $S_0$ | $A_u$ |
| $\delta_{124}$ | 1 | $S_0$ | $A_0$ |
| $\delta_{15}, \cdots, \delta_{1n}; \delta_{25}, \cdots, \delta_{2n}$ | $2n-8$ | $S_0$ | $A_j$ |
| $\delta_{125}, \delta_{126}, \delta_{12n}$ | $n-4$ | $S_1$ | $R^{(5)} \neq 0,\ R^{(j)} = 0$ |
| $\delta_{143}, \delta_{243}$ | 2 | $S_{n-5}$ | $C_i = u = 0$ |
| $\delta_{13}, \delta_{23}$ | 2 | $S_{n-4}$ | $C_i = u$ |
| $\delta_{14}, \delta_{24}$ | 2 | $S_{n-4}$ | $C_i = 0$ |
| $\delta_{12}$ | 1 | $S_{n-4}$ | $y_1^{(j)} = y_2^{(j)}$ |
| $\delta_{523}, \delta_{n23}; \delta_{153}, \delta_{1n3}$ | $2n-8$ | $S_{2n-9}$ | $y_i^{(j)} = 0$ |
| $\delta_{42n}, \delta_{14n}$ | $2n-8$ | $S_{2n-9}$ | $y_i^{(j)} = u$ |
| $\delta_{2n}$ | $n-4$ | $S_{2n-10}$ | $R^{(j)} = 0$ |
| $\delta_{4n}$ | $n-4$ | $S_{2n-10}$ | $R^{(j)} = u$ |
| $\delta_{1,n,n-1}; \delta_{2,n,n-1}$ | $(n-4)(n-5)$ | $S_{2n-10}$ | $y_i^{(j)} = y_i^{(l)}\ (j \neq l)$ |
| $\delta_{56}, \delta_{57}, \cdots, \delta_{n,n-1}$ | $\frac{(n-4)(n-5)}{2}$ | $S_{2n-10}$ | $y_1^{(j)} = y_1^{(l)}$, $y_2^{(j)} = y_2^{(l)}\ (j \neq l)$ |
| $\delta_{356}, \delta_{457}, \delta_{2,n-1,n}$ | $(n-4)(n-5)$ | Quadric | $y_1^{(j)} y_2^{(l)} = y_2^{(j)} y_1^{(l)}$ |
| $\delta_{567}, \delta_{568}, \cdots, \delta_{n-2,n-1,n}$ | $\frac{(n-4)(n-5)(n-6)}{6}$ | Quadric | $\frac{1}{u}\lvert n-2, n-1, n\rvert = 0$ |

by the origin and the sides of the skew quadrilateral. The vertex of the second is at the unit point, and the cone contains the planes joining this to the sides of the quadrilateral as well as the diagonal planes. Evidently all the reference points lie on each of the cones.

A study of the loci for the $G_7!$ reveals new points of interest. The figure in the $\Sigma_5$, $u = 0$ here consists of two planes cut by three lines in the six coördinate vertices; outside lie the unit point and the origin, as before. From $\delta_{12}$ there arises an $S_3$, and $\delta_{567}$ yields a quadric. The rest of the configuration is similar to that for the $G_{6!}$.

In the general case the $n(n^2-1)/6$ discriminant conditions of the types $\delta_{lmn} = 0$ and $\delta_{ij} = 0$ determine loci as given in Table (22). There $C$ stands for each element in turn of a column, $R$ for each of a row; the subscripts and superscripts have the same significance as above, and $i = 1, 2; j = 5, 6, \cdots, n$.

C. **Invariant linear system under $G_{6!}$.** These discriminant loci are used in determining the simplest invariant linear system for the $G_{6!}$. Since under the collineation $G_8$ any spread goes into another of the same type, it is necessary to consider only the quadratic transformations (23) and (45). Under (23)

The unit point (123) is fixed.
Directions around (0, 0, 0, 0, 1) ($\overline{124} = 0$) go into the line $\overline{134} = 0$.
Line $\overline{134} = 0$ goes into directions around $A_0$.
Line $\overline{234} = 0$ is invariant.
Line $\overline{125} = 0$ goes into the 3-way $\overline{135} = 0$.
Line $\overline{126} = 0$ goes into the 3-way $\overline{136} = 0$.

These last two then factor from the transform of a quadric on the lines $\overline{125} = \overline{126} = 0$, leaving a quadric. Now under (45)

The unit point is fixed.
Directions around (0, 0, 0, 0, 1) give the line $\overline{125} = 0$.
Line $\overline{134} = 0$ gives the 3-way $\overline{135} = 0$.
Line $\overline{234} = 0$ gives the 3-way $\overline{235} = 0$.
Line $\overline{125} = 0$ gives directions around the point $\overline{124} = 0$.
Line $\overline{126} = 0$ is invariant.

Hence any linear system of quadrics on the lines $\overline{134} = \overline{234} = 0$ is transformed into quadrics. Now the linear system on the four lines $\overline{125}$, $\overline{126}$, $\overline{134}$, $\overline{234} = 0$ is

$$au^2 + bux + cuy + duz + eut + fxt + gyz = 0.$$

Under (23) this becomes

$$ay^2t^2 + bxyt^2 + cyut^2 + dy^2zt + ey^2ut + fxyut + gyztu = 0.$$

From this $yt$ factors, leaving

$$ayt + bxt + cut + dyz + euy + fux + quz = 0.$$

Since $y = t = 0$ is not on $\overline{134} = 0$, $a$ must be zero. Since a linear system on (0, 0, 0, 0, 1) goes by (34) into a linear system on the unit point we have for the simplest invariant linear system

$$u(bx+cy+dz+et)+fxt+qyz = 0, \tag{23}$$

where

$$b+c+d+e+f+g = 0.$$

## V. Simplest invariant system under the $G_{6,2}$.

In C III it has been shown that the complex invariants lie in a linear system of dimension 9 of spreads invariant under $G_{6,2}$ and that there is no linear system of lower order in the coördinates of the points of $P_6^2$. However, the question as to whether or not there exist other linear systems of the same order invariant under the group is unanswered. We shall demonstrate the fact that there are no such systems.

A spread invariant under $A_{123}$ must be of order $5k$ with $3k$-fold points at the reference basis in $\Sigma_4$; in the simplest case, then, we have quintic spreads with triple points at the reference points. This requires that there be no terms of degree above two in a single variable. Now under $I_{23}$ the plane $t = y = 0$ corresponds to the $P$-spread $u = 0$, and the lines $z = t = u = 0$, $x = y = u = 0$ transform into the spreads $y = 0$ and $t = 0$, respectively. In order that a quintic term be invariant under two applications of $I_{23}$, the extraneous factor introduced by the transformation must be $uy^2t^2$. Then for an invariant linear system of quintics $\sum x^\alpha y^\beta z^\gamma t^\delta u^\varepsilon$ we must have

$$\begin{aligned} \gamma+\delta+\varepsilon &\geqq 2, \\ \alpha+\beta\quad\ +\varepsilon &\geqq 2, \\ \beta\quad+\delta\quad\ &\geqq 1. \end{aligned} \tag{24}$$

Hence the system contains the plane $y = t = 0$ at least singly and the lines $z = t = u = 0$ and $x = y = u = 0$ at least doubly.

Similarly under $I_{45}$ the plane $x = y = 0$ and the lines $y = t = u = 0$, $x = z = u = 0$ give $P$-spreads $u = 0$, $x = 0$, and $y = 0$, respectively. Here the extraneous factor is $ux^2y^2$, giving the conditions

$$\begin{aligned} \alpha+\beta\quad\ &\geqq 1, \\ \beta\quad+\delta+\varepsilon &\geqq 2, \\ \alpha\quad+\gamma\quad+\varepsilon &\geqq 2. \end{aligned} \tag{25}$$

and the plane $x = y = 0$ is contained singly while the lines $y = t = u = 0$ and $x = z = u = 0$ are contained at least doubly.

Moreover it is clear that the transforms of these loci under any element of the group must appear with the same multiplicities. These are the planes $z = t = 0$, $x = z = 0$, $u = x = y$, $u = x = z$, $u = y = t$, $u = z = t$, and the diagonal planes $x = y$, $z = t$; $x = z$, $y = t$. Hence we have the theorem:

*A linear system of quintics invariant under $G_{6,2}$ must have triple points at the six reference points and the unit point; it must contain at least doubly the lines $x = y = u = 0$, $z = t = u = 0$, $x = z = u = 0$, $y = t = u = 0$; and it must contain at least singly the planes*

$$x = y = 0,\quad y = t = 0,\quad z = t = 0,\quad x = z = 0,\quad u = x = y,$$
$$u = x = z,\quad u = y = t,\quad u = z = t,\quad \begin{cases} x = y & x = z \\ z = t, & y = t. \end{cases}$$

These geometric conditions greatly limit the number of quintic terms which may appear in a member of the system. Moreover the last six place conditions on the coefficients of the quintic. We shall study these latter, first, however, listing the available quintic terms in Table (27), placing beside each its general coefficient.

(27)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $u^2xyz$ | $a_1$ | $u^2xzt$ | $a_5$ | $u^2y^2z$ | $b_1$ | $u^2t^2x$ | $b_5$ |
| $u^2xyt$ | $a_2$ | $u^2yzt$ | $a_6$ | $u^2x^2t$ | $b_2$ | $u^2z^2y$ | $b_6$ |
| $t^2xyz$ | $a_3$ | $y^2xzt$ | $a_7$ | $x^2t^2z$ | $b_3$ | $y^2z^2x$ | $b_7$ |
| $z^2xyt$ | $a_4$ | $x^2yzt$ | $a_8$ | $y^2z^2t$ | $b_4$ | $x^2t^2y$ | $b_8$ |
| $ux^2yz$ | $c_1$ | $uz^2xt$ | $c_3$ | $ux^2yt$ | $d_1$ | $uz^2yt$ | $d_3$ |
| $uy^2xt$ | $c_2$ | $ut^2yz$ | $c_4$ | $uy^2xz$ | $d_2$ | $ut^2xz$ | $d_4$ |
| $ux^2zt$ | $e_1$ | $uz^2xy$ | $e_3$ | $y^2z^2u$ | $f_1$ | $xyztu$ | $g$ |
| $uy^2zt$ | $e_2$ | $ut^2xy$ | $e_4$ | $x^2t^2u$ | $f_2$ | | |

The grouping is in general according to type, transforms under $A_{123}$ being placed in the same group. Since the general spread must contain $u = x = y$ let us replace each of them by $v$. The spread is then made up of terms of the types $v^4z$, $v^4x$, $v^3xz$, $v^3z^2$, $v^2z^2x$, and $v^2x^2z$. The identical vanishing of the function thus places seven conditions on the coefficients; these are

(28)
$$\begin{aligned} a_1 + a_3 + a_5 + a_7 + d_2 + d_4 + g &= 0, \\ a_6 + b_1 + c_4 + e_2 &= 0, \\ a_2 + b_5 + c_2 + e_4 &= 0, \\ b_2 + b_8 + d_1 + f_2 &= 0, \\ b_4 + b_6 + d_3 + f_1 &= 0, \\ a_8 + b_3 + c_1 + e_1 &= 0, \\ a_4 + b_7 + c_3 + e_3 &= 0. \end{aligned}$$

By similar processes, the planes $u = x = z$, $u = y = t$, $u = z = t$, $(x = y, z = t)$ and $(x = z, y = t)$ give five sets of seven conditions; we have then all forty-two conditions on the thirty-one coefficients. An examination of these shows, however, that only 21 are linearly independent, or that of the coefficients, ten are independent. This verifies the fact that the invariant quintics form a linear system of dimension nine.

That this is the system determined by the complex invariants is readily shown. The differences like

$$\Gamma_{12,34,56} - \Gamma_{12,56,34}$$

are composed of the factor $d_2$ and one of the type $\overline{a}-\overline{d}$. $d_2 = 0$ satisfies the special geometric conditions given above, while the second factor is quadratic and contains terms in $xt$, $yz$, $ux$, $uy$, $uz$, and $ut$ only. The differences of the $\Gamma$'s then are quintics of the required type and belong to the general invariant linear system of dimension 9. We shall use this fact in the developments of the next section.

## VI. Discriminant loci from complex invariants.

Having determined the simplest invariant linear system we may determine the discriminant loci by algebraic means, as forecast in Section IV.

After factoring out a common $u$, five linearly independent quintics of the system are given by

$$\begin{aligned} &\Gamma_{12,34,56}-\Gamma_{12,56,34} = 0, \quad \Gamma_{13,45,62}-\Gamma_{13,62,45} = 0, \\ &\Gamma_{14,56,23}-\Gamma_{15,32,65} = 0, \quad \Gamma_{15,62,34}-\Gamma_{15,34,62} = 0, \\ &\Gamma_{16,23,45}-\Gamma_{16,45,23} = 0. \end{aligned} \tag{29}$$

Each of these contains the factor $d_2$ and under $A_{123}$ is transformed into another containing $\delta_{456}$. This transformation permutes the complex invariants as shown in C III (23), and if it is followed by (15) (26) leads to

$$\begin{aligned} &\Gamma_{564,312}-\Gamma_{56,12,34} = 0, \quad \Gamma_{53,41,26}-\Gamma_{532,641} = 0, \\ &\Gamma_{634,512}-\Gamma_{54,63,12} = 0, \quad -\Gamma_{54,32,61}+\Gamma_{52,64,31} = 0, \\ &\Gamma_{52,63,41}-\Gamma_{632,541} = 0. \end{aligned}$$

The common factor which these have in addition to the conditions satisfied by all members of the linear system will give the discriminant locus $\delta_{124} = 0$. A study of the above quintics shows that they have a quadruple point at (0, 0, 0, 0, 1) whereas the general quintic has only a triple point there. Hence the directions around this point correspond to the vanishing

of $\delta_{124}$. To find the loci corresponding to coincidences of the points, we again apply $A_{123}$. This gives as quintics containing the locus $\delta_{34} = 0$

$$\begin{aligned} \Gamma_{564,812} - \Gamma_{124,356} &= 0, & -\Gamma_{68,24,51} - \Gamma_{32,15,64} &= 0, \\ -\Gamma_{12,35,46} - \Gamma_{12,45,36} &= 0, & -\Gamma_{326,145} + \Gamma_{351,264} &= 0, \\ -\Gamma_{42,16,53} - \Gamma_{23,16,45} &= 0. \end{aligned}$$

Here we find the common factor $u = 0$, which is the discriminant locus $\delta_{34} = 0$.

By this method we may obtain all the discriminant loci of types $\delta_{ij} = 0$ and $\delta_{lmn}$ and verify that they coincide with those of Table (20).

## VII. Coördinates of $P$ in terms of the cubic surface.

The object of this section is to show how the coördinates of the canonical map may be expressed in terms of the coefficients of the tritangent planes of the $C^{(3)}$ and also in terms of the complex invariants. The Cremona $G_{51}$ thus furnishes an expression of the algebraic relations existing in either case.

Expressing $x, y, z, t, u$ as irrational invariants of the $P_6^2$ and dividing by $(\overline{124} \cdot \overline{356})$ we obtain

$$\begin{aligned} x &= \overline{523}\ \overline{143}\ \overline{126}\ \overline{456}, & y &= \overline{153}\ \overline{423}\ \overline{126}\ \overline{456}, \\ z &= \overline{623}\ \overline{143}\ \overline{125}\ \overline{456}, & t &= \overline{163}\ \overline{423}\ \overline{125}\ \overline{456}. \\ u &= \frac{\overline{143}\ \overline{423}\ \overline{125}\ \overline{126}\ \overline{456}}{\overline{124}}. \end{aligned} \tag{30}$$

By use of relations of the type developed in C II (59), e. g.,

$$\overline{345}\ \overline{312}\ \overline{641}\ \overline{652} \pm \overline{645}\ \overline{612}\ \overline{341}\ \overline{352} = \frac{\overline{ab}}{-d_2},$$

these may be transformed into*

$$\begin{aligned} x &= (d_2 + \overline{ab})\ (d_2 + \overline{bc}), & y &= (d_2 - \overline{de})\ (d_2 + \overline{bc}), \\ z &= (d_2 - \overline{df})\ (d_2 + \overline{bc}), & t &= (d_2 + \overline{ac})\ (d_2 + \overline{bc}), \\ u &= (d_2 - \overline{df})\ (d_2 - \overline{de}). \end{aligned} \tag{31}$$

From C III (21) we see that

$$\begin{aligned} \overline{\alpha} - \overline{\beta} &= d_2(\overline{b} - \overline{a}) + 2(\overline{b}^3 - \overline{a}^3) \\ &= \overline{b} - \overline{a}\,[a_2 + 2(\overline{a}^2 + \overline{a}\,\overline{b} + b^2)] \\ &= \overline{b} - \overline{a}\,(\overline{ab}). \end{aligned}$$

* The expressions obtained by Professor Coble (C III, p. 198) are incorrect, due to an error in the sign of $d_2$.

Substituting in (31) the resulting values for $ab$, etc., and clearing of fractions, we find

$$
\begin{aligned}
x &= [d_2(\bar{b}-\bar{a})+\bar{\alpha}-\bar{\beta}]\ [d_2(\bar{c}-\bar{b})+\bar{\beta}-\bar{\gamma}]\,(\bar{e}-\bar{d})\,(\bar{f}-\bar{d})\,(\bar{c}-\bar{a}),\\
y &= [d_2(\bar{e}-\bar{d})+\bar{\delta}-\bar{\varepsilon}]\ [d_2(\bar{c}-\bar{b})+\bar{\beta}-\bar{\gamma}]\,(\bar{b}-\bar{a})\,(\bar{f}-\bar{d})\,(\bar{c}-\bar{a}),\\
z &= [d_2(\bar{f}-\bar{d})+\bar{\delta}-\bar{\zeta}]\ [d_2(\bar{c}-\bar{b})+\bar{\beta}-\bar{\gamma}]\,(\bar{b}-\bar{a})\,(\bar{e}-\bar{d})\,(\bar{c}-\bar{a}),\\
t &= [d_2(\bar{c}-\bar{a})+\bar{\alpha}-\bar{\gamma}]\ [d_2(\bar{c}-\bar{b})+\bar{\beta}-\bar{\gamma}]\,(\bar{b}-\bar{a})\,(\bar{e}-\bar{d})\,(\bar{f}-\bar{d}),\\
u &= [d_2(\bar{e}-\bar{d})+\bar{\delta}-\bar{\varepsilon}]\ [d_2(\bar{f}-\bar{d})+\bar{\delta}-\bar{\zeta}]\,(\bar{b}-\bar{a})\,(\bar{c}-\bar{a})\,(\bar{c}-\bar{b}).
\end{aligned}
\tag{32}
$$

In terms of the complex invariants, these relations become

$$
\begin{aligned}
x &= [-\Gamma_{12,35,46}+\Gamma_{16,45,23}]\ [-\Gamma_{15,64,23}+\Gamma_{13,62,45}]\ [\Gamma_{14,63,52}-\Gamma_{14,52,63}]\\
&\quad [\Gamma_{16,24,35}-\Gamma_{16,35,24}]\ [\Gamma_{14,62,53}-\Gamma_{14,53,62}],\\
y &= [-\Gamma_{16,23,45}+\Gamma_{12,56,34}]\ [-\Gamma_{15,64,23}+\Gamma_{13,62,45}]\ [\Gamma_{15,63,24}-\Gamma_{15,24,63}]\\
&\quad [\Gamma_{16,24,35}-\Gamma_{16,35,24}]\ [\Gamma_{14,62,53}-\Gamma_{14,53,62}],\\
z &= [\Gamma_{12,56,34}-\Gamma_{13,46,52}]\ [-\Gamma_{15,64,23}+\Gamma_{13,62,45}]\ [\Gamma_{15,63,24}-\Gamma_{15,24,63}]\\
&\quad [\Gamma_{14,63,52}-\Gamma_{14,52,63}]\ [\Gamma_{16,24,35}-\Gamma_{16,35,24}],\\
t &= [\Gamma_{12,34,56}-\Gamma_{15,64,23}]\ [-\Gamma_{15,64,23}+\Gamma_{13,62,45}]\ [\Gamma_{15,63,24}-\Gamma_{15,24,63}]\\
&\quad [\Gamma_{14,63,52}-\Gamma_{14,52,63}]\ [\Gamma_{16,24,35}-\Gamma_{16,35,24}],\\
u &= [-\Gamma_{16,23,45}+\Gamma_{15,56,34}]\ [\Gamma_{12,56,34}-\Gamma_{13,46,52}]\ [\Gamma_{15,63,24}-\Gamma_{15,24,63}]\\
&\quad [\Gamma_{14,62,53}-\Gamma_{14,53,62}]\ [\Gamma_{16,34,25}-\Gamma_{16,25,34}],
\end{aligned}
\tag{33}
$$

the expressions we seek.

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# TRANSFORMATIONS OF THE KUMMER CRITERIA IN CONNECTION WITH FERMAT'S LAST THEOREM.

## (SECOND PAPER.)

By H. S. Vandiver.*

In the first article† under the present title I gave some transformations of the Kummer criteria which are useful in obtaining criteria for the solution of

$$u^p + v^p + w^p = 0, \tag{1}$$

where $u$, $v$ and $w$ are rational integers prime to $p$ of the type $m^{p-1} \equiv 1 \pmod{p^2}$. The present paper contains other transformations of the same type. There are also developed here methods for extending the known criteria for (1) of the type

$$B_{\mu-1} \equiv B_{\mu-2} \equiv B_{\mu-3} \equiv B_{\mu-4} \equiv 0 \pmod{p},$$

where $\mu = (p-1)/2$ and $B_1 = 1/6$, $B_2 = 1/30$, etc. are the numbers of Bernoulli. The Kummer criteria are as follows. If (1) is satisfied in rational integers prime to $p$, then

$$B_n f_{p-2n}(t) \equiv 0, \qquad f_{p-1}(t) \equiv 0 \pmod{p}, \tag{2}$$

where $-t = u/v, v/w, w/u, u/w, v/w, w/v$ and

$$f_a(t) = \sum_{i=1}^{p-1} i^{a-1} t^i, \qquad a > 1, \qquad n = 1, 2, \cdots, \mu - 1.$$

1. We first establish the following identity:

$$\frac{x^j - y^k}{(x-1)(y-1)} = \frac{\sum_{n=0}^{j-1} x^n y^{k-a_n}}{y-1} - \frac{\sum_{l=0}^{k-1} y^l x^{j-b_l}}{x-1}, \tag{3}$$

where $a_n = [nk/j]$, $b = [lj/k]$, $[u]$ is the greatest integer in $u$, $j$ and $k$ are any positive integers, and $x$, $y$ are indeterminates.

---

* Presented to the American Mathematical Society, December 29, 1926; received December 27, 1926. The author was enabled to complete this investigation through a grant from the Heckscher Foundation for the Advancement of Research founded by August Heckscher at Cornell University.

† Annals of Math., (2), vol. 27 (1926), pp. 171-176.

We first prove this result for the case where $j$ and $k$ are prime to each other. Assume $k<j$. We have

$$(3a)\quad \sum_{n=0}^{j-1} x^n y^{-a_n} = 1+x+\cdots+x^{b_1}+y^{-1}(x^{b_1+1}+x^{b_1+2}+\cdots+x^{b_2})$$

$$+y^{-2}(x^{b_2+1}+x^{b_2+2}+\cdots+x^{b_3})+\cdots+y^{-(k-1)}(x^{b_{k-1}+1}+\cdots+x^{j-1}),$$

whence

$$(4)\quad (x-1)\,y^k \sum_n x^n y^{-a_n} = \sum_{l=0}^{k-1} y^{k-l}\, x^{b_{l+1}+1} - \sum_{i=1}^{k-1} y^{k-i}\, x^{b_i+1} - y^k.$$

Also, by direct multiplication

$$(5)\quad (y-1)\,x^j \sum_l y^l x^{-b} = \sum_{l=0}^{k-1} y^{l+1}\, x^{j-b_l} - \sum_{f=1}^{k-1} y^f\, x^{j-b_f} - x^j.$$

We have obviously $j-1 = b_s + b_{k-s}$, whence

$$\sum_{l=0}^{k-1} y^{l+1}\, x^{j-b_l} = \sum_{l=0}^{k-1} y^{l+1}\, x^{b_{k-l}+1}.$$

Set $l+1 = k-t$. As $l$ ranges over the values 0 to $k-1$, $t$ ranges over the values in reverse order, hence

$$\sum_{l=1}^{k-1} y^{l+1}\, x^{b_{k-l}+1} = \sum_t y^{k-t}\, x^{b_{t+1}+1} = \sum_l y^{k-l}\, x^{b_{l+1}+1},$$

and similarly

$$\sum_{i=1}^{k-1} y^l\, x^{j-b_l} = \sum_{i=1}^{k-1} y^{k-i}\, x^{b_i+1}.$$

Using these relations in (4) and (5) the relation (3) easily follows for the case $k$ prime to $l$.

If $k$ is not prime to $l$, we note that in (3a), if $\delta j/k$ is an integer, then the coefficient of $y^{-s}$ arranged in ascending powers of $x$ begins with the term $x^{b_s}$ in lieu of $x^{b_s+1}$. Also in place of $j-1 = b_s + b_{k-s}$ used in transforming (5) we have, if $sj/k$ is an integer,

$$j = \frac{sj}{k} + \frac{(k-s)j}{k}.$$

Modifying the above argument in accord with these relations we obtain (3) for any $j$ and $k$.

Multiply (3) by $(x^p-1)(y^p-1)$, set $x = xe^{vk}$, $y = ye^{vj}$, divide by $e^{vjk}$, differentiate $(p-2)$ times with respect to $v$, and set $v = 0$; we have, if $f_1(x) = (x^p-1)/(x-1)$,

$$(6)\qquad \begin{aligned}&(x^j-y^k)\sum_{i=1}^{p-1}\binom{p-2}{i-1}k^{p-i-1}j^{i-1}f_{p-i}(x)f_i(y)\\ &\equiv y^k(x^p-1)\sum_{n=0}^{j-1}x^n y^{-a_n}h^{(j)}_{c_n}(y)-x^j(y^p-1)\sum_{l=0}^{k-1}y^l x^{-b_l}h^{(k)}_{d_l}(x)\pmod p,\end{aligned}$$

where $h^{(m)}_g(x)=\sum_{r=0}^{p-1}(mr+g)^{p-2}x^r$; $c_n$ is the least positive or zero residue of $nk$ modulo $j$; $d_l$ is the least positive or zero residue of $lj$ modulo $k$. Setting $x=y=t$ in (6) we get as criteria for (1) $uvw\not\equiv 0\pmod p$, using $f_n(t)f_{p-n}(t)\equiv 0\pmod p$, $n=1,2,\cdots,p-1$,

$$(7)\qquad \sum_{n=0}^{j-1}t^{n+k-a_n}h^{(j)}_{c_n}(t)\equiv\sum_{l=0}^{k-1}t^{l+j-b_l}h_{d_l}(t)\pmod p.$$

In another paper* I gave the criteria

$$f_{p-n}(t)f_n(1-t)\equiv 0\pmod p,\qquad n=2,3,\cdots,p-2,$$

and the relation is also true for $n=1$ and $p-2$ from (2). Applying these to (6) we have

$$(8)\quad (t^p-1)\sum_{n=0}^{j-1}t^n(1-t)^{k-a_n}h^{(j)}_{c_n}(1-t)\equiv -t^p\sum_{l=0}^{k-1}t^{j-b_l}(1-t)^l h_{d_l}(t)\pmod p.$$

I have verified for a number of special cases, but have not proved, the identity

$$(9)\qquad \frac{x^j-y^k}{(x-1)(y-1)}=\frac{\sum_{n=0}^{j-1}y^{k-a_n-r_n}x^n}{y-1}-\frac{\sum_{l=0}^{k-1}y^l x^{j-b_l-s_l}}{x-1},$$

where $[nk]_j=c_n$; $[lj]_k=d_l$ and

$$r_n=\left[\frac{\nu+[nk]_j}{j}\right],\qquad s_l=\left[\frac{\nu+[lj]_k}{k}\right]$$

where $\nu$ is a positive integer such that $\nu<k$ and $\nu<j$. It is possible to prove, however the identity

$$(10)\qquad \frac{x-y^k}{(x-1)(y-1)}=\frac{y^{k-\nu}}{y-1}-\frac{\sum_{l=0}^{k-1}x^{1-\left[\frac{l+\nu}{k}\right]}y^l}{x-1}$$

where $\nu$ is an integer subject to the single restriction $\nu<k$. To show this we note that

* Proc. Nat. Acad. Sci., vol. 11 (1925), pp. 292-298.

$$\sum_{l=0}^{k-1} x^{1-\left[\frac{l+\nu}{k}\right]} y^l = x(1+y+\cdots+y^{k-1-\nu})+y^{k-\nu}(1+y+\cdots+y^{\nu-1})$$
$$= x\frac{y^{k-\nu}-1}{y-1}+y^{k-\nu}\frac{y^\nu-1}{y-1}$$

from which (10) readily follows.

*Assuming that* (9) *holds,* it gives

$$(11)\qquad (x^j-y^k)f_1(x)f_1(y)$$
$$= f_1(y)(x^p-1)\sum_{n=0}^{j-1} y^{k-a_n-r_n}x^n-(y^p-1)f_1(x)y^l x^{j-b_l-s_l}.$$

Set in this relation $x = e^{2\nu k}x$, $y = e^{2\nu j}y$, multiply by $e^{\nu(k+j-2kj)}$, differentiate $(p-2)$ times, and put $\nu = 0$; we have, if we select $j$ and $k$ so that $\nu = (j+k)/2$, an integer,

$$(x^j-y^k)\sum_{i=1}^{p-1}\binom{p-2}{i-1}k^{i-1}j^{p-i-1}h_i^{(2,1)}(x)h_{p-i}^{(2,1)}(y)$$
$$(12)\qquad \equiv (x^p-1)\sum_{n=0}^{j-1}2^{p-2}y^{k-a_n-r_n}x^n h_{p-1}^{(j,\alpha_n)}(y)$$
$$-(y^p-1)\sum_{l=0}^{k-1}2^{p-2}y^l x^{j-b_l-s_l}h_{p-1}^{(k,\beta_l)}(x)\pmod p$$

where

$$\alpha_n = [\nu+[nk]_j]_j,\quad \beta_l = [\nu+[lj]_k]_k$$

and

$$\sum_{i=0}^{p-1}(\gamma i+\delta)^{i-1}x^i = h_i^{(\gamma,\delta)}(x).$$

This follows from the fact that the exponent of $e^\nu$ before differentiating in the first term of the right hand member of (11) involving $y^i$ is

$$k+j-2jk+2nk+2j(k-a_n-r_n)+2ji.$$

Setting

$$\nu+[nk]_j = jn_r+\alpha_n,\qquad nk = j\left[\frac{nk}{k}\right]+[nk]_j,$$

the exponent reduces to $2ij+2\alpha_n$.

Applying the criteria (5) of the first paper for the solution of (1) to (12) we find a set of criteria for (1) involving the $h$-functions. Use (11), take any positive integer $m$, select $j$ and $k$ so that $\nu = (\theta j+(m-\theta)k)/m$ is an integer for $\theta = 1, 2, \cdots, m-1$, and proceed in a way analogous to that in which (12) was derived. We have a relation to which we may apply the criteria (3) of the first paper for the solution of (1) and we obtain some more sets of criteria involving the $h$-functions, under the assumption that (9) holds.

2. If (1) is satisfied in integers none zero and all prime to the odd prime $p$, Kummer* proved that

$$B_{\mu-1} \equiv B_{\mu-2} \equiv 0 \pmod{p}. \tag{13}$$

By extending the method of Kummer it was shown by Mirimanoff† that

$$B_{\mu-3} \equiv B_{\mu-4} \equiv 0 \pmod{p} \tag{14}$$

hold under like conditions. His argument involves considerable numerical computation. We shall now obtain certain transformations of the Kummer criteria which enable us to simplify the derivation of (13) and (14) and to proceed further along this line.

Consider the obvious identity

$$f_1(x)f_1(y) = \frac{y(x^p-1)f_1(y)}{x-y} = \frac{x(y^p-1)f_1(x)}{x-y}.$$

Set $x = e^v x$, $y = e^v y$, differentiate $m$ times with respect to $v$, and put $v = 0$; we have, after multiplying by $(x^p-y^p)/(x-y)$,

$$(x^p-y^p)\left(\sum_{n=1}^{m+1}\binom{m}{n-1}f_n(x)f_{m+2-n}(y)\right)$$

$$\equiv -(y^p-1)xf_{m+1}(x)\frac{x^p-y^p}{x-y} + y(x^p-1)f_{m+1}(y)\frac{x^p-y^p}{x-y} \pmod{p}.$$

In this relation put $x = xe^v$, differentiate $(p-2-m)$ times with respect to $v$, put $v = 0$, and write

$$f_1(x, y) = \frac{x^p-y^p}{x-y}y$$

and for $k>1$

$$f_k(x, y) = \sum_{s=1}^{p-1} s^{k-1}\, y^{p-s}\, x^s;$$

we obtain

$$(x^p-y^p)\sum_{n=1}^{m+1}\binom{m}{n-1}f_{n+p-2-m}(x)f_{m+2-n}(y)$$

$$\equiv (x^p-1)f_{m+1}(y)f_{p-1-m}(x, y) - (y^p-1)\sum_{n=1}^{p-1-m}\binom{p-2-m}{n-1}f_{p-n}(x)f_n(y, x) \pmod{p}.$$

* Berlin. Abhandlungen, 1857, pp. 41–74.

† Journ. für Math. (Crelle), vol. 128 (1905), pp. 45–54.

In this relation set $y = t$, $x = -1$; then the Kummer criteria give for $m < p-2$

$$(15)\quad (t^p-1)\sum_{n=1}^{p-1-m}\binom{p-2-m}{n-1} f_{p-n}(-1)f_n(-t) \equiv -2f_{m+1}(t)f_{p-1-m}(-t) \pmod p$$

as criteria for (1) in Case I, since

$$(16)\qquad f_k(-1) \equiv \frac{2-2^{k+1}}{k} b_k \pmod p$$

where $b_1 = -1/2$, $b_{2s} = (-1)^{s-1} B_s$, $b_{2s+1} = 0$, $s > 0$.

In (15) set $m = p-3$; we have

$$(17)\quad (t^p-1)f_{p-1}(-1)f_1(-t) + f_{p-2}(-1)f_2(-t) \equiv -2f_{p-2}(t)f_2(-t) \pmod p.$$

Relation (16) becomes for $k = p-1$

$$f_{p-1}(-1) \equiv \frac{1-2^{p-1}}{p} \pmod p.$$

If (1) is satisfied in Case I, it is known that

$$2^{p-1} \equiv 1 \pmod{p^2};$$

hence $f_{p-1}(-1) \equiv 0 \pmod p$. Also $f_{p-2}(-1) \equiv 0 \pmod p$ by (16), so that (17) reduces to

$$f_{p-2}(t)f_2(-t) \equiv 0 \pmod p.$$

Take $m = p-5$ in (15) and multiply the resulting congruence by $B_2$; we have by (2), since $f_{p-4}(-1) \equiv 0 \pmod p$,

$$B_2 f_{p-3}(-1)f_3(-t) \equiv 0 \pmod p.$$

In a similar way we find, using (2) and $f_{p-a}(-1) \equiv 0 \pmod p$, for $a$ even,

$$B_2 B_3 f_{p-5}(-1)f_5(-t) \equiv 0 \pmod p.$$

We shall now assume for any $a < \mu - 1$,

$$(18)\qquad B_2 B_3 \cdots B_{a+1} f_{p-(2a+1)}(-1)f_{2a+1}(-t) \equiv 0 \pmod p$$

and prove therefrom that

$$(18a)\qquad B_2 B_3 \cdots B_{a+2} f_{p-(2a+3)}(-1)f_{2a+3}(-t) \equiv 0 \pmod p.$$

Take (15) for $m = p-(2a+5)$ and multiply by $B_2 B_3 \cdots B_{a+2}$; we obtain by (2) and (18) the relation (18a) for any $a < \mu$. Hence by induction we have

THEOREM. *If* $u^p+v^p+w^p=0$ *is satisfied in integers none zero and all prime to the odd prime* $p$, *then*

$$(2^{p-(2a+1)}-1)\prod_{s=2}^{a+1} B_s B_{\mu-a} f_{2a+1}(-t) \equiv 0 \pmod{p}, \quad a=1,2,\cdots,\mu-1.$$

This is equivalent to a result which I gave without proof in a previous paper.*

We shall now employ the theorem in connection with the Bernoulli number criteria of Kummer and Mirimanoff already referred to. Since we may take $p>7000$, it is easily seen that it yields, for small values of $a$,

$$B_{\mu-a} f_{2a+1}(-t) \equiv 0 \pmod{p}. \tag{19}$$

But by (2) we also know that this solution holds for $t$ in lieu of $-t$. Hence

$$B_{\mu-a} f_{2a+1}(z) \equiv 0 \pmod{p} \tag{19a}$$

holds for

$$z = t,\ \frac{1}{t},\ 1-t,\ \frac{1}{1-t},\ \frac{t-1}{t},\ \frac{t}{t-1}, \tag{20}$$

and also for

$$z = -t,\ -\frac{1}{t},\ t-1,\ \frac{1}{t-1},\ \frac{1-t}{t},\ \frac{t}{1-t}. \tag{20a}$$

As is known, the six values (20) are incongruent modulo $p$ unless $t^2-t+1 \equiv 0$ or $t=-1, 2, 1/2$. The first condition cannot hold, however, since Pollaczek† by a very ingenious argument showed that the congruence $t^2-t+1 \equiv 0 \pmod{p}$ is inconsistent with (2). The same method gives immediately

$$t^2+1 \not\equiv 0 \pmod{p}. \tag{21}$$

No member of the set (20) is congruent to any member of the set (20a) unless the relation

$$z^2-z-1 \equiv 0 \pmod{p} \tag{22}$$

holds for $z$ one of the set (20), using (21). The congruence (22) is not satisfied unless 5 is a quadratic residue of $p$.

It is known‡ that there exists a polynomial $R_i(z)$ with integral coefficients and of degree $i-1$ such that

$$(z-1)^{p-i} R_i(z) \equiv f_i(z) \pmod{p}.$$

* Bull. Amer. Math. Soc., vol. 28 (1922), p. 404. In error the factor $2^{p-(2a+1)}-1$ was omitted.

† Wiener Ber., Abt. IIa, vol. 126 (1917), pp. 45–59.

‡ Cf. Mirimanoff, loc. cit., p. 57.

The use of this relation together with (19a) and (22) simplifies considerably the computation given by Mirimanoff (pp. 51–53) for obtaining (14) and renders practicable the examination of $R_{11}(z)$ and $R_{13}(z)$ with a possible derivation of $B_{\mu-5} \equiv B_{\mu-6} \equiv 0 \pmod{p}$ in view. The writer* showed that

$$1+\frac{1}{2}+\cdots+\frac{1}{\left[\frac{p}{5}\right]} \equiv 0 \pmod{p}$$

and also

$$\sum_r (5r+a)^{p-2} \equiv 0 \mod p) \tag{23}$$

where $r$ ranges so that $(5r+a)$ assumes all values $<p$ of this form. Let $p \equiv 1 \pmod 5$ and assume that $k$ is a primitive root of $x^5 \equiv 1 \pmod{p}$; then (23) gives easily $f_{p-2}(k) \equiv 0 \pmod{p}$ or

$$\frac{(k-1)^p - k^p + 1}{p} \equiv 0 \pmod{p}.$$

It is possible to choose $k$ so that $k^p \equiv k \pmod{p^2}$; if this is done, we have

$$(k-1)^{p-1} \equiv 1 \pmod{p^2} \tag{24}$$

for $k$ any root of the congruence $x^5 \equiv 1 \pmod{p}$, $p \equiv 1 \pmod 5$, selected so that $k^{p-1} \equiv 1 \pmod{p^2}$.

Frobenius† proved results equivalent to the following

$$\sum_r (ir+a)^{p-2} \equiv 0 \pmod{p},$$

$r$ ranging over the integers 1, 2, $\cdots$, such that $ir+a<p$ and provided $i$ has various small integral values. If for a particular $i$ we assume $p \equiv 1 \pmod{i}$ and $j$ is a root of $x^i \equiv 1 \pmod{p}$ with $j^{p-1} \equiv 1 \pmod{p^2}$, then we find

$$(j-1)^{p-1} \equiv 1 \pmod{p^2}$$

in the same way that (24) was obtained.

* Journ. für Math. (Crelle), vol. 144 (1914), pp. 314–318.

† Berlin. Sitzber., 1914, pp. 653–681.

# THEOREMS ON DEDUCIBILITY.

## (SECOND PAPER.)*

By C. H. Langford.

Questions of deducibility arise especially in connection with sets of defining properties for types of order. A set of properties $p_1, \cdots, p_n$ may be related to a class of properties $Q$ in such a way that, if $q$ be any member of $Q$, one of the properties $q$ or $\sim q$ follows from $p_1, \cdots, p_n$ jointly. A previous paper has been concerned with the class of all first-order functions which can be formulated on the base K, $R_2$; and sets of defining properties for three types of dense series have been studied in relation to this class of functions.† It has been shown that each of these sets of properties is sufficient to determine the truth-value of every first-order function on K, $R_2$, in the sense that one or the other of every pair of mutually contradictory functions on this base follows from the set. The present paper will be concerned with this same class of functions, namely, the class of all first-order functions on K, $R_2$, in relation to a set of properties for discrete series, which have a first but no last element, and which are such that every element but one has an immediate predecessor. It will be shown that every first-order function on K, $R_2$ has its truth-value determined by this set.

In the first paper, we have had occasion to show that every first-order function is equivalent to some function having one of the forms

$$(1) \qquad (x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) :. (\exists x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) : \cdots f(x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, \cdots)$$

or

$$(2) \qquad (\exists x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) :. \quad (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) : \cdots f(x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, \cdots),$$

in which $f$ is an elementary function. It has also been pointed out that any elementary function $f$, on K, $R_2$, can be expanded with respect to K, R, I,‡ in such a way that $f$ is a disjunctive function of conjunctive functions of the primary propositional constituents on K, R, I, and such that, if $p$ be any primary propositional constituent on K, R, or I, formed of the variables $x_1, \cdots$ in $f$, either $p$ or $\sim p$ occurs in each conjunctive function.

* Presented to the American Mathematical Society, December 28, 1926, received January 29, 1927.

† Some theorems on deducibility, Annals of Math., vol. 28 (1926), p. 16.

‡ Identity.

Accordingly, any first-order function on K, $R_2$ can be expressed in one or the other of these forms, and in such a way that the elementary function $f$ is in expanded form with respect to K, R, I.

The following properties characterize linear discrete series which have a first but no last element and in which every element but one has an immediate predecessor.

1. $(x) . \sim \mathrm{R}xx.$
2. $(x, y) : x, y \varepsilon \mathrm{K} . x \neq y \;.\supset.\; \mathrm{R}xy \vee \mathrm{R}yx.$
3. $(x, y) : x, y \varepsilon \mathrm{K} . x \neq y \;.\supset.\; \sim \mathrm{R}xy \vee \sim \mathrm{R}yx.$
4. $(x, y, z) :. x \neq y . y \neq z . x \neq z . x, y, z \varepsilon \mathrm{K} \;.\supset:\; \mathrm{R}xy . \mathrm{R}yz \;.\supset.\; \mathrm{R}xz.$
5. $(x) ::: (\exists y) ::. (z) :: x \varepsilon \mathrm{K} :.\supset:. y \varepsilon \mathrm{K} . \mathrm{R}xy :. z \varepsilon \mathrm{K} :\supset:\; \mathrm{R}zx . \mathrm{R}zy \;.\vee.\; \mathrm{R}yz . \mathrm{R}xz \;.\vee.\; x = z \vee y = z.$
6. $(\exists x) :. (y) : x \varepsilon \mathrm{K} : y \varepsilon \mathrm{K} . x \neq y \;.\supset.\; \mathrm{R}xy.$

These properties are especially relevant to elements belonging to the class K; properties 7, 8 take account of elements not in K.

7. $(x, y) :. \sim x \varepsilon \mathrm{K} \;.\vee.\; \sim y \varepsilon \mathrm{K} :\supset:\; \sim \mathrm{R}xy.$
8. $(\exists x_1, x_2, \cdots) . x_1 \neq x_2 \cdots \sim x_1 \varepsilon \mathrm{K} . \sim x_2 \varepsilon \mathrm{K} \cdots.$

The properties of the set 1–8 will be studied by means of a subsidiary set of properties, 1′–8′, on the base K, K′, $R_2$.

1′. $(x) : x \varepsilon \mathrm{K} \vee x \varepsilon \mathrm{K}' \;.\supset.\; \sim \mathrm{R}xx.$
2′. $(x, y) : x \varepsilon \mathrm{K} . y \varepsilon \mathrm{K} . x \neq y \;.\supset.\; \mathrm{R}xy \vee \mathrm{R}yx.$
3′. $(x, y) : x \varepsilon \mathrm{K} . y \varepsilon \mathrm{K} . x \neq y \;.\supset.\; \sim \mathrm{R}xy \vee \sim \mathrm{R}yx.$
4′. $(x, y, z) :. x \varepsilon \mathrm{K} . y \varepsilon \mathrm{K} . z \varepsilon \mathrm{K} . x \neq y . y \neq z . x \neq z :\supset:\; \mathrm{R}xy . \mathrm{R}yz \;.\supset.\; \mathrm{R}xz.$
5′. $(x) ::: (\exists y) ::. (z) :: x \varepsilon \mathrm{K} :.\supset:. y \varepsilon \mathrm{K} . \mathrm{R}xy :. z \varepsilon \mathrm{K} :\supset:\; \mathrm{R}xz . \mathrm{R}yz \;.\vee.\; \mathrm{R}zx . \mathrm{R}zy \;.\vee.\; x = z \vee y = z.$
6′. $(\exists x) : (y) . x \varepsilon \mathrm{K} . y \varepsilon \mathrm{K} \supset \mathrm{R}xy.$
7′. $(x, y) : x \varepsilon \mathrm{K}' \vee y \varepsilon \mathrm{K}' \;.\supset.\; \sim \mathrm{R}xy.$
8′. $(\exists x_1, x_2, \cdots) . x_1 \neq x_2 \cdots x_1 \varepsilon \mathrm{K}' . x_2 \varepsilon \mathrm{K}' \cdots.$

Since there are at least two elements in K, it follows from 2′, 7′ that $\sim (\exists x) . x \varepsilon \mathrm{K} . x \varepsilon \mathrm{K}'$. K and K′ are non-overlapping classes $((x) : x \varepsilon \mathrm{K} . \supset . \sim x \varepsilon \mathrm{K}')$, and it is, of course, easy to show that K has at least $n$ elements. The relation of the set 1′–8′ to the set 1–8 is seen when the value $\sim \mathrm{K}$ is assigned to K′; that is, when $(x) : x \varepsilon \mathrm{K} \;.\equiv.\; \sim x \varepsilon \mathrm{K}'$. In this case, 1′–8′ reduces to 1–8; the hypothesis in 1′ becomes logically necessary, and so vanishes; and the property $\sim (\exists x) . x \varepsilon \mathrm{K} . x \varepsilon \mathrm{K}'$ is

certifiable on logical grounds alone. The set 1′–8′ places no restriction whatever on elements which are neither in K nor in K′. In the arguments which follow, the set on K, K′, $R_2$ will be under consideration; and then the results established will be applied, in particular, to the set on K, $R_2$.

Just as any elementary function on K, $R_2$ can be expressed in expanded form with respect to K, R, I, so any elementary function on K, K′, $R_2$ can be expressed in expanded form with respect to K, K′, R, I; so that every first-order function on K, K′, $R_2$ can be expressed so as to involve a single complex quantifier applied to an elementary function in expanded form with respect to K, K′, R, I. In what follows we shall not be concerned with functions which place any restriction on elements which are neither in K nor in K′. Let $F(x_1, \cdots, x_\mu, y_1, \cdots, y_\nu)$ be a function on K, K′, $R_2$ which involves a single complex quantifier, such that the variables $x_1, \cdots, x_\mu$ are quantified universally and the variables $y_1, \cdots, y_\nu$ are quantified particularly; and let $f$ be the elementary function in $F$, expressed in expanded form with respect to K, K′, R, I. Denote by $f'$ an elementary function derived from $f$ by omitting from $f$ all those alternatives in which at least one of the variables $x_1, \cdots, x^\mu, y_1, \cdots, y_\nu$ is such that it is assigned both to $\sim$K and to $\sim$K′. Let K″ be the class such that $(x) : x \varepsilon \mathrm{K}'' \,.\equiv.\, \sim x \varepsilon \mathrm{K} \,.\, \sim x \varepsilon \mathrm{K}'$; then the three classes K, K′, K″ are mutually exclusive and severally exhaustive. Consider the elementary function

(a) $$f'(x_1, \cdots, x_\mu, y_1, \cdots, y_\nu) \,.\mathrm{V}.\, x_1 \varepsilon \mathrm{K}'' \,.\mathrm{V}.\, \cdots \,.\mathrm{V}.\, x_\mu \varepsilon \mathrm{K}'';$$

either the function $f'$ holds, or one of the elements $x_1, \cdots, x_\mu$ is neither in K nor in K′. In an obvious sense, (a) is about elements in K or K′ only; and if $F$ is to place no restriction on elements in K″, $f$ must be equivalent to the function (a). When K″ is null, that is, when $(x) : x \varepsilon \mathrm{K} \,.\equiv.\, \sim x \varepsilon \mathrm{K}'$, the alternatives $x_1 \varepsilon \mathrm{K}'', \cdots, x_\mu \varepsilon \mathrm{K}''$ are impossible, and so vanish; so that $f$ reduces to $f'$.

If $\mathrm{R}(a_1 a_2) \,.\, \mathrm{R}(a_2 a_3) \,.\, \mathrm{R}(a_1 a_3)$, then we may write simply $\mathrm{R}(a_1 a_2 a_3)$; and in general, $\mathrm{R}(a_1 a_2 \cdots)$ is to mean that any two elements $a_i$, $a_j$ which occur in the order from left to right are such that $\mathrm{R}(a_i a_j)$ holds. In view of 1′, if none of the elements $a_1, \cdots$ belongs to K″, and if $\mathrm{R}(a_1 a_2 \cdots)$ holds, then $a_1, \cdots$ must all be distinct. If $a_1, a_2, \cdots$ are in K and $\mathrm{R}(a_1 a_2 \cdots)$ holds, then any ordered dyad of distinct elements, $a_i a_j$, which can be formed of the elements $a_1, \cdots$ and which is not asserted in $\mathrm{R}(a_1 a_2 \cdots)$, is such that $\mathrm{R}(a_j a_i)$ fails. We may denote "$\mathrm{R}(a_1 a_2 \cdots)$, and R fails for every other ordered dyad of distinct elements" by $\mathrm{R}'(a_1 a_2 \cdots)$. If $a_1, a_2, \cdots$ are in K, they are also such that $\sim \mathrm{R}(a_1 a_1) \,.\, \sim \mathrm{R}(a_2 a_2) \cdots$. We may denote "$\mathrm{R}'(a_1 a_2 \cdots)$ holds, and $\sim \mathrm{R}(a_1 a_1) \,.\, \sim \mathrm{R}(a_2 a_2) \cdots$" by $\mathrm{R}''(a_1 a_2 \cdots)$.

If $b_1, b_2, \cdots$ are distinct and belong to K′, then $\sim \mathrm{R}(b_1 b_2) . \sim \mathrm{R}(b_2 b_1) \cdots \sim \mathrm{R}(b_1 b_1) . \sim \mathrm{R}(b_2 b_2) \cdots$, from 1′, 7′. $\mathrm{S}(b_1 b_2 \cdots)$ is to mean "$b_1, b_2, \cdots \varepsilon \mathrm{K}'$, and R fails for every dyad of distinct or identical elements among $b_1, \cdots$, and $b_1, \cdots$ are all distinct."

If $a_1, a_2, \cdots$ are in K and $b_1, b_2, \cdots$ are in K′, then R fails for every ordered dyad which can be formed of one element from the set $a_1, \cdots$ and one from the set $b_1, \cdots$, from 7′; and we may express the failure of R for these dyads by writing $\mathrm{T}(a_1 a_2 \cdots; b_1 b_2 \cdots)$.

It has been shown in the first paper, what is immediately obvious, that, as a consequence of 1′–4′, any $m$ distinct elements, all in K, are such that for some permutation of these elements, $a_1, \cdots, a_m$, $\mathrm{R}''(a_1 \cdots a_m)$ holds. But for any $n$ distinct elements, $b_1, \cdots, b_n$, all in K′, $\mathrm{S}(b_1 \cdots b_n)$ holds. Accordingly, any $m+n$ elements, $m$ in $K$ and $n$ in $K'$, are such that, for some permutation of the $m$ elements in K,

$$\mathrm{R}''(a_1 \cdots a_m) . \mathrm{S}(b_1 \cdots b_n) . \mathrm{T}(a_1 \cdots a_m; b_1 \cdots b_n).$$

When $n$ vanishes this expression reduces to $\mathrm{R}''(a_1 \cdots a_m)$, and when $m$ vanishes it reduces to $\mathrm{S}(b_1 \cdots b_n)$.

Since

(3) $\quad (x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) (\exists x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \cdots f(x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, \cdots)$

is equivalent to

(4) $\quad \sim(\exists x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \cdots \sim f(x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, \cdots),$

if it is established that every function of the form (3) has its truth-value determined by 1′–8′, then every function of the form (4) has its truth-value determined, and conversely; for the truth-value of a function determines the truth-value of its contradictory. It is sufficient to establish deducibility in at least one of these cases.

If $f$ is of the form (a), $\sim f$ is

$$\sim f'(x_1, \cdots, x_\mu, y_1, \cdots, y_\nu) . \sim x_1 \varepsilon \mathrm{K}'' \cdots \sim x_\mu \varepsilon \mathrm{K}'',$$

in which $\sim f'$ is equivalent to a disjunctive function which involves all those alternatives on $x_1, \cdots, x_\mu, y_1, \cdots, y_\nu$ which are not among the alternatives in $f'$. But every alternative which involves at least one of the variables $x_1, \cdots, x_\mu$ assigned to K″ fails; so that these alternatives vanish. If the elementary function which results be denoted by $f''$, then $f''$ involves all those alternatives on $x_1, \cdots, x_\mu, y_1, \cdots, y_\nu$ in which every variable is assigned to K or to K′ and which do not occur in $f'$, and also every possible alternative in which at least one of the variables $y_1, \cdots, y_\nu$ are

assigned to K″. Since all possible alternatives of the kind in question are involved in this latter set, and since there are at least $n$ elements in K and at least $n$ elements in K′, the disjunctive function formed of all these latter alternatives, when it occurs in a function of the form (2), is equivalent to

$$y_1 \varepsilon \mathsf{K}'' \vee \cdots \vee y_\nu \varepsilon \mathsf{K}''.$$

In a function of the form (2), $y_1, \cdots, y_\nu$ will, of course, be quantified universally. It follows that any function of the form (1), in which the elementary function is of the form (a), is such that its contradictory can be expressed in the form (2), and such that the elementary function is of the form (a); and conversely. In dealing with functions of this kind, it is sufficient to establish deducibility for at least one of every pair of contradictory functions.

It will be shown that every singly quantified first-order function on K, K′, $\mathsf{R}_2$, in which the elementary function can be expressed in the form (a), has its truth-value determined by properties 1′–8′. Any singly quantified first-order function has one of the forms

$$(x_1, \cdots, x_n) . f(x_1, \cdots, x_n) \text{ or } (\exists x_1, \cdots, x_n) . f(x_1, \cdots, x_n);$$

and we may confine attention to functions of the second kind. Note that in functions of the second kind in which the elementary function is of the form (a), $f = f'$. Every set of elements in K is such that, for some permutation, $\mathsf{R}''(a_1\ a_2 \cdots)$; and every set of elements in K′ is such that $\mathsf{S}(b_1\ b_2 \cdots)$. If the function is to hold, $f'$ must involve at least one alternative; and any alternative in $f'$ which should involve conditions on identity which are contradictory ($x = y . x \neq y$) is impossible; and any alternative which should involve a condition on identity, $x = y$, where $x \varepsilon \mathsf{K}$ and $y \varepsilon \mathsf{K}'$, must fail, since these classes have no members in common. All of these alternatives may be discarded without altering the truth-value of the function. In the function so reconstituted, let $p$ by any alternative. In general, some of the variables which occur in $p$ must take the same value, due to a condition on identity. In each set of variables all taking the same value, drop all of the variables but one. Call the alternative which results $p'$; so that in $p'$ all variables take distinct values. If $p'$ is of the form

$$\mathsf{R}''(a_1\ a_2 \cdots) . \mathsf{S}(b_1\ b_2 \cdots) . \mathsf{T}(a_1\ a_2 \cdots; b_1\ b_2 \cdots),$$

then it follows from 1′–8′ that $(\exists x_1, \cdots, x_n) . f(x_1, \cdots, x_n)$ holds; so that, if there is at least one alternative $p$ which reduces to this form, the function follows from the set. Clearly, the condition is also necessary,

since every set of distinct elements must have this form. Derivatively, if we assign to K′ the value ∼K, it is clear that *every singly quantified first-order function on* K, $R_2$ *has its truth-value determined by properties* 1–8.

We wish to establish a similar result in the case of $n$-tuply quantified functions; and this will require a somewhat detailed preliminary analysis of some properties of general propositions, which is, however, not without interest on its own account. Consider the elementary function

$$f(\cdots, x_1^{(i)}, \cdots, x_n^{(i)}, \cdots), \tag{5}$$

where $f$ is in expanded form with respect to K, K′, R, I; and let $p_j, p_k$ $(j \neq k)$ be any two alternatives in $f$. If we discard all of the variables from $p_j, p_k$ except $x_1^{(i)}, \cdots, x_n^{(i)}$, the resulting functions of $x_1^{(i)}, \cdots, x_n^{(i)}$ on K, K′, R, I may be the same or different. Let the alternatives in $f$ be collected into sets $P_1, \cdots, P_t$, such that $P_i = p_{i\alpha} \vee \cdots \vee p_{i\varepsilon}$, where $p_{i\alpha}, \cdots, p_{i\varepsilon}$ are all those alternatives in $f$ involving some one and the same function of $x_1^{(i)}, \cdots, x_n^{(i)}$ on K, K′, R, I. Then $f = P_1 \vee \cdots \vee P_t$; so that we may express an $n$-tuply quantified function in the form

$$\cdots (\exists x_1^{(i)}, \cdots, x_n^{(i)}) \cdots . P_1 \vee \cdots \vee P_t. \tag{6}$$

This function is equivalent to

$$\cdots : (\exists x_1^{(i1)}, \cdots, x_n^{(i1)}) \cdots P_1 \,.\vee.\, \cdots .\vee. (\exists x_1^{(it)}, \cdots, x_n^{(it)}) \cdots P_t, \tag{7}$$

since no set of elements $a_1, \cdots, a_n$, substituted for $x_1^{(i)}, \cdots, x_n^{(i)}$, which can satisfy $x_1^{(i)}, \cdots, x_n^{(i)}$ in $P_j$, can also satisfy these variables in $P_k$, if $j \neq k$. In view of the relation of (6) to (7), if $F$ be a function which involves a single complex quantifier, and which is such that the elementary function $f$ is in expanded form with respect to K, K′, R, I, then $F$ can be expressed as a function which is such that, (i) if $t$ be any variable to which the applicative "every" attaches, and if we consider all those alternatives in which $t$ occurs, and let $y_1, \cdots, y_\lambda$ be the variables in those alternatives of wider scope than $t$, then $y_1, \cdots, y_\lambda$ involve one and the same function on K, K′, R, I in all of these alternatives; and (ii), if $u$ be any variable to which the applicative "some" attaches, and if we consider all those alternatives in which $u$ occurs, and let $y_1, \cdots, y_\mu$ be the variables in these alternatives which are not of narrower scope than $u$, which of course includes $u$ and those variables having the same scope as $u$, then $y_1, \cdots, y_\mu$ involve one and the same function on K, K′, R, I in all of these alternatives.*

* There are, of course, variables which do not occur in any alternative in which $u$ occurs; the scopes of these variables are independent of the scope of $u$, and they are not in question in (i) and (ii).

For want of a better term, a function expressed in this way will be said to be in analytic form. It will be seen that a function

$$(8) \qquad (\exists x_1, \cdots, x_m) \cdots f(x_1, \cdots, x_m, \cdots),$$

when expressed in analytic form, breaks up into an elementary disjunctive function,

$$(9) \quad (\exists x_1^{(1)}, \cdots, x_m^{(1)}) \cdots f_1(x_1^{(1)}, \cdots, x_m^{(1)}, \cdots) \mathbin{\mathsf{V}} \cdots \mathbin{\mathsf{V}} (\exists x_1^{(t)}, \cdots, x_m^{(t)}) \cdots f_t(x_1^{(t)}, \cdots, x_m^{(t)}, \cdots);$$

and since the truth-value of an elementary function is determined if the truth-values of the constituent functions are determined, we may confine attention to functions having the form of the alternatives in (9).

Let $f(\cdots, u_1, \cdots, u_n, \cdots)$ be an elementary function on $\mathsf{K}, \mathsf{K}', \mathsf{R}_2$ in expanded form with respect to $\mathsf{K}, \mathsf{K}', \mathsf{R}, \mathsf{I}$ (which is not of the form (a)), and let $p$ be any alternative in $f$. For each pair of variables $x, y$ in $f$, $p$ involves $x \mathsf{I} y$ or it involves $\sim x \mathsf{I} y$, and not both; so that $p$ contains, as a part, a conjunctive function of $u_1, \cdots, u_n$ on $\mathsf{I}$. This function will, in general, assert identity for some pairs of variables and non-identity for others; and in general, some of the alternatives $p_1, \cdots, p_s$ in $f$ will involve the same function of $u_1, \cdots, u_n$ on identity, and others will involve different functions in this respect. The alternatives in $f$ can be collected into sets, such that any two alternatives belonging to the same set involve the same function of $u_1, \cdots, u_n$ on identity, and any two alternatives belonging to different sets involve different functions of these variables on identity; so that $f$ can be expressed in the form $f_1, \cdots, f_r$, in which $f_i$ is a disjunctive function involving all those and only those alternatives belonging to one and the same set. We may denote the function of $u_1, \cdots, u_n$ on $\mathsf{I}$ which occurs in $f_1$ by $\mathsf{I}_1$ and that which occurs in $f_r$ by $\mathsf{I}_r$; so that $\mathsf{I}_1, \cdots, \mathsf{I}_r$ are all different.

Consider a function,

$$(10) \qquad (\exists x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) : (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \,.\, F_1 \mathbin{\mathsf{V}} \cdots \mathbin{\mathsf{V}} F_t,$$

in analytic form, in which in general $F_i$ is not an elementary function. The variables $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}$ may, of course, be supposed to involve the same function on $\mathsf{K}, \mathsf{K}', \mathsf{R}, \mathsf{I}$ throughout $F_1, \cdots, F_t$. We may collect $F_1, \cdots, F_t$ into sets, determined by the function of $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}$ on $\mathsf{I}$ which occurs in them. It is clear that only one function of these variables on identity can occur in any function $F_i$. Then $F_1 \mathbin{\mathsf{V}} \cdots \mathbin{\mathsf{V}} F_t$ can be expressed in the form $F_1' \mathbin{\mathsf{V}} \cdots \mathbin{\mathsf{V}} F_r'$, in which $F_j'$ is a disjunctive function made up of functions from among $F_1, \cdots, F_t$, and such that each alternative which occurs in $F_j$ involves the same function of $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}$ on $\mathsf{I}$. Let

the functions of $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}$ on I which occur in $F_1', \cdots, F_r'$ be denoted by $\mathbf{I}_1, \cdots, \mathbf{I}_r$ respectively. Then (10) is equivalent to

$$(11)\quad (\exists x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) : (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \,.\, F_1' \,\mathsf{V} \sim \mathbf{I}_1 : \cdots : (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \,.\, F_r' \,\mathsf{V} \sim \mathbf{I}_r.$$

Similarly, consider any constituent of (11) of the form

$$(12)\qquad (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \,.\, F_j' \,\mathsf{V} \sim \mathbf{I}_j,$$

in which $F_j'$ will, in general, be of the form

$$(13)\quad \begin{array}{l} (\exists x_1^{\mathrm{III}(\alpha)}, \cdots, x_p^{\mathrm{III}(\alpha)}) : (x_1^{\mathrm{IV}(\alpha)}, \cdots, x_q^{\mathrm{IV}(\alpha)}) \,.\, F_{j_{\alpha 1}} \,\mathsf{V} \cdots \mathsf{V}\, F_{j_{\alpha a}} : \mathsf{V} : \cdots : \mathsf{V} : \\ (\exists x_1^{\mathrm{III}(\varepsilon)}, \cdots, x_p^{\mathrm{III}(\varepsilon)}) : (x_1^{\mathrm{IV}(\varepsilon)}, \cdots, x_q^{\mathrm{IV}(\varepsilon)}) \,.\, F_{j_{\varepsilon 1}} \,\mathsf{V} \cdots \mathsf{V}\, F_{j_{\varepsilon k}}. \end{array}$$

In each of the alternatives

$$(14)\qquad (\exists x_1^{\mathrm{III}(\gamma)}, \cdots, x_p^{\mathrm{III}(\gamma)}) : (x_1^{\mathrm{IV}(\gamma)}, \cdots, x_q^{\mathrm{IV}(\gamma)}) \,.\, F_{j_{\gamma 1}} \,\mathsf{V} \cdots \mathsf{V}\, F_{j_{\gamma h}} \qquad (h = a, \cdots, k;\ \gamma = a, \cdots, \varepsilon),$$

any one of the functions $F_{j_{\gamma i}}$ $(i = 1, \cdots, h)$ may or may not be an elementary function, and is such that the elementary alternatives which occur in it involve the same function of $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, x_1^{\mathrm{III}(\gamma)}, \cdots, x_p^{\mathrm{III}(\gamma)}, x_1^{\mathrm{IV}(\gamma)}, \cdots, x_q^{\mathrm{IV}(\gamma)}$ on K, K′, R, I. This follows from the fact that (10) is in analytic form. And so, as previously, $F_{j_{\gamma 1}}, \cdots, F_{j_{\gamma h}}$ may be collected into sets; and this analysis may be continued until all universal variables have been taken into account. In a function expressed in this way, if $t_1, \cdots, t_\nu$ be any apparent variables all having the same scope, and if $s_1, \cdots, s_\mu$ be those variables of wider scope than $t_1, \cdots, t_\nu$, such that $t_1, \cdots, t_\nu$ are within their scope, then any two alternatives within the scope of $t_1, \cdots, t_\nu$ involve the same function of $s_1, \cdots, s_\mu, t_1, \cdots, t_\nu$ on identity.

It follows from these considerations that, if we have a function in analytic form which is, further, expressed in such a way that any set of variables in the function involve the same function on identity in every alternative in which they all occur, then there is an equivalent function obtained by taking any two variables $x$, $y$, such that the function asserts $x \,\mathsf{I}\, y$, and discarding that variable which has the narrower scope, or, in case they have the same scope, that which occurs first in the quantifier. This is possible for the reason that, if the function involves $x \,\mathsf{I}\, y$ in one alternative, it involves $x \,\mathsf{I}\, y$ in every alternative in which both $x$ and $y$ occur; and that, if $x$ is of wider scope than $y$, or if they have the same scope, $x$ occurs in every alternative in which $y$ occurs. $y$ may, accordingly, be discarded; which carries with it every elementary constituent $z \neq y$,

$y \varepsilon \mathrm{K}$, $\mathrm{R}zy$, $\cdots$ in which $y$ occurs. This applies also, of course, to any function on identity $\mathrm{I}j$ in which $y$ occurs. In a function expressed in this way, distinct variables denote distinct elements.

The foregoing arguments are concerned with a function in which the elementary function is not of the form $(a)$; and we wish to extend these results to a function in which the elementary function is of this form. Consider a function

$$\begin{aligned}(15)\quad &\cdots (x_1^{(i)}, \cdots, x_\lambda^{(i)})\,(\exists x_1^{(j)}, \cdots, x_\mu^{(j)})\,(x_1^{(k)}, \cdots, x_\nu^{(k)}) \cdots . P_1 \vee \cdots \vee P_t \\ &\qquad \cdots \vee x_1^{(i)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(i)} \varepsilon \mathrm{K}'' \vee x_1^{(k)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\nu^{(k)} \varepsilon \mathrm{K}'' \cdots,\end{aligned}$$

and express (15) so that

$$\cdots \vee x_1^{(i)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(i)} \varepsilon \mathrm{K}''$$

shall be outside the scope of $(\exists x_1^{(j)}, \cdots, x_\mu^{(j)})\,(x_1^{(k)}, \cdots, x_\nu^{(k)}) \cdots$,

$$\begin{aligned}(16)\quad &\cdots (x_1^{(i)}, \cdots, x_\lambda^{(i)}) : (\exists x_1^{(j)}, \cdots, x_\mu^{(j)})\,(x_1^{(k)}, \cdots, x_\nu^{(k)}) \cdots . P_1 \\ &\vee \cdots \vee P_t \vee x_1^{(k)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\nu^{(k)} \varepsilon \mathrm{K}'' \cdots .\vee. \cdots x_1^{(i)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(i)} \varepsilon \mathrm{K}''.\end{aligned}$$

This function is equivalent to

$$\begin{aligned}(17)\quad &\cdots (x_1^{(i)}, \cdots, x_\lambda^{(i)}) :. (\exists x_1^{(j1)}, \cdots, x_\mu^{(j1)})\,(x_1^{(k1)}, \cdots, x_\nu^{(k1)}) \cdots . P_1 \vee x_1^{(k1)} \varepsilon \mathrm{K}'' \vee \\ &\cdots \vee x_\nu^{(k1)} \varepsilon \mathrm{K}'' \cdots .\vee. \cdots .\vee. (\exists x_1^{(jt)}, \cdots, x_\mu^{(jt)})\,(x_1^{(kt)}, \cdots, x_\nu^{(kt)}) \cdots . P_t \\ &\vee x_1^{(kt)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\nu^{(kt)} \varepsilon \mathrm{K}'' \cdots :\vee: \cdots \vee x_1^{(i)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(i)} \varepsilon \mathrm{K}''.\end{aligned}$$

In view of the relation of (16) to (17), any function can be expressed in such a way that, if we exclude alternatives of the form $x \varepsilon \mathrm{K}''$, properties i, ii (page 464) characterize the function. We shall say, in this case also, that the function is in analytic form.

Consider any constituent of a function in analytic form,

$$\begin{aligned}(18)\quad &(x_1, \cdots, x_\lambda) : F_1(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee \cdots \vee F_t(\cdots, x_1, \cdots, x_\lambda, \cdots) \\ &\qquad .\vee. x_1 \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \varepsilon \mathrm{K}'',\end{aligned}$$

where $F_1, \cdots, F_t$ are such that, in each of them, every alternative involves the same function of $x_1, \cdots, x_\lambda$, together with every variable of wider scope, on K, K′, R, I. The functions $F_1, \cdots, F_t$ may be grouped according to the functions of $x_1, \cdots, x_\lambda$ on I which their alternatives involve, and in such a way that any two alternatives belonging to functions in the same group involve the same function of $x_1, \cdots, x_\lambda$ on identity, and any two alternatives belonging to functions in different groups involve different functions of $x_1, \cdots, x_\lambda$ on identity. The groups of functions themselves

form disjunctive functions; and we may denote these disjunctive functions by $F'_1, \cdots, F'_s$. Then the function just given may be written

$$(19)\quad (x_1, \cdots, x_\lambda) : F'_1(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee \cdots \vee F'_s(\cdots, x_1, \cdots, x_\lambda, \cdots) \\ .\vee. x_1 \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \varepsilon \mathrm{K}'',$$

which is equivalent to

$$(12')\quad (x_1, \cdots, x_\lambda) : F'_1(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee x_1 \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \varepsilon \mathrm{K}'' \\ .\vee. \cdots .\vee. F'_s(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee x_1 \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \varepsilon \mathrm{K}''.$$

The constituents of this function,

$$F'_j(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee x_1 \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \varepsilon \mathrm{K}'', \qquad j = 1, \cdots, s,$$

are such that, any set of elements substituted for $x_1, \cdots, x_\lambda$ satisfies at most one of them or at least all of them. If none of the elements which are substituted are in K″, and they satisfy one of the $F''$s, then at most one of the $F''$s can be satisfied; if one of the elements substituted is in K″, every function is satisfied. It follows that the function is equivalent to

$$(13')\quad (x_1^{(1)}, \cdots, x_\lambda^{(1)}) : F'_1(\cdots, x_1^{(1)}, \cdots, x_\lambda^{(1)}, \cdots) \vee x_1^{(1)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(1)} \varepsilon \mathrm{K}'' .\vee. \sim \mathrm{I}_1 \\ :. \cdots :. (x_1^{(s)}, \cdots, x_\lambda^{(s)}) : F'_s(\cdots, x_1^{(s)}, \cdots, x_\lambda^{(s)}, \cdots) \vee x_1^{(s)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(s)} \varepsilon \mathrm{K}'' .\vee. \sim \mathrm{I}_s.$$

When a function in analytic form has been expanded in this way, it may also be expressed in such a way that distinct variables require distinct values, as we have shown.

There are certain conditions under which the scope of a variable to which the applicative "some" attaches can be increased, without altering the truth-value of the function. To use a simple illustration, consider an elementary alternative $\mathrm{R}''(a\,b\,x)$, in which $x$ is of wider scope than $a$, and $b$ of wider scope then $x$; and let $a$, $b$ be particular and $x$ universal. The variable $a$ is separated from the universal variable, within whose scope it falls, by a particular variable of still wider scope. Now a particular variable of wider scope has the force of "some one and the same", and a particular variable of narrower scope has the force of "some one or other". Then, in respect of this alternative alone, $a$ can be given a wider scope than $x$; for, in view of 1′–8′,

$$(\exists\, b) :. (x) : (\exists\, a) . \mathrm{R}''(bx) \supset \mathrm{R}''(abx) :.\equiv:. \\ (\exists\, b) :. (\exists\, a) : (x) . \mathrm{R}''(bx) \supset \mathrm{R}''(abx).$$

Note that it is indifferent whether $x$ is universal, or whether it is assigned to K, as in this illustration; and that if this relation among $a$, $b$, $x$

holds in every alternative in which $a$ occurs, the scope of $a$ can be increased.

Let $F$ be a function in analytic form whose variables of widest scope are particular, and in which the elementary function is of the form (a). Express $F$ in such a way that the function on identity of any set of universal variables having the same scope is constant, and such that distinct variables of interdependent scopes require distinct values. Let the variables of widest scope in the function be $x_1, \cdots, x_m, x_1^{\mathrm{I}}, \cdots, x_n^{\mathrm{I}}$, where $x_1, \cdots, x_m$ are assigned to K, and $x_1^{\mathrm{I}}, \cdots, x_n^{\mathrm{I}}$ to K′. Now if $t_1, \cdots, t_p, t$ are variables in the function having interdependent scopes, and if $t_1, \cdots, t_p$ are of wider scope than $t$, then the function of $t_1, \cdots, t_p$ on K, K′, R, I is constant throughout every alternative in which $t$ occurs. We are to confine attention to those alternatives in $F$ which are such that, if $z$ be any universal variable in one of them, then $\mathrm{R}(x_i z)$, $i = 1, \cdots, m$, are asserted in the alternative—every universal variable follows all of the variables $x_1, \cdots, x_m$. In these alternatives, consider any variable $u_1$ which is, (i) particular, (ii) other than $x_1, \cdots, x_m$, (iii) asserted to precede some variable $x_1, \cdots, x_m$, (iv) not within the scope of any particular variable, other than $x_1, \cdots, x_m$, which is asserted to precede some variable $x_1, \cdots, x_m$. Then $u_1$ may be given the same scope as $x_1, \cdots, x_m$. In general, $u_1$ now has within its scope variables of which it was independent; but since it does not occur in any alternative in which these variables occur, it remains in effect independent of them. In like manner, $u_2$ may be given the same scope as $x_1, \cdots, x_m, u_1$, and so on, until no variable of this kind remains in the function. Let these particular variables be $u_1, \cdots, u_k$; so that $u_1, \cdots, u_k$, although they have the widest scope in $F$, do not in general occur in every alternative—the function is not expanded with respect to them. Among themselves, some have independent scopes, and some are within the scopes of others; two of them may be independent and yet be within the scope of some one other.

Consider $F$ before any of the variables $u_1, \cdots, u_k$ have been assigned a wider scope; and let $u_1$ be given the widest scope in $F$. Now there may not be any element $u_1$, as demanded in the alternatives in which $u_1$ occurs; but the mere existence-demand can, in any case, be satisfied by identifying $u_1$ with, say, $x_1$. Let the function obtained by identifying $u_1$ with $x_1$ in every alternative be denoted by $F_1$. Then $F_1$ implies $F$. Again, in every alternative in which $u_1$ does not occur, assign it to $K$ and assign to it the relations to $x_1, \cdots, x_m$, in terms of R, I, which it has in those alternatives in which it does occur; and call this function $F_2$. Then $F_2$ implies $F$. But $F$ implies $F_1 \vee F_2$; so that $F$ has been separated into two functions; but in each of these functions there remain only variables

$u_2, \cdots, u_k$, corresponding to $u_1, \cdots, u_k$ in $F$, to be given an increased scope. In $F_1$, $u_1$ may be discarded, being identical with $x_1$; so that $u_2$ may be given the widest scope in $F_1$, and $F_1$ expressed in the form $F_{11} \vee F_{12}$, in the manner just indicated. In $F_2$, give $u_2$ the widest scope; and identify $u_2$ with $x_1$ in every alternative, and call this function $F_{21}$. Then form another function by giving $u_2$ the relation to $x_1, \cdots, x_m$, in every alternative, which it has in those alternatives in which it occurs. But in this case a new situation arises; $u_1$ is among the variables of widest scope in $F_2$. If $u_2$ is within the scope of $u_1$, or has the same scope, give $u_2$ the relation to $u_1$, in terms of R, I, in every alternative, which it has in those alternatives in which $u_2$ occurs. Here we get $F_2 .\equiv. F_{21} \vee F_{22}$, without difficulty. Suppose that $u_1$ and $u_2$ are of independent scope, but that, as they occur among $x_1, \cdots, x_m$, they are separated by at least one of these variables. In this case the relations of $u_1, u_2$ are determined by their relations to $x_1, \cdots, x_m$, and by the properties of serial relations; and in this case also, we have $F_2 .\equiv. F_{21} \vee F_{22}$. Suppose that $u_1, u_2$ both fall between some pair of neighboring terms among $x_1, \cdots, x_m$, or both precede all these terms; then $Ru_1u_2$ may be true, or $Ru_2u_1$, or $u_1 \mathrm{I} u_2$. Call the functions corresponding to these three cases $F_{22}, F_{23}, F_{24}$. Then $F_2 .\equiv. F_{21} \vee F_{22} \vee F_{23} \vee F_{24}$. The generalization is obvious. Now every alternative into which $F$ is resolved involves $u_1, \cdots, u_k$ with widest scope; but there may be some conditions of identity among $u_1, \cdots, u_k$. In this case, discard all but one of any set of variables which are identified. Then any function $F^{\mathrm{I}}$, into which $F$ is resolved, is such that, (i) any two variables having interdepedent scopes require distinct values, and in particular, all variables of widest scope require distinct values among themselves, and distinct from the values of all other variables; (ii) the variables of widest scope are particular; (iii) these variables exhibit a completely determinate function on K, K′, R, I, the same in every alternative; (iv) in general, the relation of $u_1, \cdots, u_k$, in terms of R, I, to variables other than $x_1, \cdots, x_m$, $x_1^1, \cdots, x_n^1$ is not determinate in certain alternatives; but it is determinate in every alternative in which every variable $t$, assigned to K and not of widest scope, follows $x_1, \cdots, x_m$. For let $x_m$ be such that $\mathrm{R}x_ix_m$ $(i = 1, \cdots, m-1)$; then $\mathrm{R}u_jx_m$ $(j = 1, \cdots, k)$; so that $\mathrm{R}u_jt$. If we should discard from $F^{\mathrm{I}}$ all alternatives except those in which every variable $t$ follows $x_1, \cdots, x_m$, $F^{\mathrm{I}}$ would be in analytic form, and, further, expressed in such a way that distinct variables require distinct values.

We wish to show that the truth-value of $F^{\mathrm{I}}$ is determined by properties 1′–8′. Let the degree of quantification of $F$, when expressed in expanded form, be $n$; then the degree of quantification of $F^{\mathrm{I}}$, in expanded form, cannot be more than $n$, though it can be less. We wish to show that,

if the truth-value of every function of degree $n-1$ is determined, the truth-value of $F^{\mathrm{I}}$ is determined; and we may, of course, assume that $F^{\mathrm{I}}$ is of degree $n$. Let $x_1, \cdots, x_m$ be such that $\mathrm{R}''(x_1, \cdots, x_m)$; let $a_1, \cdots, a_m$ be distinct elements in K, such that $\mathrm{R}''(a_1, \cdots, a_m)$, and let $b_1, \cdots, b_n$ be distinct elements in K′; substitute $a_1, \cdots, a_m, b_1, \cdots, b_n$ for $x_1, \cdots, x_m, x_1^{\mathrm{I}}, \cdots, x_n^{\mathrm{I}}$, and call the resulting function $F_1^{\mathrm{I}}$. Discard from $F_1^{\mathrm{I}}$ all those elementary alternatives in which some universal variable is asserted to precede $a_m$, and call this function $F_1^{\mathrm{II}}$; so that $F_1^{\mathrm{II}} \supset F_1^{\mathrm{I}} \supset F^{\mathrm{I}}$. Let $\mathrm{R}_\alpha x_1 x_2$ mean $\mathrm{R}x_1x_2 \,.\, \mathrm{R}a_m x_1 \,.\, \mathrm{R}a_m x_2$; $x \varepsilon \mathrm{K}_\alpha$ mean $x \varepsilon \mathrm{K} \,.\, \mathrm{R}a_m x$; $x \varepsilon \mathrm{K}'_\alpha$ mean $x \varepsilon \mathrm{K}' \,.\, x \neq b_1, \cdots, b_n$. It is clear that $\mathrm{K}_\alpha$, $\mathrm{K}'_\alpha$, $\mathrm{R}_\alpha$ have all of the properties 1′–8′; so that the truth-value of every function of degree $n-1$ on $\mathrm{K}_\alpha$, $\mathrm{K}'_\alpha$, $\mathrm{R}_\alpha$ is determined. Form $F_1^{\mathrm{III}}$ from $F_1^{\mathrm{II}}$ by discarding $a_1, \cdots, a_m, b_1, \cdots, b_n$; and form $F_{1\alpha}^{\mathrm{III}}$ from $F_1^{\mathrm{III}}$ by substituting $\mathrm{K}_\alpha$, $\mathrm{K}'_\alpha$, $\mathrm{R}_\alpha$ for K, K′, R. Then $F_{1\alpha}^{\mathrm{III}}$ is of degree $n-1$. Choose $a_1, \cdots, a_m$ so that they are the first $m$ elements in K. Then $F_{1\alpha}^{\mathrm{III}} \supset F_1^{\mathrm{III}} \supset F_1^{\mathrm{II}}$*; and it is to be noted that, if $F_{1\alpha}^{\mathrm{III}}$ follows from 1′–8′ for one choice of $x_m$, it follows for every choice.

We wish to show that $\sim F_{1\alpha}^{\mathrm{III}} \supset \sim F^{\mathrm{I}}$. $F^{\mathrm{I}}$ entails that, for some $x_1, \cdots, x_m$, $F_1^{\mathrm{I}}$; and we may show that $\sim F_{1\alpha}^{\mathrm{III}} \supset \sim F_1^{\mathrm{I}}$ for every $x_1, \cdots, x_m$. $F_1^{\mathrm{I}}$ differs from $F_{1\alpha}^{\mathrm{III}}$ only in that certain elementary conjunctive functions, occuring in functions $f'$, do not occur in $F_{1\alpha}^{\mathrm{III}}$, and in that alternatives of the form $t \varepsilon \mathrm{K}''$ are replaced by $t \varepsilon \mathrm{K}'' \vee \mathrm{R}t x_m$. These elementary conjunctive functions are such as involve $\mathrm{R}t x_m$, for some universal variable $t$. But any elementary conjunctive function which involves $\mathrm{R}t x_m$ can be introduced into $F_{1\alpha}^{\mathrm{III}}$, since every function of this kind is implicit in the alternative $t \varepsilon \mathrm{K}'' \vee \mathrm{R}t x_m$. For $t \varepsilon \mathrm{K}'' \vee \mathrm{R}t x_m$ can be expanded into $t \varepsilon \mathrm{K}'' \vee \cdots$, in which a disjunctive function, consisting of all possible elementary functions which involve $\mathrm{R}t x_m$, replaces $\mathrm{R}t x_m$. If we expand every alternative of the form $t \varepsilon \mathrm{K}'' \vee \mathrm{R}t x_m$, which occurs in $F_{1\alpha}^{\mathrm{III}}$, in this way, then $F_{1\alpha}^{\mathrm{III}}$ is identical with $F_1^{\mathrm{I}}$ in every respect, except that, in general, $F_{1\alpha}^{\mathrm{III}}$ involves some elementary conjunctive functions which do not occur in $F_1^{\mathrm{I}}$; so that $\sim F_{1\alpha}^{\mathrm{III}} \supset \sim F_1^{\mathrm{I}}$. It follows that every first-order function on K, K″, $\mathrm{R}_2$ has its truth-value determined by properties 1′–8′; and derivatively, that *every first-order function on the base* K, $\mathrm{R}_2$ *has its truth-value determined by properties* 1–8.

* See property (i), p. 470.

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# ON SOME NEW TYPES OF NON-DIFFERENTIABLE FUNCTIONS.*

By Avadhesh Narayan Singh.

## Introduction.

It is well known that any number in the interval (0,1) can be uniquely represented as

$$\frac{a_1}{b_1}+\frac{a_2}{b_1 b_2}+\frac{a_3}{b_1 b_2 b_3}+\cdots,$$

where $b_1, b_2, b_3, \cdots$, is a sequence of positive integers which may be periodic, and the $a$'s are positive integers such that

$$0 \leqq a_n \leqq b_n - 1 \qquad (n = 1, 2, 3, \cdots).$$

Some new types of continuous non-differentiable functions can be easily defined by the help of such a representation of the points in the interval (0,1).† The object of the present paper is to give two such types of simple non-differentiable functions.

## First type.

1. Let $b_1, b_2, b_3, \cdots, b_m$ denote the $m$ odd numbers 3, 5, 7, $\cdots$, $(2m+1)$ respectively. Then any number in the interval (0,1) can be represented as

$$t = \frac{a_1}{3}+\frac{a_2}{3\cdot 5}+\frac{a_3}{3\cdot 5\cdot 7}+\cdots+\frac{a_m}{3\cdot 5\cdots(2m+1)}$$
$$+\frac{a_{m+1}}{3^2\cdot 5\cdot 7\cdots(2m+1)}+\cdots+\frac{a_{2m}}{3^2\cdot 5^2\cdots(2m+1)^2}+\cdots$$

where the $a$'s are defined as in the above introduction.

* Received March 31, 1927.

† Non differentiable functions have been defined by Peano, Math. Annalen, vol. 36, p. 159, by the help of the representation of the numbers of a given interval in radix fractions with an odd base. Recently K. Petr has defined such functions by using the representation of the numbers of a given interval in radix fractions with an even base, (Casopis pro pestovani matematiky fysiky, 1920). I use the representation of the points of an interval in the form $\frac{a_1}{b_1}+\frac{a_2}{b_1 b_2}+\frac{a_3}{b_1 b_2 b_3}+\cdots$. It is believed that no previous writer has used this procedure to define non-differentiable functions. My best thanks are due to Prof. Ganesh Prasad for his kind encouragement and interest.

Let the quantities $x_1, x_2, \cdots, x_m$ be defined as follows:

$$x_1 = \varphi_1(t) = \frac{c_{1,1}}{3} + \frac{c_{1,2}}{3^2} + \frac{c_{1,3}}{3^3} + \cdots,$$

$$x_2 = \varphi_2(t) = \frac{c_{2,1}}{5} + \frac{c_{2,2}}{5^2} + \frac{c_{2,3}}{5^3} + \cdots,$$

. . . . . .

$$x_m = \varphi_m(t) = \frac{c_{m,1}}{(2m+1)} + \frac{c_{m,2}}{(2m+1)^2} + \frac{c_{m,3}}{(2m+1)^3} + \cdots.$$

Here the $c$'s are defined as below:

$$c_{1,1} = a_1, \qquad c_{1,2} = P_1^{a_2+a_3+\cdots+a_m}(a_{m+1}),$$

$$c_{1,3} = P_1^{a_2+\cdots+a_m+a_{m+2}+\cdots a_{2m}}(a_{2m+1}), \cdots,$$

$$c_{2,1} = P_2^{a_1}(a_2), \qquad c_{2,2} = P_2^{a_1+a_3+a_4+\cdots+a_{m+1}}(a_{m+2}), \quad \cdots,$$

. . . . . .

$$c_{m,1} = P_m^{a_1+a_2+\cdots+a_{m-1}}(a_m), \quad c_{m,2} = P_m^{a_1+\cdots+a_{m-1}+a_{m+1}+\cdots+a_{2m-1}}(a_{2m}), \quad \cdots.$$

where $P_r^k(a)$ denotes $a$ or $(2r-a)$ according as $k$ is even or odd $(r = 1, 2, \cdots, m)$.

2. It can be easily shown that all the functions $x_1, x_2, \cdots, x_m$ are continuous functions of $t$, and that none of them possesses either a finite or an infinite differential coefficient for any value of $t$ in the interval (0,1). The method of proof is illustrated by the following example.

Let any point $t$ in the interval (0,1) be represented as

$$t = \frac{a_1}{3} + \frac{a_2}{3 \cdot 5} + \frac{a_3}{3^2 \cdot 5} + \frac{a_4}{3^2 \cdot 5^2} + \cdots,$$

where the $a$'s are positive integers such that $a_{2r} \leqq 4$ and $a_{2r+1} \leqq 2$ $(r = 0, 1, 2, \cdots)$.

Corresponding to $t$ let a number $x$ be defined as

$$x \equiv \varphi(t) = \frac{c_1}{3} + \frac{c_2}{3^2} + \frac{c_3}{3^3} + \cdots,$$

where

$$c_1 = a_1, \; c_2 = P_1^{a_2}(a_3), \; \cdots, \; c_n = P_1^{a_2+a_4+\cdots+a_{2n}}(a_{2n+1}), \; \cdots.$$

I shall first prove that $\varphi(t)$ is a continuous function of $t$ and then prove that it is non-differentiable.

3. THEOREM 1. *$x = \varphi(t)$ is a continuous function of $t$.*

The numbers $t$ may be divided into two classes:

(1) Those which are capable of double representation.

(2) Those which have a single representation only.

If $t$ be a number of the second class, $x$ is uniquely defined.

If $t$ be a number of the first class, whose ending representation runs up to an odd number of terms, then, since

$$\frac{a_1}{3}+\frac{a_2}{3\cdot 5}+\cdots+\frac{a_{2n+1}}{3^{n+1}\cdot 5^n}+\left(\frac{4}{3^{n+1}\cdot 5^{n+1}}+\frac{2}{3^{n+2}\cdot 5^{n+1}}\right)+\left(\frac{4}{3^{n+2}\cdot 5^{n+2}}+\frac{2}{3^{n+3}\cdot 5^{n+2}}\right)+\cdots \text{ to infinity}$$
$$=\frac{a_1}{3}+\frac{a_2}{3\cdot 5}+\cdots+\frac{a_{2n+1}+1}{3^{n+1}\cdot 5^n},$$

$t$ can be represented by the finite series or by the infinite series; and if $a_2+a_4+\cdots+a_{2n}$ is even, we see that the values obtained by applying the definition of $x$ to the two modes of representation of $t$ are

$$\frac{c_1}{3}+\frac{c_2}{3^2}+\cdots+\frac{c_n}{3^n}+\frac{c_{n+1}}{3^{n+1}}+\frac{2}{3^{n+2}}+\frac{2}{3^{n+3}}+\cdots \text{ to infinity}$$

and

$$\frac{c_1}{3}+\frac{c_2}{3^2}+\cdots+\frac{c_n}{3^n}+\frac{c'_{n+1}}{3^{n+1}},$$

where $c_{n+1}=a_{2n+1}$, and $c'_{n+1}=a_{2n+1}+1$, so that, the same value of $x$ is obtained for both representations of $t$. Similarly if $a_2+a_4+\cdots+a_{2n}$ is odd, it can be seen that $x$ has the same value for both representations of $t$. The case when the ending representation of $t$ runs up to an even number of terms may be similarly treated.

$x=\varphi(t)$ is thus a single-valued function of $t$. It is continuous, for if $t$ and $t'$ are identical as regards their first $2n$ terms, the corresponding $x$ and $x'$ are identical as regards their first $n$ terms and, therefore, when $t'$ tends to $t$ with incresing $n$, $x'$ tends to $x$.

4. THEOREM 2. *$x=\varphi(t)$ is a non-differentiable function.*

For proving the non-differentiability of $\varphi(t)$, the numbers $t$ may be divided into two classes (A) and (B).

First consider class (A), i. e., the class of numbers in which $a_{2r}\leqq 2$ for infinitely many values of $r$.

If $t_1$ be a point of this class, we see that the addition of $\frac{2}{3^r\cdot 5^r}$ to $t_1$ does not make any change in the value of $\varphi(t_1)$. Since $a_{2r}$ becomes $a_{2r}+2$

and $k$ remains even or odd as before, while the addition of $\frac{1}{3^{r+1}\cdot 5^r}$ to $t_1$ does so, so that

$$\lim \left| \frac{\varphi\left(t_1+\frac{2}{3^r\cdot 5^r}\right) - \varphi(t_1)}{\frac{2}{3^r\cdot 5^r}} \right| = 0$$

and

$$\overline{\lim} \left| \frac{\varphi\left(t_1+\frac{1}{3^{r+1}\cdot 5^r}\right) - \varphi(t_1)}{\frac{1}{3^{r+1}\cdot 5^r}} \right| = \infty$$

where the limits are taken as $r$ tends to infinity assuming those values for which the inequality $a_{2r} \leqq 2$ is satisfied.

Thus at the points of class (A) the differential coefficient is non-existent.

Now consider class (B), i. e. those numbers $t$ in which $a_{2r} > 2$ for infinitely many values of $r$.

If $t_2$ be a point of this class, we see that the subtraction of $\frac{2}{3^r\cdot 5^r}$ from $t_2$ does not change the value of $\varphi(t_2)$, while the subtraction of $\frac{1}{3^{r+1}\cdot 5^r}$ from $t_2$ does so, so that

$$\lim \left| \frac{\varphi\left(t_2-\frac{2}{3^r\cdot 5^r}\right) - \varphi(t_2)}{\frac{2}{3^r\cdot 5^r}} \right| = 0$$

and

$$\overline{\lim} \left| \frac{\varphi\left(t_2-\frac{1}{3^{r+1}\cdot 5^r}\right) - \varphi(t_2)}{\frac{1}{3^{r+1}\cdot 5^r}} \right| = \infty$$

where the limits are taken as $r$ tends to infinity assuming those values for which the inequality $a_{2r} > 2$ is satisfied.

Thus at the points of class (B) the differential coefficient is non-existent.

$x = \varphi(t)$ is, therefore, a non-differentiable function in the interval (0,1).

Every function of this type can, by a similar treatment, be shown to be devoid of a differential coefficient at any point in (0,1).

## Second type.

5. Another type of continuous non-differentiable functions may be defined of which the following is a particular case.

Let any point $t$ in the interval (0,1) be represented as

$$t = \frac{a_1}{3} + \frac{a_2}{3 \cdot 5} + \frac{a_3}{3^2 \cdot 5} + \frac{a_4}{3^2 \cdot 5^2} + \cdots.$$

Corresponding to $t$ let a number $y$ be defined as

$$y = f(t) = \frac{b_1}{3} + \frac{b_2}{3 \cdot 5} + \frac{b_3}{3^2 \cdot 5} + \frac{b_4}{3^2 \cdot 5^2} + \cdots,$$

where $b_1 = Q^{a_1+a_2}(a_3)$, $b_2 = Q^{a_1+a_2+a_4+a_5}(a_6)$, and in general

$$b_r = Q^{a_1+a_2+a_4+a_5+\cdots a_{3r-1}}(a_{3r}).$$

Here if $r$ is odd $Q^k(a_{3r}) = a_{3r}$ or $2 - a_{3r}$ according as $k$ is even or odd, and if $r$ is even $Q^k(a_{3r}) = a_{3r}$ or $4 - a_{3r}$ according as $k$ is even or odd.

By a method similar to the one employed in § 1, it can be shown that $f(t)$ is a continuous non-differentiable function of $t$ in the interval (0,1).

THE UNIVERSITY OF CALCUTTA,
March, 1927.

# ON THE PROBLEM OF COLORING MAPS IN FOUR COLORS, II.*

By C. N. Reynolds, Jr.

1. **Introduction.** In this paper we shall continue the investigation of the implications of the published reductions of the four color problem which we began in the earlier paper of the same title, published in this journal, volume 28, pp. 1–15, to be refered to hereafter as (I).† The notation used here will include that of (I). The two papers may be considered as a treatment of the arithmetic geometry of the irreducible map in so far as the properties of such maps may be found by summing over such maps the numerical attributes of neighborhoods of regions.

More specifically, we shall strengthen and arithmetize two qualitative propositions, proven by Wernicke and Franklin, which read as follows:

(a) Every irreducible map must contain either two adjacent pentagons, or a pentagon adjacent to a hexagon.‡

(b) Every irreducible map must contain either a pentagon adjacent to two other pentagons, a pentagon adjacent to a pentagon and to a hexagon, or a pentagon adjacent to two hexagons (cf. reduction (6) in section 2). We shall also establish arithmetically other existence theorems of the same generic type in order to suggest ways of avoiding the dificulty which seems inherent in the reduction of pentagons.

The method characteristic of this paper is that of setting up a set of functions defined over a discrete set of points on a simply connected closed surface bearing an irreducible map. For our set of points we choose an arbitrary point within each of the regions of our map: we may call this point the center of our region. Such a region will be bounded by a sequence of regions having $i_1, i_2, \cdots, i_n$ sides respectively. Then we let $\Delta a(i_1 \cdots i_n)$ be a function, associated with the circular permutation enclosed by the

* Presented, in part, to the American Mathematical Society, December 29, 1926; received February 11, 1927.

† This paper should be corrected by the insertion of a term "$-2a$" in the right hand side of (14); and the imposition of the condition "$n \geqq 9$" on (20). A part of the material in (I) was presented, by title, to the mathematical section of the *Association Française pour l'Avancement des Sciences*, July 31, 1926; accordingly a four page illustrated abstract of (I) appeared in the Comptes Rendus (1926) of that Association, pp. 88–91.

‡ P. Wernicke, Über den kartographischen Vierfarbensatz, Math. Annalen, vol. 58 (1904), p. 419.

parentheses, defined at the "center" of each region as 1, or 0, according as the region is, or is not, bounded by a set of regions of the type described by the given circular permutation. Summing this function over the central points of our map we have $a(i_1 \cdots i_n)$, the number of regions in the map bounded by a sequence of regions described by the given circular permutation. Other functions whose arguments are cyclic permutations, e. g. $f(i_1 \cdots i_n)$, may by summed over those particular regions of our map which are bounded by the given permutation by first multiplying by $\Delta a(i_1 \cdots i_n)$ and then summing over the entire map, the result being indicated by the symbol $f(i_1 \cdots i_n)\, a(i_1 \cdots i_n)$. We shall often have to deal with the result of a second summation, indicated by $\sum f(i_1 \cdots i_n)\, a(i_1 \cdots i_n)$, in which the result of our first summation is summed over all the cyclic permutations which describe sequences of polygons which may bound a single polygon in an irreducible map.

Our final results are existence theorems in which certain configurations are shown to exist in an irreducible map in such numbers as to satisfy certain equations which are necessary but not sufficient conditions for the irreducibility of a map of a simply connected closed surface.

Before presenting our proofs we shall present, in section two, a detailed and classified list of the reductions of our problem which have been published (February 1927); and, in section three, we shall present a bibliography of the four color problem and the related theorems of Euler, Petersen and Tait. This bibliography, covering the years 1921 to 1926 inclusive, will supplement the very complete bibliography published by Errera in 1921.

2. **The reductions of our problem.** In this section we shall present a detailed list, with references to proofs, of all the known properties of those maps of simply connected closed surfaces, known as irreducible maps, which if they exist, (1) may not be colored in fewer than five colors, and which (2) have as few regions as any other map satisfying (1). Hereafter we shall refer to any reduction by giving its number in this list. Each reduction will be followed by a reference to its proof, the author of the article to which reference is made being responsible for the reduction unless the contrary is stated.

(1) Each vertex belongs to three and only three regions. Due to A. B. Kempe. Cf. G. D. Birkhoff, The Reducibility of Maps, Amer. Journ. Math., 35 (1913), pp. 115–128. In later citations this paper will be refered to as "Birkhoff".

(2) There are no multiply connected sets of fewer than six regions except in the case of five regions surrounding a pentagon. This proposition summarizes the following propositions:

(2a) There are no multiply connected regions. Due to Kempe. Cf. Birkhoff, p. 115.

(2b) There are no multiply connected sets of two or three regions. Due to Kempe. Cf. Birkhoff, p. 115.

(2c) There are no regions of fewer than five sides. Due to Kempe. Cf. Birkhoff, p. 116.

(2d) There are no multiply connected sets of four regions. Cf. Birkhoff, p. 123.

(2e) There are no multiply connected sets of five regions except about a single region. Cf. Birkhoff, p. 123.

We shall introduce the phrase "Errera ring" to refer to any ring of polygons, no three of which meet at a point, provided that it consists of either (a) an even number, possibly zero, of hexagons and zero or more even sequences of pentagons, (b) an even number of pentagons and two consecutive polygons having arbitrary numbers of sides, or (c) an even number of pentagons and one polygon of an arbitrary number of sides. Rings of this type were first considered by Alfred Errera in the twelfth paper of our bibliography. Errera's treatment of these rings involves the erasure of certain boundaries of the polygons forming an Errera ring. He shows that unless these erasures lead to a set of boundaries forming a linear graph containing an isthmus, these rings are reducible. With the exception of a few cases the reducibility of any given Errera ring can only be determined by experiment. These exceptional cases, however, generalize many of the earlier reductions.

(3) No Errera ring may be present if its reduction in the manner indicated by Errera introduces no isthmus. Cf. Errera (12) paragraph (3, 3). The hypothesis of this reduction is satisfied in the following cases (among many others):

(3a) When eight hexagons surround a pair of hexagons and a pentagon which meet at a point. Cf. (12) paragraph (2, 4, 1).

(3b) When six hexagons surround a pair of adjacent pentagons. Cf. (12) paragraph (2, 4, 1).

(3c) When eight hexagons surround the figure formed by a pair of pentagons each of which is in contact with opposite sides of a hexagon lying between them and inside of the ring. Cf. (12) paragraph (2, 4, 1).

(4) No Errera ring may surround a single region. This is an immediate consequence of (3) and the specific nature of Errera's process of reduction, which he does not mention explicitly, and which generalizes and summarizes the following ealier reductions:

(4a) No hexagon may be surrounded by pentagons. Cf. Birkhoff, p. 126.

(4b) No polygon may be surrounded by pentagons. Cf. Birkhoff, pp. 127, 128.

(4c) No polygon of an even number of sides may be surrounded by hexagons. Cf. Birkhoff, p. 128.

(4d) No polygon of an even number of sides may be surrounded by hexagons and some (or no) pairs of adjacent pentagons. Cf. Franklin (14), p. 232.

(4e) No polygon of an even number of sides may be surrounded by a sequence of pentagons and two other adjacent regions. Cf. Franklin (14), p. 232.

(4f) No region of an odd number of sides may be surrounded by a sequence of pentagons and one other region. Cf. Franklin (14), p. 232.

(4g) No region of $n$ sides may be in contact with $n-1$ pentagons. Cf. Franklin (14), p. 232.

(5) No boundary of a region of fewer than seven sides may be enclosed by this region and three pentagons. This proposition summarizes the following reductions:

(5a) No boundary may be surrounded by four pentagons. Cf. Birkhoff, p. 126.

(5b) No side of a hexagon may be surrounded by the hexagon and three pentagons. Cf. Franklin (14), p. 229.

(5c) No pentagon may be in contact with three pentagons and a hexagon. Cf. Franklin (14), p. 230.

(5d) No pentagon may be contact with two pentagons and three hexagons. Cf. Franklin (14), p. 231.

(5e) No hexagon may be surrounded by four pentagons and two hexagons. Cf. Franklin (14), p. 232.

(6) At least one pentagon of an irreducible map must be in contact with two or more polygons of fewer than seven sides each. Cf. Franklin (14), p. 227.

(7) Every irreducible map must contain at least one region of more than six sides. Cf. Errera (12) paragraph (4, 1).

These reductions, the analytic relations established by the author in I, culminating in the proposition that any irreducible map must contain at least twenty-eight regions, and the results of this paper, represent the present state of our knowledge concerning irreducible maps, in so far as that knowledge has been published.

With the exception of reduction (3), due to Errera, and (7) which is dependent upon (3), these reductions are restraints upon the immediate neighborhoods of regions. In this paper we shall use only these reductions *im Kleinen*, viz. the reductions (1), (2), (4), (5) and their respective subordinate reductions, as listed above, together with reduction (3b). This is the same list which we used in (I). When we speak of an irreducible map we shall refer to a map the reducibility of which is not implied by

these reductions. Therefore our propositions are necessary conditions for the existence of any map which is irreducible with respect to the reductions known at present or *to be developed in the future.* It may be possible to find the implications of such future reductions by eliminating the rows of our matrices corresponding to types of regions which may be found reducible. In this sense this sequence of papers is an attempt to construct a prolegomenon to at least one future study of the four color problem.

**3. Bibliography of papers on the four color problem and the associated theorems of Euler, Petersen and Tait for the years 1921–1926.** For the earlier papers on these topics we refer to the very complete bibliography by Errera in the sixth paper listed below.*

(1) Brahana, H. R., Riemann surfaces and the map problem (Abstract of an unpublished paper), Bull. Amer. Math. Soc., 29 (1923), p. 198.
(2) — The four color problem, Amer. Math. Mon., 30 (1923), pp. 234–243.
(3) Chuard, J., Quelques propriétés des réseaux cubiques tracés sur une sphère, Comptes Rendus, 176 (1923), pp. 73–75.
(4) — Le problème des quatre couleurs, l'Ens. Math., 22 (1923), pp. 373–374.
(5) — Sur un théorème relatif à certains réseaux cubiques sur une sphère, l'Ens. Math., 23 (1923), p. 209.
(6) Errera, A., Du coloriage des cartes et de quelques questions d'analysis situs. Thesis. Gauthier-Villars, Paris (1921) 66 pp.
(7) — Une démonstration du théorème de Petersen, Mathesis, 36 (1922), pp. 56–61.
(8) — Le problème des quatre couleurs, l'Ens. Math., 23 (1923), pp. 95–96.
(9) — Un théorème sur les liaisons, Comptes Rendus, 177 (1923), pp. 489–491.
(10) — Sur le problème des quatre couleurs, Comptes Rendus de l'Assoc. Franç. pour l'Avancement des Sciences (1924), pp. 95–96.
(11) — Quelques remarques sur le problème des quatre couleurs (Abstract of (12)), Abstracts of the Int. Math. Congress at Toronto (1924), p. 57.
(12) — Une contribution au problème des quatre couleurs, Bul. Soc. Math. de France, 53 (1925), pp. 42–55.
(13) Fitting, F., Über das Problem der Rundreise und einen damit im Zusammenhang stehenden Satz von Tait, Nieuw Archief (2), 13 (1921), pp. 348–360.

* There is also available an unpublished dissertation by Dr. J. P. Ballantine (Ph. D., Chicago, 1923) entitled "A Postulational Introduction to the Four Color Problem", 48 pp. + 5 p. abstract. This manuscript is deposited in the University of Chicago Library, Book number: QA 999, B 22.

(14) Franklin, P., The four color problem. Am. J. Math., 44 (1922),pp. 225–236.
(15) Frink, O., A proof of Petersen's theorem, Annals of Math. (2), 27 (1926), pp. 491–493.
(16) Hunziker, G., Über das Kartenfärbungsproblem. Thesis. Zürich (1924), 62 pp.
(17) Kubota, T., Partitioning of the plane by polygons, Tôhoku Math. J., 24 (1925), pp. 273–276.
(18) Lebesgue, H., Remarques sur les deux premières démonstrations du théorème d'Euler relativ aux polyèdres, Bul. Soc. Math. de France, 42 (1924), pp. 315–336.
(19) Maroni, A., Il teorema Descartes-Eulero relativo ai poliedri, Periodico di Mat. (4), 1 (1921), pp. 337–346.
(20) Reynolds, C. N., Note on the map coloring problem, Bull. Amer. Math. Soc., 30 (1924), p. 220.
(21) — On the map coloring problem, with particular reference to connected sets of pentagons (Abstract of a part of (22)), Bull. Amer. Math. Soc., 31 (1925), pp. 297–298.
(22) — On the problem of coloring maps in four colors, I. Annals of Math. (2), 28 (1926), pp. 1–15.
(23) Ste. Laguë, A., Les réseaux, Comptes Rendus, 176 (1923), pp. 1202–1204.
(24) — Les réseaux unicursaux et bicursaux, Comptes Rendus, 182 (1926), pp. 747–750.
(25) — Les réseaux (ou graphes), Mémorial des Sciences Math., fasc. 18, Paris (1926), 64 pp.

**4. The classification of polygons.** We shall have occasion to classify polygons according as they are bounded by various circular permutations of pentagons, hexagons, and $n$-gons $(n>6)$. For our immediate purposes two permutations will be regarded as equivalent if one may be obtained from the other by replacing an $n$-gon $(n>6)$ by an $n'$-gon $(n'>6)$ even though $n$ and $n'$ may be unequal. We shall indicate the number of polygons, in an irreducible map, which are bounded by the circular permutation $(i, i', i'', \cdots)$ by the symbol $a(i, i', i'', \cdots)$. Reductions (4) and (5) eliminate all but twenty-seven homeomorphically distinct types of pentagons, and all but seventy-four homeomorphically distinct types of hexagons.

We shall divide the vertices of an irreducible map, $v$ in number, into ten classes according to the number of pentagons, hexagons and $n$-gons meeting at any vertex. If we have $v_{555}$ vertices in which three pentagons meet, $\cdots$, and $v_{nnn}$ vertices in which three $n$-gons meet (the $n$'s being greater than six, and possibly unequal), then we shall have:

$$(1)\quad v = v_{555} + v_{556} + v_{55n} + v_{566} + v_{56n} + v_{5nn} + v_{666} + v_{66n} + v_{6nn} + v_{nnn}.$$

Multiplying this relation by three, we shall have the number of "corners of polygons" in our map, five of which are contributed by each pentagon, six by each hexagon, etc. These "corners of polygons" will be apportioned among our ten classes of vertices in the following manner:

$$5a_5 = 3v_{555} + 2v_{556} + 2v_{55n} + v_{566} + v_{56n} + v_{5nn} \tag{2.1}$$

$$6a_6 = 3v_{666} + 2v_{566} + 2v_{66n} + v_{556} + v_{56n} + v_{6nn} \tag{2.2}$$

$$\sum_7 n a_n = 3v_{nnn} + 2v_{5nn} + 2v_{6nn} + v_{55n} + v_{56n} + v_{66n}. \tag{2.3}$$

Here as in I, $a_n$ is the number of $n$-gons in our map. Any particular type of polygon will make a certain contribution to each term on the right hand side of these relations. For example, a pentagon of the type $(5, 5, 6, n, 6)$ contributes 1 to $3v_{555}$, 2 to $2v_{556}$, 2 to $v_{56n}$, and 0 to each of the other $v$'s. These contributions may be derived from figures, or from the symbols for circular permutations, two consecutive 5's in a symbol for a pentagon indicating a unit contribution toward $3v_{555}$, a 5 and an adjacent 6 indicating a unit contribution toward $2v_{556}$, etc., it being remembered that, since our permutations are cyclic, the first and last numbers in each symbol are to be considered consecutive. Summing these contributions over an entire irreducible map we can expand the quantities $v_{i,j,k}$ linearly in terms of the quantities $a(i, i', i'', \cdots)$. The coefficients of these linear expansions are given in matrices $M_5$, $M_6$, $M_n$, and $M_r$.

In $M_5$, and $M_6$, the pentagons, and hexagons, are classified at the left in terms of their associated circular permutations. These permutations are alphabetized with respect to the sequence $5, 6, n$, as an alphabet, each circular permutation being so cut, in writing its symbol, as to place it as early in the list as possible. The headings of the columns are the various quantities $v_{i,j,k}$, together with certain linear combinations of them to be defined later. Any element of the matrix is the coefficient of the quantity $a(i, i', i'', \cdots)$ associated with the permutation at the left hand end of its row in the expansion of the quantity at the top of its column. To classify the heptagons in the same way would involve 193 circular permutations, the number of permutations increasing rapidly with the number of sides of our polygons. For brevity we have, in $M_n$, combined the pentagons and the hexagons of the permutations surrounding $n$-gons in a single group to be referred to as $k$-gons. Strictly speaking the matrix $M_n$ has an infinite number of rows. We have presented the upper end of this matrix, the portion in which the heptagons and the octagons of our map are classified by means of the circular permutations of the letters $k$ and $n$, repetitions being allowed.

An alternative method of avoiding a large number of types of heptagons and octagons is presented in matrix $M_r$ where hexagons, heptagons and octagons are classified according as they are bounded by various circular permutations of pentagons and $r$-gons ($r>5$). This classification differs from that used in the matrix $M_n$, since the hexagons bounding a given polygon have been grouped with the $n$-gons ($n>6$) to form the group of $r$-gons ($r>5$). Matrix $M_r$ is constructed just as $M_n$ was constructed except that (1) we provide columns for vertices of types $V_{55r}$, $V_{5rr}$, and $V_{rrr}$; and (2) we provide columns for the computation of the numbers, $y_n$ (cf. I), of instances in which a polygon of more than five sides comes in contact with $n$ consecutive pentagons of a connected configuration of pentagons. The last columns of $M_r$ are linear combinations of earlier columns, to be defined later. The results to be obtained from $M_r$ are closely related to the propositions given in (I) and may be combined with them to advantage.

*The Matrix $M_5$.*

| $i_1 i_2 \cdots i_5$ | $3v_{555}$ | $2v_{556}$ | $2v_{55n}$ | $v_{566}$ | $v_{56n}$ | $v_{5nn}$ | $\varphi_5$ | $\psi_5$ |
|---|---|---|---|---|---|---|---|---|
| (5566*n*) | 1 | 1 | 1 | 1 | 1 | 0 | 3 | 10 |
| (556*n*6) | 1 | 2 | 0 | 0 | 2 | 0 | 4 | 10 |
| (556*nn*) | 1 | 1 | 1 | 0 | 1 | 1 | 7 | 10 |
| (55*n*5*n*) | 1 | 0 | 4 | 0 | 0 | 0 | 4 | 4 |
| (55*n*6*n*) | 1 | 0 | 2 | 0 | 2 | 0 | 6 | 10 |
| (55*nnn*) | 1 | 0 | 2 | 0 | 0 | 2 | 10 | 10 |
| (5665*n*) | 0 | 2 | 2 | 1 | 0 | 0 | 2 | 8 |
| (56666) | 0 | 2 | 0 | 3 | 0 | 0 | 0 | 14 |
| (5666*n*) | 0 | 1 | 1 | 2 | 1 | 0 | 3 | 14 |
| (566*n*6) | 0 | 2 | 0 | 1 | 2 | 0 | 4 | 14 |
| (566*nn*) | 0 | 1 | 1 | 1 | 1 | 1 | 7 | 14 |
| (56*n*5*n*) | 0 | 1 | 3 | 0 | 1 | 0 | 5 | 8 |
| (56*n*6*n*) | 0 | 1 | 1 | 0 | 3 | 0 | 7 | 14 |
| (56*nn*6) | 0 | 2 | 0 | 0 | 2 | 1 | 8 | 14 |
| (56*nnn*) | 0 | 1 | 1 | 0 | 1 | 2 | 11 | 14 |
| (5*n*5*nn*) | 0 | 0 | 4 | 0 | 0 | 1 | 8 | 8 |
| (5*n*66*n*) | 0 | 0 | 2 | 1 | 2 | 0 | 6 | 14 |
| (5*n*6*nn*) | 0 | 0 | 2 | 0 | 2 | 1 | 10 | 14 |
| (5*nnnn*) | 0 | 0 | 2 | 0 | 0 | 3 | 14 | 14 |
| (66666) | 0 | 0 | 0 | 5 | 0 | 0 | 0 | 20 |
| (6666*n*) | 0 | 0 | 0 | 3 | 2 | 0 | 4 | 20 |
| (666*nn*) | 0 | 0 | 0 | 2 | 2 | 1 | 8 | 20 |
| (66*n*6*n*) | 0 | 0 | 0 | 1 | 4 | 0 | 8 | 20 |
| (66*nnn*) | 0 | 0 | 0 | 1 | 2 | 2 | 12 | 20 |
| (6*n*6*nn*) | 0 | 0 | 0 | 0 | 4 | 1 | 12 | 20 |
| (6*nnnn*) | 0 | 0 | 0 | 0 | 2 | 3 | 16 | 20 |
| (*nnnnn*) | 0 | 0 | 0 | 0 | 0 | 5 | 20 | 20 |

*The Matrix $M_6$.*

| $(i_1 i_2 \cdots i_6)$ | $3v_{666}$ | $2v_{566}$ | $2v_{66n}$ | $v_{556}$ | $v_{55n}$ | $v_{6nn}$ | $\varphi_6$ | $\psi_6$ |
|---|---|---|---|---|---|---|---|---|
| (55655*n*) | 0 | 2 | 0 | 2 | 2 | 0 | 0 | 0 |
| (556566) | 1 | 4 | 0 | 1 | 0 | 0 | 0 | 2 |
| (55656*n*) | 0 | 3 | 1 | 1 | 1 | 0 | 1 | 3 |
| (5565*n*6) | 0 | 3 | 1 | 1 | 1 | 0 | 1 | 3 |
| (5565*nn*) | 0 | 2 | 0 | 1 | 2 | 1 | 4 | 6 |
| (55665*n*) | 1 | 2 | 0 | 1 | 2 | 0 | 0 | 2 |
| (55666*n*) | 2 | 1 | 1 | 1 | 1 | 0 | 1 | 7 |
| (5566*n*6) | 1 | 2 | 2 | 1 | 0 | 0 | 2 | 8 |
| (5566*nn*) | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 11 |
| (556*n*5*n*) | 0 | 1 | 1 | 1 | 3 | 0 | 1 | 3 |
| (556*n*6*n*) | 0 | 1 | 3 | 1 | 1 | 0 | 3 | 9 |
| (556*nn*6) | 0 | 2 | 2 | 1 | 0 | 1 | 6 | 12 |
| (556*nnn*) | 0 | 1 | 1 | 1 | 1 | 2 | 9 | 15 |
| (55*n*55*n*) | 0 | 0 | 0 | 2 | 4 | 0 | 0 | 0 |
| (55*n*56*n*) | 0 | 1 | 1 | 1 | 3 | 0 | 1 | 3 |
| (55*n*5*nn*) | 0 | 0 | 0 | 1 | 4 | 1 | 4 | 6 |
| (55*n*66*n*) | 1 | 0 | 2 | 1 | 2 | 0 | 2 | 8 |
| (55*n*6*nn*) | 0 | 0 | 2 | 1 | 2 | 1 | 6 | 12 |
| (55*nnnn*) | 0 | 0 | 0 | 1 | 2 | 3 | 12 | 18 |
| (565656) | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 |
| (56565*n*) | 0 | 4 | 0 | 0 | 2 | 0 | 0 | 0 |
| (565666) | 2 | 4 | 0 | 0 | 0 | 0 | 0 | 4 |
| (56566*n*) | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 5 |
| (5656*n*6) | 0 | 4 | 2 | 0 | 0 | 0 | 2 | 6 |
| (5656*nn*) | 0 | 3 | 1 | 0 | 1 | 1 | 5 | 9 |
| (565*n*5*n*) | 0 | 2 | 0 | 0 | 4 | 0 | 0 | 0 |
| (565*n*6*n*) | 0 | 2 | 2 | 0 | 2 | 0 | 2 | 6 |
| (565*nnn*) | 0 | 2 | 0 | 0 | 2 | 2 | 8 | 12 |
| (566566) | 2 | 4 | 0 | 0 | 0 | 0 | 0 | 4 |
| (56656*n*) | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 5 |
| (5665*nn*) | 1 | 2 | 0 | 0 | 2 | 1 | 4 | 8 |
| (56665*n*) | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 4 |
| (566666) | 4 | 2 | 0 | 0 | 0 | 0 | 0 | 8 |
| (56666*n*) | 3 | 1 | 1 | 0 | 1 | 0 | 1 | 9 |
| (5666*n*6) | 2 | 2 | 2 | 0 | 0 | 0 | 2 | 10 |
| (5666*nn*) | 2 | 1 | 1 | 0 | 1 | 1 | 5 | 13 |
| (566*n*5*n*) | 1 | 1 | 1 | 0 | 3 | 0 | 1 | 5 |
| (566*n*66) | 2 | 2 | 2 | 0 | 0 | 0 | 2 | 10 |
| (566*n*6*n*) | 1 | 1 | 3 | 0 | 1 | 0 | 3 | 11 |
| (566*nn*6) | 1 | 2 | 2 | 0 | 0 | 1 | 6 | 14 |
| (566*nnn*) | 1 | 1 | 1 | 0 | 1 | 2 | 9 | 17 |
| (56*n*56*n*) | 0 | 2 | 2 | 0 | 2 | 0 | 2 | 6 |
| (56*n*5*n*6) | 0 | 2 | 2 | 0 | 2 | 0 | 2 | 6 |
| (56*n*5*nn*) | 0 | 1 | 1 | 0 | 3 | 1 | 5 | 9 |

| $(i_1 i_2 \cdots i_6)$ | $3v_{666}$ | $2v_{566}$ | $2v_{665}$ | $v_{556}$ | $v_{565}$ | $v_{655}$ | $\varphi_6$ | $\psi_6$ |
|---|---|---|---|---|---|---|---|---|
| (56*n*65*n*) | 0 | 2 | 2 | 0 | 2 | 0 | 2 | 6 |
| (56*n*66*n*) | 1 | 1 | 3 | 0 | 1 | 0 | 3 | 11 |
| (56*n*6*n*6) | 0 | 2 | 4 | 0 | 0 | 0 | 4 | 12 |
| (56*n*6*nn*) | 0 | 1 | 3 | 0 | 1 | 1 | 7 | 15 |
| (56*nn*5*n*) | 0 | 1 | 1 | 0 | 3 | 1 | 5 | 9 |
| (56*nn*6*n*) | 0 | 1 | 3 | 0 | 1 | 1 | 7 | 15 |
| (56*nnn*6) | 0 | 2 | 2 | 0 | 0 | 2 | 10 | 18 |
| (56*nnnn*) | 0 | 1 | 1 | 0 | 1 | 3 | 13 | 21 |
| (5*n*5*n*5*n*) | 0 | 0 | 0 | 0 | 6 | 0 | 0 | 0 |
| (5*n*5*n*6*n*) | 0 | 0 | 2 | 0 | 4 | 0 | 2 | 6 |
| (5*n*5*nnn*) | 0 | 0 | 0 | 0 | 4 | 2 | 8 | 12 |
| (5*n*666*n*) | 2 | 0 | 2 | 0 | 2 | 0 | 2 | 10 |
| (5*n*66*nn*) | 1 | 0 | 2 | 0 | 2 | 1 | 6 | 14 |
| (5*n*6*n*6*n*) | 0 | 0 | 4 | 0 | 2 | 0 | 4 | 12 |
| (5*n*6*nnn*) | 0 | 0 | 2 | 0 | 2 | 2 | 10 | 18 |
| (5*nn*5*nn*) | 0 | 0 | 0 | 0 | 4 | 2 | 8 | 12 |
| (5*nn*6*nn*) | 0 | 0 | 2 | 0 | 2 | 2 | 10 | 18 |
| (5*nnnnn*) | 0 | 0 | 0 | 0 | 2 | 4 | 16 | 24 |
| (66666*n*) | 4 | 0 | 2 | 0 | 0 | 0 | 2 | 14 |
| (6666*nn*) | 3 | 0 | 2 | 0 | 0 | 1 | 6 | 18 |
| (666*n*6*n*) | 2 | 0 | 4 | 0 | 0 | 0 | 4 | 16 |
| (666*nnn*) | 2 | 0 | 2 | 0 | 0 | 2 | 10 | 22 |
| (66*n*66*n*) | 2 | 0 | 4 | 0 | 0 | 0 | 4 | 16 |
| (66*n*6*nn*) | 1 | 0 | 4 | 0 | 0 | 1 | 8 | 20 |
| (66*nnnn*) | 1 | 0 | 2 | 0 | 0 | 3 | 14 | 26 |
| (6*n*6*n*6*n*) | 0 | 0 | 6 | 0 | 0 | 0 | 6 | 18 |
| (6*n*6*nnn*) | 0 | 0 | 4 | 0 | 0 | 2 | 12 | 24 |
| (6*nn*6*nn*) | 0 | 0 | 4 | 0 | 0 | 2 | 12 | 24 |
| (6*nnnnn*) | 0 | 0 | 2 | 0 | 0 | 4 | 18 | 30 |
| (*nnnnnn*) | 0 | 0 | 0 | 0 | 0 | 6 | 24 | 36 |

## *The Matrix* $M_n$

| $(i_1 i_2 \cdots i_n)$ | $v_{kkn}$ | $2v_{knn}$ | $3v_{nnn}$ |
|---|---|---|---|
| (*kkkkkkk*) | 7 | 0 | 0 |
| (*kkkkkkn*) | 5 | 2 | 0 |
| (*kkkkknn*) | 4 | 2 | 1 |
| (*kkkknkn*) | 3 | 4 | 0 |
| (*kkkknnn*) | 3 | 2 | 2 |
| (*kkknkkn*) | 3 | 4 | 0 |
| (*kkknknn*) | 2 | 4 | 1 |
| (*kkknnnn*) | 2 | 2 | 3 |
| (*knkknnn*) | 2 | 4 | 1 |
| (*knknkn*) | 1 | 6 | 0 |
| (*knknnnn*) | 1 | 4 | 2 |

| $(i_1 i_2 \cdots i_n)$ | $v_{kkn}$ | $2v_{knn}$ | $3v_{nnn}$ |
|---|---|---|---|
| (*kknnknn*) | 1 | 4 | 2 |
| (*kknnnnn*) | 1 | 2 | 4 |
| (*knknknn*) | 0 | 6 | 1 |
| (*knknnnn*) | 0 | 4 | 3 |
| (*knnknnn*) | 0 | 4 | 3 |
| (*knnnnnn*) | 0 | 2 | 5 |
| (*nnnnnnn*) | 0 | 0 | 7 |
| (*kkkkkkkk*) | 8 | 0 | 0 |
| (*kkkkkkkn*) | 6 | 2 | 0 |
| (*kkkkkknn*) | 5 | 2 | 1 |

| $(i_1\, i_2 \cdots i_n)$ | $v_{kkn}$ | $2v_{knn}$ | $3v_{nnn}$ |
|---|---|---|---|
| (*kkkkknkn*) ..... | 4 | 4 | 0 |
| (*kkkkknnn*) ..... | 4 | 2 | 2 |
| (*kkkknkkn*) ..... | 4 | 4 | 0 |
| (*kkkknknn*) ..... | 3 | 4 | 1 |
| (*kkkknnnn*) ..... | 3 | 2 | 3 |
| (*kkknkkkn*) ..... | 4 | 4 | 0 |
| (*kkknkknn*) ..... | 3 | 4 | 1 |
| (*kkknknkn*) ..... | 2 | 6 | 0 |
| (*kkknknnn*) .... | 2 | 4 | 2 |
| (*kkknnknn*) ..... | 2 | 4 | 2 |
| (*kkknnnnn*) ..... | 2 | 2 | 4 |
| (*kknkknkn*) ..... | 2 | 6 | 0 |
| (*kknkknnn*) ..... | 2 | 4 | 2 |
| (*kknknknn*) ..... | 1 | 6 | 1 |
| (*kknknnnn*) ..... | 1 | 4 | 3 |

| $(i_1\, i_2 \cdots i_n)$ | $v_{kkn}$ | $2v_{knn}$ | $3v_{nnn}$ |
|---|---|---|---|
| (*kknnkknn*) ..... | 2 | 4 | 2 |
| (*kknnknnn*) ..... | 1 | 4 | 3 |
| (*kknnnnnn*) .... | 1 | 2 | 5 |
| (*knknknkn*) ..... | 0 | 8 | 0 |
| (*knknknnn*) .... | 0 | 6 | 2 |
| (*knknnknn*) ..... | 0 | 6 | 2 |
| (*knknnnnn*) ..... | 0 | 4 | 4 |
| (*knnknnnn*) ..... | 0 | 4 | 4 |
| (*knnnknnn*) .... | 0 | 4 | 4 |
| (*knnnnnnn*) .... | 0 | 2 | 6 |
| (*nnnnnnnn*) .... | 0 | 0 | 8 |
| (*kkkkkkkkk*) .... | 9 | 0 | 0 |

. . . . . .

## *The Matrix $M_r$.*

| $(i_1\, i_2 \cdots i_r)$ | $v_{55r}$ | $2v_{5rr}$ | $3v_{rrr}$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $3\pi_r$ | $\varrho_r$ |
|---|---|---|---|---|---|---|---|---|---|---|
| (55*r*55*r*) ..... | 2 | 4 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | −1 |
| (55*r*5*rr*) ..... | 1 | 4 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| (55*rrrr*) ..... | 1 | 2 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| (5*r*5*r*5*r*) ..... | 0 | 6 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| (5*r*5*rrr*) ..... | 0 | 4 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 1 |
| (5*rr*5*rr*) ..... | 0 | 4 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 1 |
| (5*rrrrr*) ..... | 0 | 2 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 2 |
| (*rrrrrr*) ..... | 0 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| (55555*rr*) ..... | 4 | 2 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| (5555*r*5*r*) ..... | 3 | 4 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| (5555*rrr*) ..... | 3 | 2 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| (555*r*55*r*) ..... | 3 | 4 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| (555*r*5*rr*) ..... | 2 | 4 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| (555*rrrr*) ..... | 2 | 2 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| (55*r*55*rr*) ..... | 2 | 4 | 1 | 0 | 2 | 0 | 0 | 0 | 1 | 0 |
| (55*r*5*r*5*r*) ..... | 1 | 6 | 0 | 2 | 1 | 0 | 0 | 0 | 1 | 0 |
| (55*r*5*rrr*) ..... | 1 | 4 | 2 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| (55*rr*5*rr*) ..... | 1 | 4 | 2 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| (55*rrrrr*) ..... | 1 | 2 | 4 | 0 | 1 | 0 | 0 | 0 | 1 | 2 |
| (5*r*5*r*5*rr*) ..... | 0 | 6 | 1 | 3 | 0 | 0 | 0 | 0 | 1 | 1 |
| (5*r*5*rrrr*) ..... | 0 | 4 | 3 | 2 | 0 | 0 | 0 | 0 | 1 | 2 |
| (5*rr*5*rrr*) ..... | 0 | 4 | 3 | 2 | 0 | 0 | 0 | 0 | 1 | 2 |
| (5*rrrrrr*) ..... | 0 | 2 | 5 | 1 | 0 | 0 | 0 | 0 | 1 | 3 |
| (*rrrrrrr*) ..... | 0 | 0 | 7 | 0 | 0 | 0 | 0 | 0 | 1 | 4 |

| $(i_1 i_2 \cdots i_r)$ | $v_{55r}$ | $2v_{5rr}$ | $3v_{rrr}$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $3\pi_r$ | $\varrho_r$ |
|---|---|---|---|---|---|---|---|---|---|---|
| (55555r5r) .... | 4 | 4 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| (55555rrr) .... | 4 | 2 | 2 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| (5555r55r) .... | 4 | 4 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| (5555r5rr) .... | 3 | 4 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| (5555rrrr).... | 3 | 2 | 3 | 0 | 0 | 0 | 1 | 0 | 1 | 2 |
| (555r555r) .... | 4 | 4 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 |
| (555r55rr) .... | 3 | 4 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| (555r5r5r) .... | 2 | 6 | 0 | 2 | 0 | 1 | 0 | 0 | 1 | 1 |
| (555r5rrr)..... | 2 | 4 | 2 | 1 | 0 | 1 | 0 | 0 | 1 | 2 |
| (555rr5rr)..... | 2 | 4 | 2 | 1 | 0 | 1 | 0 | 0 | 1 | 2 |
| (555rrrrr)..... | 2 | 2 | 4 | 0 | 0 | 1 | 0 | 0 | 1 | 3 |
| (55r55r5r) .... | 2 | 6 | 0 | 1 | 2 | 0 | 0 | 0 | 2 | 0 |
| (55r55rrr)..... | 2 | 4 | 2 | 0 | 2 | 0 | 0 | 0 | 2 | 1 |
| (55r5r5rr)..... | 1 | 6 | 1 | 2 | 1 | 0 | 0 | 0 | 2 | 1 |
| (55r5rrrr)..... | 1 | 4 | 3 | 1 | 1 | 0 | 0 | 0 | 2 | 2 |
| (55rr5rr)..... | 2 | 4 | 2 | 0 | 2 | 0 | 0 | 0 | 2 | 1 |
| (55rr5rrr)..... | 1 | 4 | 3 | 1 | 1 | 0 | 0 | 0 | 2 | 2 |
| (55rrrrrr)..... | 1 | 2 | 5 | 0 | 1 | 0 | 0 | 0 | 2 | 3 |
| (5r5r5r5r)..... | 0 | 8 | 0 | 4 | 0 | 0 | 0 | 0 | 2 | 1 |
| (5r5r5rrr)..... | 0 | 6 | 2 | 3 | 0 | 0 | 0 | 0 | 2 | 2 |
| (5r5rr5rr)..... | 0 | 6 | 2 | 3 | 0 | 0 | 0 | 0 | 2 | 2 |
| (5r5rrrrr)..... | 0 | 4 | 4 | 2 | 0 | 0 | 0 | 0 | 2 | 3 |
| (5rr5rrrr)..... | 0 | 4 | 4 | 2 | 0 | 0 | 0 | 0 | 2 | 3 |
| (5rrr5rrr)..... | 0 | 4 | 4 | 2 | 0 | 0 | 0 | 0 | 2 | 3 |
| (5rrrrrrr)..... | 0 | 2 | 6 | 1 | 0 | 0 | 0 | 0 | 2 | 4 |
| (rrrrrrrr)..... | 0 | 0 | 8 | 0 | 0 | 0 | 0 | 0 | 2 | 5 |
| (5555555rr) ... | 6 | 2 | 1 | . | . | . | . | . | . | . |
| etc. | | | | | | | | | | |

5. **Identities derived from matrices $M_5$, $M_6$, $M_n$, and $M_r$.** Some numerical characteristics of irreducible maps may be expanded by means of one of our matrices and only one. The symbol representing such a quantity will contain a subscript indicating the matrix which is to be used in its expansion. The expansion of such a quantity $f_x$ by means of the matrix $M_x$, in terms of the $y$-gons of our map will be indicated by the symbol

$$\sum f_x(i_1 \cdots i_y)\, a(i_1 \cdots i_y).$$

When a quantity, such as $v_{556}$, may be expanded by means of two or more matrices, the particular matrix, $M_x$, being used will be indicated thus

$$\sum v_{556,x}(i_1 \cdots i_y)\, a(i_1 \cdots i_y).$$

Since two expansions of one quantity derived from each of two matrices are equal, we have the following relations from $M_5$ and $M_6$:

$$(3.1)\quad \sum v_{556,5}(i_1 \cdots i_5)\,a(i_1 \cdots i_5) = \sum v_{556,6}(i_1 \cdots i_6)\,a(i_1 \cdots i_6),$$

$$(3.2)\quad \sum v_{566,5}(i_1 \cdots i_5)\,a(i_1 \cdots i_5) = \sum v_{566,6}(i_1 \cdots i_6)\,a(i_1 \cdots i_6),$$

$$(3.3)\quad \sum v_{56n,5}(i_1 \cdots i_5)\,a(i_1 \cdots i_5) = \sum v_{56n,6}(i_1 \cdots i_6)\,a(i_1 \cdots i_6).$$

These may be combined linearly with (1) to obtain a variety of expressions for $3v$. For example, if we let

$$\varphi_5 \equiv 2v_{55n} + 2v_{56n} + 4v_{5nn}$$

and

$$\varphi_6 \equiv 2v_{66n} + 4v_{6nn},$$

then (1), (2.1), (2.2), (3.1), (3.2), (3.3) imply that

$$(4.1)\quad 6v = 10a_5 + \sum \varphi_5(i_1 \cdots i_5)\,a(i_1 \cdots i_5) + 12a_6 + \sum \varphi_6(i_1 \cdots i_6)\,a(i_1 \cdots i_6) + 6\sum v_{nnn}(i_1 \cdots i_n)\,a(i_1 \cdots i_n),$$

where the $\varphi$'s are expanded from the seventh columns of matrices $M_5$ and $M_6$. Since $3v = \sum_5 n a_n$ and $a_5 = 12 + \sum_7 (n-6)\,a_n$, cf. (10a) of (I), we have, eliminating $a_7$,

$$3v = 12a_5 + 6a_6 - 84 - 6\sum_7 (n-7)\,a_n.$$

This relation, with (4.1), implies that

$$(4.2)\quad \sum (14 - \varphi_5(i_1 \cdots i_5))\,a(i_1 \cdots i_5) = 168 + 12\sum (n-7)a_n + \sum \varphi_6(i_1 \cdots i_6)\,a(i_1 \cdots i_6) + 6\sum v_{nnn}(i_1 \cdots i_n)\,a(i_1 \cdots i_n).$$

It will be found from matrix $M_5$ that $\varphi_5(i_1 \cdots i_5)$ it less than 14 in all cases except these three: $\varphi_5(5nnnn) = 14$, $\varphi_5(6nnnn) = 16$, and $\varphi_5(nnnnn) = 20$. Transposing the corresponding zero or negative terms on the left hand side of (4.2) they become zero or positive. The remaining terms correspond to those types of pentagons which are in contact with at least two polygons of fewer than seven sides each. Franklin has proven, reduction (6), that an irreducible map contains at least one such pentagon. After the above transposition the right hand side of (4.2) is at least 168 and the highest coefficients on the left are $14 - \varphi_5(56666) = 14 - \varphi_5(66666) = 14$. Therefore, incidentally, there are at least twelve of the pentagons considered by Franklin, and there are more than twelve unless each pentagon is entirely surrounded by pentagons and hexagons and every hexagon is of a type for which $\varphi_6(i_1 \cdots i_6) = 0$. The relation (4.2) is, however, a more powerful restraint upon the numbers

of various types of pentagons in an irreducible map than any simple corrolary would be. It implies

THEOREM I. *In any irreducible map there exists a set of at least twelve pentagons, each of which is in contact with at least two polygons of fewer than seven sides, of such types and, with respect to each type, in such numbers as to satisfy equation* (4.2).

If we let

$$\psi_5 = 2v_{556} + 2v_{55n} + 4v_{566} + 4v_{56n} + 4v_{5nn}$$

and

$$\psi_6 = 6v_{666} + 6v_{66n} + 6v_{6nn}$$

then by a sequence of steps similar to that used in deriving (4.1) and (4.2) we find that

$$6v = 10a_5 + \sum \psi_5(i_1 \cdots i_5)\, a(i_1 \cdots i_5) + \sum \psi_6(i_1 \cdots i_6)\, a(i_1 \cdots i_6) + 6\sum v_{nnn}(i_1 \cdots i_n)\, a(i_1 \cdots i_n). \tag{5.1}$$

Upon introducing the notation $p_n$ ($n = 1, 2, 3, 4, 5$), the numbers of various types of pentagons defined in (I), and equation (10a) as given in (I) we have

$$\begin{aligned} &p_2 + 5p_3 + 6p_4 + 8p_5 + 6a_6 \\ &= 120 + \sum\nolimits_7 (9_n - 60)\, a_n + \frac{1}{2}\sum \psi_6(i_1 \cdots i_6)\, a(i_1 \cdots i_6) \\ &\qquad + 3\sum v_{nnn}(i_1 \cdots i_n)\, a(i_1 \cdots i_n). \end{aligned} \tag{5.2}$$

Here the first four terms on the left, if positive, imply by reductions (3 *b*), (5), and (5 *a*) that there exist polygons of more than six sides in our irreducible map. If, therefore, an irreducible map had no polygons of more than six sides it would have to have a least twenty hexagons. Such a map of thirty two regions, irreducible with respect to the reductions which we are considering but reducible with respect to the reduction (7), may be formed by cutting off the twenty vertices of a regular dodecahedron by hexagonal sections. In any case we have

THEOREM II. *The polygons of an irreducible map must be of such types and, with respect to each type, must occur in such numbers as to satisfy equation* (5.2).

Turning now to the matrix $M_r$ we find that if we define $\pi_r$ by the identity

$$3\pi_r \equiv 3v_{rrr} + 2y_1 + 3y_2 + \sum\nolimits_3 (n-2)\, y_n + 2(n-9)\, a_n,$$

every term in this expression is positive or zero except the last which is negative for polygons of fewer than nine sides. From our matrix,

however, we find that no type of polygon making a negative contribution toward the last term of $\pi_r$ does in fact make a negative contribution toward $\pi_r$ itself. $\sum \pi_r(i_1 \cdots i_r)\, a(i_1 \cdots i_r)$ is therefore positive or zero for every irreducible map.

Upon counting the corners of polygons of more than five sides we find three at each vertex $V_{rrr}$, $n-1$ at the vertices $V_{55r}$, dividing each $Y_n$ contact among $n$ pentagons, and two at each vertex $V_{5rr}$ dividing two $Y_n$ contacts from each other. Since each $n$-gon has $n$ corners we have

$$\sum_6 n\, a_n = 3\, v_{rrr} + \sum_1 (n+1)\, y_n. \tag{6}$$

Eliminating $v_{rrr}$ between this relation and the formula for $\pi_r$ we find that

$$\sum_7 (n-6)\, a_n = \sum_8 y_n + \sum \pi_r(i_1 \cdots i_r)\, a(i_1 \cdots i_r). \tag{7.1}$$

The first nine identities of (I) enable us to write this in many ways, e. g.

$$p_1 + p_5 + y_2 = 12 + \sum \pi_r(i_1 \cdots i_r)\, a(i_1 \cdots i_r). \tag{7.2}$$

Eliminating $y_2$ from our definition of $\pi_r$ and (7.2) and simplifying the result by means of equations (1) to (9) from (I) we find that

$$\begin{aligned} 2p_5 + 6a_6 + 3a_7 = {} & 48 + 3\sum_8 (n-8)\, a_n + 6p_1 + 3p_2 + p_3 \\ & + 3\sum v_{rrr}(i_1 \cdots i_r)\, a(i_1 \cdots i_r). \end{aligned} \tag{7.3}$$

Every term on the right hand side of this relation being positive or zero we have

THEOREM III. *In any irreducible map there must exist a set of at least eight polygons each one of which is either a pentagon in contact with three other pentagons, a hexagon or a heptagon, of such types, and, with respect to each type, in such numbers as to satisfy relation* (7.3).

If we let

$$\varrho_r \equiv (n-3)\, a_n - y_1 - y_2 - \sum_1 (n-1)\, y_n$$

then from (6) and $M_r$ we find that the only negative contribution to the sum of the values of this function taken over our irreducible map is made by hexagons of the type $(55r55r)$, each of which contributes a minus one. This type of hexagon contributes plus two toward $y_2$. Therefore from

$$\begin{aligned} & 3a_6 + 2a_7 + a_8 \\ = {} & 24 + 3p_1 + p_2 + \sum_9 (n-9)\, a_n + \sum (\varrho_r(i_1 \cdots i_r) + y_2(i_1 \cdots i_r))\, a(i_1 \cdots i_r) \end{aligned} \tag{8}$$

we conclude that

THEOREM IV. *In any irreducible map there exists a set of at least eight polygons of either six, seven or eight sides, of such types, and, with respect to each type, in such numbers as to satisfy the relation* (8).

An examination of matrix $M_r$ shows that the only type of octagon which does not contribute at least one to the last term of (8) is the type (55555$r$5$r$). Therefore our theorem implies the existence of either hexagons, heptagons, or octagons *of this particular type.* Similarly it would be possible to state more or less involved corollaries of our earlier theorems.

6. **Conclusion.** It is apparent, I believe, that the four theorems enunciated in this paper are merely specimen implications of the matrices which have been introduced. Whether we shall use one or another of the functions $\varphi_5, \varphi_6, \cdots, \varrho_r$, or other similarly constructed functions depends upon the immediate object which we have in view. Here our immediate purpose has been the developement of existence proofs; with the more remote goal of devising approaches to the four color problem by which one might evade the often attempted reduction of at least one pentagon of any irreducible map. Theorem IV, for example, implies that if it were possible to reduce at least one region of either six, seven or eight sides, then the four color problem would be solved.

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# DIVISION ALGEBRAS OF ORDER SIXTEEN.

By Raymond Garver.

Section 1 of this paper is a summary of certain results obtained in Dickson's *New Division Algebras*, Transactions of the American Mathematical Society, vol. 28, 207–234. The methods used in section 8, which are described only briefly in this paper, are extensions of those used by Cecioni in his *Sopra un tipo di algebre prive di divisori dello zero* in the Rendiconti del Circolo Matematico di Palermo, vol. 47, 209–254.

1. **Introduction.** Consider a quartic equation irreducible in the field of rational numbers and having the Galois group $G_4 = 1, (12)(34), (13)(24), (14)(23)$. Since the group is Abelian, the equation is Abelian, three of its roots, $\theta_1, \theta_2, \theta_3$, are expressible rationally in terms of the fourth, $i$, and the following relations hold between the roots:

$$
\begin{aligned}
&\theta_1[\theta_1(i)] = \theta_2[\theta_2(i)] = \theta_3[\theta_3(i)] = i,\\
&\theta_1[\theta_2(i)] = \theta_2[\theta_1(i)] = \theta_3(i),\\
&\theta_1[\theta_3(i)] = \theta_3[\theta_1(i)] = \theta_2(i),\\
&\theta_2[\theta_3(i)] = \theta_3[\theta_2(i)] = \theta_1(i).
\end{aligned} \tag{1}
$$

Then if we define multiplication by

$$(a+bj_1)(c+dj_1) = ac+bd(\theta_1)g+[ad+bc(\theta_1)]j_1, \tag{2}$$

where $a$, $b$, $c$, $d$ and $g$ are in the field $R(i)$, the elements $a+bj_1$ form an algebra $\Sigma$ of order 8 over $R$, the field of rational numbers. We may take as its 8 basal units $1, i, i^2, i^3, j_1, ij_1, i^2j_1$, and $i^3j_1$, where $j_1^2 = g$. (In (2) take $a = c = 0$, $b = d = 1$.)

This algebra $\Sigma$ is an associative division algebra if and only if the following two conditions are satisfied:

$A_1$ $\qquad g = g(\theta_1)$,

$N_1$ $\qquad g \neq z(\theta_1)z$, for any $z$ in $R(i)$.

If we now consider elements $A_0+A_1j_2$, $B_0+B_1j_2$, where $A_0$, $A_1$, $B_0$, and $B_1$ are in $\Sigma$, and if we define multiplication of these elements by

$$(A_0+A_1j_2)(B_0+B_1j_2) = A_0B_0+A_1B_1'\gamma+(A_0B_1+A_1B_0')j_2, \tag{3}$$

($\gamma$ in $R(i)$, $A'$ defined to be $a(\theta_2)+b(\theta_2)\alpha j_1$, with $\alpha$ in $R(i)$, assuming $A = a+bj_1$) then the elements $A_0+A_1j_2$ form an algebra $\Gamma$ of order 2 over $\Sigma$, or of order 16 over $R$. Considered as an algebra of order 16, its basal units may be taken to be $i^r j_s$ ($r, s = 0, 1, 2, 3$), where $j_0$ is 1, and $j_3 = j_1 j_2$. As special cases of (3) we obtain $j_2^2 = \gamma$, $j_2 j_1 = \alpha j_3$.

This algebra $\Gamma$ is an associative division algebra, provided that $\Sigma$ is one, if and only if

$$A_2 \qquad \gamma = \gamma(\theta_2),$$

$$A_3 \qquad \alpha\alpha(\theta_1)g = g(\theta_2),$$

$$A_4 \qquad \alpha\alpha(\theta_2)\gamma(\theta_1) = \gamma,$$

$$N_2 \qquad \gamma \neq X'X, \text{ for any } X \text{ in } \Sigma.$$

We thus have six conditions, four of which we shall refer to as associativity conditions (since they determine the associativity of $\Sigma$ and $\Gamma$), and the other two as norm conditions, though $N_2$ obviously involves a generalized norm. We wish to show that by proper choice of a quartic equation and the polynomials in $i$ which we have called $g$, $\gamma$, $\alpha$ we can actually satisfy these six conditions and thus define an extensive class of division algebras of order 16.

**2. The quartic equation.** It has been shown that every quartic with rational coefficients, irreducible in the field of rational numbers and having the Galois group $G_4$, can be transformed by a rational Tschirnhaus transformation into one of the form $x^4+px^2+n^2 = 0$, where $p$ and $n$ are rational.* We therefore choose our quartic in this form. The roots may be written

$$(4)\quad i,\ \theta_1 = -i,\quad \theta_2 = -\frac{n}{i} = \frac{i^3+pi}{n},\quad \theta_3 = \frac{n}{i} = \frac{-i^3-pi}{n},$$

where we may take $n$ as the positive square root of $n^2$. These roots obviously satisfy the relations (1), We also note the following arithmetic relations, which are useful in the later work:

$$(5)\qquad \begin{aligned} i\theta_2 &= -n,\\ \theta_2^2 &= -i^2-p \text{ (this is } \textit{not}\ \theta_2[\theta_2(i)], \text{ which is } i),\\ \theta_2^3 &= -\frac{p}{n}i^3+\frac{(n^2-p^2)}{n}i. \end{aligned}$$

The quartic $x^4+px^2+n^2 = 0$ is irreducible in the field of rational numbers provided none of $p^2-4n^2$, $2n-p$, $-2n-p$, is a perfect square.

* I have obtained this result directly with a quadratic transformation. See also Wiman, Arkiv Mat. Fis., vol. 3, no. 28.

Hence we consider $p$ and $n$ chosen so as to satisfy this condition. We now proceed to the consideration of the other six conditions.

3. **The first two associativity conditions.** It is easily seen that $A_1$ and $A_2$ can be satisfied by taking $g$ and $\gamma$ in the form

$$g = c_1(i^2+1) \quad (c_1 \text{ rational}), \tag{6}$$

$$\gamma = c_2(\theta_2+i)+c_3 \quad (c_2 \text{ and } c_3 \text{ rational}). \tag{7}$$

In fact as we have chosen $\gamma$, it is the most general polynomial in $i$ (with rational coefficients) which satisfies $A_2$, while in the case of $g$ we have made an additional restriction in taking the coefficients of $i^2$ and 1 equal. Some step of this kind seems to be necessary to the carrying through of the later work.

4. **The third associativity condition.** If we attempt to work directly from $A_3$ and $A_4$ the work becomes involved. This suggests that, if possible, we choose $\alpha$ so that at least one simpler condition can be derived from $A_3$ and $A_4$. A convenient choice is to set $\alpha = \dfrac{x}{g\gamma(\theta_1)}$, where $x$ is in $R(i)$ but otherwise as yet undetermined. Making this substitution, $A_3$ and $A_4$ become

$$xx(\theta_1) = gg(\theta_2)\gamma\gamma(\theta_1), \tag{8}$$

$$xx(\theta_2) = gg(\theta_2)\gamma\gamma(\theta_1). \tag{9}$$

We then obviously have the condition

$$x(\theta_1) = x(\theta_2), \tag{$A_3'$}$$

and may replace $A_3$ and $A_4$ by the equivalent pair of conditions $A_3'$ and (9) or $A_4'$, where $\alpha$ is then determined as above. We satisfy $A_3'$ by choosing $x$ in the form

$$x = d(\theta_2 - i). \tag{10}$$

The most general $x$ satisfying $A_3'$ would differ from this by an additive constant, but we make this additional restriction for the same reason as in section 3.

5. **Two types of diophantine equations.** We need to know, for our later work, conditions under which certain equations are solvable or not solvable in rational numbers. The equations we shall meet are of two types:

$$ax^2+by^2+cz^2+du^2 = 0, \tag{11}$$

$$ax^2+by^2+cz^2 = 0, \tag{12}$$

Concerning the first, we quote from Dickson, History of the Theory of Numbers, vol. II, 432–433.

"A. Meyer gave criteria for the solvability in integers of his

$$ax^2+by^2+cz^2+du^2 = 0, \tag{7}$$

where $a$, $b$, $c$, $d$ are integers not zero, without square factors, and such that no three have a common factor.

"Write $(a, b)$ for the positive g. c. d. of $a$, $b$, and set

$$a = (a, b)\,(a, c)\,(a, d)\,\alpha, \quad b = (b, a)\,(b, c)\,(b, d)\,\beta,$$
$$c = (c, a)\,(c, b)\,(c, d)\,\gamma, \quad d = (d, a)\,(d, b)\,(d, c)\,\delta.$$

Then necessary conditions for the solvability of (7) in integers not all zero are I. $a, b, c, d$ are not all of the same sign, II. $-(a, c)\,(a, d)\,(b, c)\,(b, d)\,\gamma\delta$ is a quadratic residue of $(a, b)$ with five similar conditions derived by permuting the letters."

If the equation is of the particular form

$$ax^2+ay^2+z^2+u^2 = 0, \tag{13}$$

where $a$ is different from zero and without square factors, the necessary conditions become I. $a<0$, II. $-1$ a quadratic residue of $a$. Hence, if $-1$ is *not* a quadratic residue of $a$, (13) is not solvable in integers not all zero, and hence not in rational numbers. If $a$ contains a square factor, we may incorporate it in the $x^2$ and $y^2$ terms, and proceed similarly.

In discussing equations of type (12) we use conditions proved by Gauss. (See Dickson, ibid., 422–423.)

"If no two of $a$, $b$, $c$ have a common factor and if each is neither zero nor divisible by a square then $ax^2+by^2+cz^2 = 0$ has integral solutions not all zero *if and only if* $-bc$, $-ac$, and $-ab$ are quadratic residues of $a$, $b$, $c$ respectively, and $a$, $b$, $c$ are not of the same sign.

"If $a$, $b$, $c$ are arbitrary integers, let $\alpha^2$, $\beta^2$, $\gamma^2$ be the largest squares dividing $bc$, $ac$, $ab$ respectively, and set

$$\alpha a = \beta\gamma A, \quad \beta b = \alpha\gamma B, \quad \gamma c = \alpha\beta C.$$

Then the former equation is solvable if and only if $AX^2+BY^2+CZ^2 = 0$ is solvable, and the latter falls under the above theorem, since $A$, $B$, $C$ are relatively prime in pairs and have no square factors. For $bc/\alpha^2 = BC$ is an integer without square factors, so that $B$ and $C$ are relatively prime and without square factors."

If the equation (12) is of the particular form

$$x^2+by^2+cz^2=0, \tag{14}$$

and if $b$ and $c$ are relatively prime, and if neither is zero nor divisible by a square, the necessary and sufficient conditions become

(15) I. $b$ and $c$ not both positive,

II. $-b$ a quadratic residue of $c$, $-c$ a quadratic residue of $b$.

If $b$ and $c$ are arbitrary, proceed again as above. Now $\alpha^2$, $\beta^2$, $\gamma^2$ are the largest squares dividing $bc$, $c$, $b$ respectively. Then $b=b_1f\gamma^2$, $c=c_1f\beta^2$, where $b_1$ and $c_1$ are relatively prime, and $\alpha^2=\beta^2\gamma^2f^2$. We then have $A=\alpha a/\beta\gamma=f$, $B=\beta b/\alpha\gamma=b_1$, $C=\gamma c/\alpha\beta=c_1$.

We may then apply the conditions to

$$fX^2+b_1Y^2+c_1Z^2=0. \tag{16}$$

6. **The fourth associativity condition.** If we substitute in $A_4'$ or (9) from (6), (7), and (10), and reduce by means of (5), we arrive at

$$-d^2(2n-p)=c_1^2(n^2+1-p)[c_3^2+c_2^2(2n+p)]. \tag{17}$$

If we put $d=kc_1$, and multiply through by $(n^2+1-p)$ we have an equation of form (14):

$$X^2+(2n+p)Y^2+(2n-p)(n^2+1-p)Z^2=0. \tag{18}$$

For the case $n=1$, which we discuss in section 9, $(2-p)$ is a factor in (17). Hence for $p\neq 2$ the equation becomes

$$c_3^2+k^2=-(p+2)c_2^2. \tag{19}$$

Since we wish conditions that insure the satisfaction of $A_4'$ and hence (18) or (19), we apply section 5. This covers the case when $p$ and $n$ are integral. Restricting our discussion to this case involves no loss in generality, since if our original quartic does not have $p$ and $n$ integral, we can readily transform it into one which does. The application of section 5 to the present case is immediate; we have merely to read the last two paragraphs with the proper changes in notation.

For the special case $n=1$, conditions for the solvability of (19) in integers, and hence in rational numbers, are known to be

(20) I. $p<-2$,

II. $(p+2)$, apart from square factors and the possible factor 2, is a product of odd primes of the form $4n+1$.

In this connection, see Dickson, History of the Theory of numbers, vol. II, viii.

7. **The first norm condition.** We now seek to restrict $p$ and $n$ so that $g$ of form (6) shall not be equal to $z(\theta_1)z$, for any $z$ in $R(i)$. The most general $z$ is then of the form $z = a+bi+ci^2+di^3$, where $a$, $b$, $c$, $d$ are rational. We then have $z(\theta_1) = a-bi+ci^2-di^3$, and their product reduces to

$$(21)\quad (a^2-c^2n^2+2bdn^2-d^2pn^2)+i^2(2ac-c^2p-b^2+d^2n^2+bdp-d^2p^2).$$

If (21) is to equal $c_1(i^2+1)$, we must obviously have the coefficient of $i^2$ in (21) equal to term free of $i$. If we set up this equation, and make certain transformations, we find that it can have a rational, non-zero solution in $a$, $b$, $c$, $d$ only in case the equation

$$(22)\qquad (p-1-n^2)x^2+(p-1-n^2)y^2+z^2+u^2 = 0$$

has a rational solution in $x$, $y$, $z$, $u$ not all zero.

But (22) is of type (13), and has no integral, and hence no rational, solution other than the zero solution provided $-1$ is not a quadratic residue of $(p-1-n^2)$, provided this has no square factors, or provided $-1$ is not a quadratic residue of $(p-1-n^2)$ with square factors divided out, in case it contain such factors. Hence the first norm condition will certainly be satisfied if we take $(p-1-n^2)$ to be a prime of the form $4n+3$.

For the special case $n = 1$ this amounts to taking $(p-2)$ a prime of the form $4n+3$. Indeed it is sufficient if $(p-2)$ has a single prime factor of this form, to an odd power.

8. **The second norm condition.** I shall give only a brief outline of the methods used in seeking further restrictions on $p$ and $n$ which are sufficient to insure that $\gamma \neq X'X$, for any $X$ in $\Sigma$. If $X = a+bj_1$ where $a$ and $b$ are polynomials in $i$, $X'$ is, by definition, $a(\theta_2)+b(\theta_2)\alpha j_1$, and $X'X$ is equal to

$$(23)\qquad a\,a(\theta_2)+b(\theta_1)b(\theta_2)\alpha g+[a(\theta_2)b+a(\theta_1)b(\theta_2)\alpha]j_1.$$

We first consider the case $b = 0$, that is where $X$ itself is a polynomial in $i$ with rational coefficients. If $X$ is of this form, $X'$ is the same polynomial with $i$ replaced by $\theta_2$. We can thus form $X'X$, and reduce it to a linear combination of $(\theta_2+i)$ and 1. If this were to equal a $\gamma$ of form (7), we then have 2 equations to satisfy.

If we make a number of transformations, and use the fact that $c_2$ and $c_3$ are supposed to be subject to the additional conditions necessary to the satisfying of the fourth associativity condition, (that is, they must satisfy

the equation (17), which is possible under the restrictions on $p$ and $n$ given in section 6) we find that the two equations cannot be satisfied rationally provided the equation

$$x^2+(p-2n)y^2+(p-1-n^2)z^2 = 0 \tag{24}$$

has no rational solution other than the zero solution. This equation is of form (14), and the discussion following that equation in section 5 applies with the necessary changes in notation.

For the case $n = 1$, the two coefficients in (24) become equal, and we may write it

$$(p-2)X^2+Y^2+Z^2 = 0. \tag{25}$$

This equation will obviously not be satisfied if $p$ is subject to the restriction stated at the end of section 7.

It might be well to add that there are three special cases which do not lead to (24), but they can be handled without introducing additional conditions.

The next case to be treated is when $a = 0$, that is, $X$ is of the form $bj_1$, with $b$ in $R(i)$. This leads to another equation of form (14) (except again for special cases, which must be treated separately):

$$x^2+(p+2n)y^2-(p-2n)z^2 = 0. \tag{26}$$

Again we wish conditions which insure that (26) has no integral, and hence no rational, non-zero solution in $x$, $y$, $z$, and section 5 applies as before.

For the case $n = 1$, section 5 says that (26) will have no solution provided

(27) $-(p+2)$ is not a quadratic residue of $(2-p)$. In fact it is clearly sufficient for $(2-p)$ to have one odd, prime factor, to an odd power, of which this is true, Or

$(p-2)$ is not a quadratic residue of $-(p+2)$, with a similar additional statement.

These statements imply that the coefficients in (26) are relatively prime and lack square factors. Otherwise we should state them somewhat differently.

Finally, the general case, where neither $a$ nor $b$ is zero, can be shown to involve no further restrictions.

9. **The Case $n = 1$.** We shall show in conclusion that the various conditions can be satisfied easily by taking $n = 1$. We can satisfy the following conditions; which form a sufficient set:

1) $p^2-4$, $2-p$, $-2-p$, are none of them perfect squares, (section 1),
2) $g$, $\gamma$, $\alpha$ must satisfy the four associativity conditions. Thus $p$ must satisfy the conditions of section 6, (20). We may take $p<-2$, and $(p+2)$ a product of odd primes of the form $4n+1$,
3) $p-2$ contains a prime factor, to an odd power, of the form $4n+3$, (section 7, also section 8, first part),
4) $p-2$ contains a prime factor, to an odd power, with respect to which $-(p+2)$ is not a quadratic residue. (This is one of the alternate restrictions of (27), section 8.)

Values of $p$ which satisfy these conditions are $-19$, $-31$, $-55$, $-67$, $-75$. Conditions 1 and 2 are satisfied. The factor of condition 3 is, in the respective cases, 3 or 7, 3 or 11, 3 or 19, 3 or 23, and 7 or 11. The factor of condition 4 may be taken as 3 in every case but the last, in which we take 7. For these values of $p$, and for values of the constants of multiplication satisfying the associativity conditions, we have thus set up division algebras of order 16.

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# NOTE ON A THEOREM CONCERNING CONTINUOUS CURVES.*

By W. L. Ayres.

A point set which is closed, connected and connected im kleinen is said to be a continuous curve. This is sometimes called a generalized continuous curve. In a recent paper†, we proved this result:

Theorem A. *If $M$ and $N$ are bounded plane continuous curves and $N$ is a subset of $M$, then, for any positive number $\varepsilon$, $M-N$ contains only a finite number of maximal connected subsets of diameter greater than $\varepsilon$.*

It is the purpose of this note to indicate how the proof of Theorem A may be modified so as to prove the following generalization. We have found this generalization useful in a number of recent applications.

Theorem B. *Theorem* A *remains true if the condition that $M$ be bounded is removed.*

Suppose the collection $[d]$ of maximal connected subsets of $M-N$ of diameter greater than $\varepsilon$ is infinite. If $d$ is any set of $[d]$, $d$ contains a point $A_d$, such that $A_d$ is within a distance $\varepsilon$ of some point of $N$, and $d$ contains a point $B_d$, such that the distance from $A_d$ to $B_d$ is greater than $\frac{1}{2}\varepsilon$ and less than $\varepsilon$. Since $N$ is bounded and each point of the sets $[A_d]$ $[B_d]$ lies within a distance $2\varepsilon$ of some point of $N$, there exist two points $\bar{A}$ and $\bar{B}$ and two sequences $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ such that (1) the points $A_1, A_2, A_3, \cdots$ are distinct points of the set $[A_d]$ and $\bar{A}$ is the sequential limit point of this sequence, (2) the points $B_1, B_2, B_3, \cdots$ are distinct points of the set $[B_d]$ and $\bar{B}$ is the sequential limit point of this sequence, (3) for every $i$, the points $A_i$ and $B_i$ belong to the same set of the collection $[d]$.

Evidently $d(\bar{A}, \bar{B}) \geqq \frac{1}{2}\varepsilon$. There exists an arc $A_i B_i$ which lies entirely in the set of $[d]$ containing $A_i$ and $B_i$. Let $C_1$, $C_2$, $C^{\mp}$ be circles with center at $\bar{A}$ and radii $\frac{1}{8}\varepsilon$, $\frac{3}{8}\varepsilon$ and $\frac{1}{4}\varepsilon$ respectively. There exists an integer $k$ such that if $i > k$, $A_i$ lies within $C_1$ and $B_i$ lies without $C_2$. The points $X_i$, $Y_i$, $\bar{X}_i$, $\bar{Y}_i$ are selected as in Theorem A except that the statement "and every other point of $\bar{X}_i \bar{Y}_i$ lies outside both circles" is replaced by

* Presented to the American Mathematical Society, February 26, 1927; received March 21, 1927.

† Concerning continuous curves and correspondences, Annals of Math., vol. 28 (1927), pp. 396–418.

"and every other point of $\bar{X}_i \bar{Y}_i$ lies between the two circles". Each arc $\bar{X}_i \bar{Y}_i$ contains a point $Z_i$ on the circle $C^{\mp}$. From this point on, the proof follows exactly that of Theorem $A$ except that every number of the form $r\varepsilon$ is replaced by $\frac{1}{2}r\varepsilon$,

The theorem no longer holds true if the condition that $N$ is bounded is removed, as may be seen by the following example. Let $N$ be the line $y = 0$. Let $M$ consist of the line $y = 0$ and the lines $x = i$ ($i = 0, \pm 1, \pm 2, \cdots$). Then $M$ and $N$ are continuous curves but $M - N$ contains an infinite number of maximal connected subsets of diameter greater than $\varepsilon$, if $\varepsilon$ is any positive number whatever.

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# DEVELOPMENT OF FUNCTIONS IN A SYSTEM OF APPROXIMATELY ORTHOGONAL FUNCTIONS.*

By W. D. Cairns.

Numerous writers† have proved that the terms of the expansion $(1+1)^{2n}$, used as equally spaced ordinates of a "polygon", approximate to the normal curve $y = ke^{-h^2x^2}$ for large values of $n$. In the author's treatment (except for a change in the constant factor for the sake of present convenience) the $k$th term of $(n/2)^{1/2}(1+1)^{2n}/2^{2n}$ was assigned to the position whose abscissa $x$ is $k\Delta x$; $\Delta x$, the interval between successive ordinates, was so chosen that the term for which the ordinates of the polygon decrease most rapidly takes the position whose abscissa $\sigma$ is $(n/2)^{1/2}\Delta x$. ($\sigma$ is here chosen as unity in order to simplify the presentation). Under these conditions the polygon was shown to approach uniformly the curve $y = e^{-x^2/2}/(2\pi)^{1/2}$.

## The system of functions.

We choose as our first function the positive square root of this $k$th term and denote it by $F(k, n)$, so that

$$[F(k,n)]^2 = \frac{(n/2)^{1/2}}{2^{2n}}\binom{2n}{n+k}. \tag{1}$$

The first $k$-difference of this $k$th term is

$$-\frac{(n/2)^{1/2}(2k+1)}{2^{2n}(n+k+1)}\binom{2n}{n+k};$$

the gradient of the polygon is this difference divided by the length of the interval, $\Delta x$. Since $k$ is ordinarily a large number but small compared with $n$, we base the second function upon

$$\frac{(n/2)^{1/2}\cdot 2k}{2^{2n}n\Delta x}\binom{2n}{n+k} = \frac{k}{(n/2)^{1/2}}[F(k,n)]^2,$$

and we choose as our second function $p_1(k) = k/(n/2)^{1/2}\cdot F(k,n)$. Similar functions obtained from the second, third, $\cdots$ differences, by disregarding the preceding signs, are successively

* Read before the American Mathematical Society, April 2, 1927; received April 9, 1927.

† See, in effect, Czuber, Wahrscheinlichkeitsrechnung, 2. Aufl., 1908, vol. 1, p. 116; Poincaré, Calcul des prob., 1912, p. 88; see also A. A. Bennett, Bull. Amer. Math. Soc., vol. 26 (1920), p. 477; W. D. Cairns, ibidem, vol. 26 (1919), pp. 105–108.

$$p_2(k) = \frac{k^2 - n/2}{n/2} F(k, n), \qquad p_3(k) = \frac{k^3 - 3k \cdot n/2}{(n/2)^{3/2}} F(k, n), \cdots,$$

and for completeness we add $p_0(k) = F(k, n)$.

The coefficients of $F(k,n)$ in these functions are the well-known Hermitian* polynomials $H_m(x)$, being here functions of $x$, where $x$ represents $k \cdot (n/2)^{-1/2}$ except when it is necessary to express these in terms of $k$ and $n$ in making certain summations; thus,

$$p_m(x) = H_m(x)\ F(k, n). \tag{2}$$

THEOREM: *The set of functions $p_m(x)$ is orthogonal to within an error involving terms in $n^{-1}$, that is,*

$$\begin{aligned} \sum_{k=-n}^{n} [p_m(k)]^2 \Delta x &= m! - \frac{m(m-1)m!}{4n} + \frac{\varphi(m)}{n^2} - \cdots \\ \sum_{k=-n}^{n} p_m(k)\, p_l(k) \Delta x &= 0 - \frac{\varphi_1(l, m)}{n} + \frac{\varphi_2(l, m)}{n^2} - \cdots (m \neq l) \end{aligned} \tag{3}$$

*where $\varphi(m)$, $\varphi_1(l, m)$, $\varphi_2(l, m)$, $\cdots$ are polynomials in $m$ or $l$ and $m$ except for a single factorial multiplier.*

We have

$$H_m(x) = x^m - \frac{m(m-1)}{2} x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2 \cdot 4} x^{m-4} - \cdots.$$

Whether $m$ is odd $(2i+1)$ or even $(2i)$, this will be a polynomial of $i+1$ terms, as is seen by writing the general term.

1. To prove the first orthogonal property, we square this polynomial, multiply the result by $[F(k, n)]^2 \Delta x$, and sum for $k$. If this be done *in extenso* as an array with $i+1$ rows and colums, the $(j+1)$th column will be

$$\begin{aligned} \sum_k (-1)^j \frac{m!/(m-2j)!}{2 \cdot 4 \cdots 2j \cdot 2^{2n}} \Big[ & x^{2m-2j} - \frac{m(m-1)}{2} x^{2m-2j-2} \\ & + \frac{m(m-1)(m-2)(m-3)}{2 \cdot 4} x^{2m-2j-4} - \cdots \Big] \binom{2n}{n+k}. \end{aligned} \tag{4}$$

The author has previously† investigated summations of the sort here involved, and (using the present notation) has proved that the leading term in $\sum_k \binom{2n}{n+k} x^{2p}$ is $1 \cdot 3 \cdot 5 \cdots (2p-1) \cdot 2^{2n}$. The leading term in this $(j+1)$th column is therefore, after slight modification,

* Hermite, Œuvres, vol. 2, p. 301.

† Bull. Amer. Math. Soc., vol. 26 (1920), pp. 160–164.

$$(5)\qquad (-1)^j \frac{m!/(m-2j)!}{2\cdot 4\cdots 2j}\, 1\cdot 3\cdot 5\cdots(2m-2j-1)\left[1-\frac{m(m-1)}{2(2m-2j-1)}+\frac{m(m-1)(m-2)(m-3)}{2\cdot 4(2m-2j-1)(2m-2j-3)}-\cdots\right].$$

It can be proved that the expression in brackets vanishes for all values of $j$ from 1 to the given value $i$ and equals

$$\frac{m!\,2^m}{2m(2m-1)\cdots(m+1)}$$

for $j=0$. This results from the fact that the $k$th derivative of the expression $(x^2-1)^n$, which after development of the binomial may be found to be

$$(6)\qquad 2n(2n-1)\cdots(2n-k+1)\left[x^{2n-k}-\frac{(2n-k)(2n-k-1)}{2(2n-1)}x^{2n-k-2}+\frac{(2n-k)(2n-k-1)(2n-k-2)(2n-k-3)}{2\cdot 4(2n-1)(2n-3)}x^{2n-k-4}-\cdots\right]$$

has $(x^2-1)^{n-k}$ as a factor and hence vanishes for $x=1$ if $k=0,1,2,\cdots\cdots,n-1$: while the $n$th derivative for $x=1$ is $n!\,2^n$, being $n!$ times the coefficient of $(x-1)^n$ in the development of $(x^2-1)^n\equiv(x-1)^n[(x-1)+2]^n$ in terms of $x-1$. The substitution of $2n-k=m$, $i-k=j$, $n=i$, and $x=1$ gives our desired result, for while $k$ assumes the values $0,1,\cdots,n$, $j$ assumes successively the values $i, i-1,\cdots,1,0$, whether $m$ is even or odd. Insertion of these values in (5) gives at once the result that the leading term in the first column is $m!$ and in all the others is zero.

An extension of the article last mentioned shows us that the second term in $\sum\binom{2n}{n+k}x^{2p}$ is $-1\cdot 3\cdot 5\cdots(2p-1)\,\frac{p(p-1)}{6}\,\frac{2^{2n}}{n}$ and that the further terms are of higher powers of $1/n$. The second term in the $(j+1)$th column of (4) is therefore

$$(7)\qquad -(-1)^j\frac{m!/(m-2j)!}{2\cdot 4\cdots 2j}\,1\cdot 3\cdots(2m-2j-1)\,\frac{(m-j)(m-j-1)}{6n}\times\left[1-\frac{m(m-1)}{2}\,\frac{m-j-2}{(2m-2j-1)(m-j)}+\frac{m(m-1)(m-2)(m-2)}{2\cdot 4}\times\frac{(m-j-2)(m-j-3)}{(2m-2j-1)(2m-2j-3)(m-j)(m-j-1)}-\cdots\right].$$

The evaluation of the expression in brackets follows the same line as before, except that we use $x^4(x^2-1)^{n-2}$ instead of $(x^2-1)^n$. The algebraic work is complicated but entirely straightforward. The $k$th derivative

$$2n(2n-1)\cdots(2n-k+1)\left[x^{2n-k}-\frac{(2n-k)(2n-k-1)}{2}\right. \\ \left.\times\frac{n-2}{(2n-1)n}x^{2n-k-2}+\cdots\right] \tag{8}$$

has $(x^2-1)^{n-k-2}$ as a factor and hence vanishes when $x=1$ so long as $k<n-2$, as is seen by the fact that the terms of degree $0, 1, \cdots, n-3$ are lacking in the development of $x^4(x^2-1)^n$ in terms of $x-1$; for $k=n-2$, $n-1$, and $n$, this derivative has in turn the value $(n-2)!\,2^{n-2}$, $(n-1)!(n+6)2^{n-3}$ and $n!(n^2+11n+22)2^{n-5}$. Using the substitutions $2n-k=m$, $n-k=j$, and $n$ in succession equal to $m-2$, $m-1$ and $m$, it is found that the third, second and first columns ($j=2, 1, 0$) have as values for the expression in brackets in (7)

$$\frac{m!(m-4)!\,2^{m-4}}{(2m-4)!}, \quad \frac{m!(m-2)!(m+5)2^{m-4}}{(2m-2)!}, \quad \frac{(m!)^2(m^2+11m+22)2^{m-3}}{(2m)!}$$

all the other columns vanishing. Insertion of these values in (7) gives as the value of our desired second term

$$-\frac{m(m-1)m!}{4n}.$$

The further terms of the original summation are polynomials in $m$ divided by $n^2$ or higher powers of $n$.

The simplicity of the result obtained would tend to indicate the possibility of a simpler proof, but no such procedure is as yet evident.

2. To prove the second orthogonal property, we multiply $H_m(x)$ by $H_l(x)$, multiply by $[F(k,n)]^2\Delta x$ and sum for values of $k$ ranging from $-n$ to $n$. This gives, when fully written, an array with $i+1$ columns and $i'+1$ rows, where $m$ is $2i+1$ or $2i$, and $l$ is $2i'+1$ or $2i'$. If $m+l$ is odd, the whole sum vanishes, being an odd function of $k$; hence we consider only the case where $m+l$ is even and, for the sake of a slight advantage, we let $m$ be the smaller integer. The $(j+1)$th column will be

$$\sum(-1)^j\frac{m!/(m-2j)!}{2\cdot4\cdots2j\cdot2^{2n}}\left[x^{m+l-2j}-\frac{l(l-1)}{2}x^{m+l-2j-2}\right. \\ \left.+\frac{l(l-1)(l-2)(l-3)}{2\cdot4}x^{m+l-2j-4}-\cdots\right]\binom{2n}{n+k}. \tag{9}$$

The leading term in this expression is

$$(10)\qquad (-1)^j \frac{m!/(m-2j)!}{2\cdot 4\cdots 2j}\,1\cdot 3\cdots(m+l-2j-1)\Big[1-\frac{l(l-1)}{2(m+l-2j-1)} + \frac{l(l-1)(l-2)(l-3)}{2\cdot 4(m+l-2j-1)(m+l-2j-3)}-\cdots\Big].$$

Now in (6) of the first part of the proof we use the substitution $2n-k=l$, $m-k=2j$, whence $2n=m+l-2j$, and obtain for the $k$th derivative

$$\frac{(m+l-2j)!}{l!}\Big[x^l-\frac{l(l-1)}{2(m+l-2j-1)}x^{l-2}+\cdots\Big].$$

By the argument there used, this vanishes when $x=1$ for $k=0, 1, \cdots \cdots, (n-1)$, and equals $n!2^n$ for $k=n$. With the changed letters this means that it vanishes for $j=0, 1, \cdots, i$ so long as $k$ runs through the values $n-1, n-2, \cdots, 1, 0$; that is, so long as $m$ runs through the values $l+2j-2, l+2j-4, \cdots, 2j$. This is just what is needed in order to make the leading term in the first, second, $\cdots$, $(j+1)$th columns vanish, for the coefficient of the $(j+1)$th bracket in (9) vanishes for $m=2j-1$, $2j-2, \cdots, 1, 0$; hence the leading term in the $(j+1)$th column vanishes for $m=0, 1, \cdots, l+2j-2$. Hence when $x=1$ it vanishes for $m=0$, $1, \cdots, l+2j$ except that the first column ($j=0$) does not do so for $m=l$; in that event, the expression in brackets becomes $m!$ when $x=1$, as proved in the first part.

The second term in (9) is similarly

$$(11)\qquad -(-1)^j \frac{m!/(m-2j)!}{2\cdot 4\cdots 2j}\,1\cdot 3\cdots(m+l-2j-1)\,\frac{(m+l-2j)(m+l-2j-2)}{24n} \times\Big[1-\frac{l(l-1)}{2}\,\frac{m+l-2j-4}{(m+l-2j-1)(m+l-2j)}+\cdots\Big].$$

In the expression (8) for the $k$th derivative of $x^4(x^2-1)^{n-2}$, use the substitution $2n-k=l$, $m-k=2j$, whence $2n=m+l-2j$, and we obtain

$$(12)\quad (m+l-2j)!/l!\Big[x^l-\frac{l(l-1)}{2}\,\frac{m+l-2j-4}{(m+l-2j-1)(m+l-2j)}\,x^{l-2}+\cdots\Big].$$

Since by the previous argument (8) vanishes when $x=1$ for $k=n-3$, $n-4, \cdots, 1, 0$, it follows from our substitution that (12) vanishes when $x=1$ for $m<l+2j-4$, that is, for $m=l+2j-6, l+2j-8, \cdots, 2j$. Hence, recalling that $m$ is less than $l$, the second term involved in the first

column ($j = 0$) in (9) vanishes when $x = 1$ for $m = 0, 2, \cdots, l-6$; that involved in the second column for $m = 2, 4, \cdots, l-4$; that involved in the third column for $m = 4, 6, \cdots, l-2$; that involved in the $(j+1)$th column for all further values of $j$ for $m = 2j, 2j+2, \cdots$ up to $l$.

Special investigation of the non-vanishing cases shows that for $j = 0$, $m = l-4$, $x = 1$, the first column of (9) gives $-l!/(24n)$; for $j = 0$, $m = l-2$, $x = 1$, it gives $-l!(l-2)(l+5)/(48n)$; for $j = l$, $m = l-2$, $x = 1$, the second column of (9) gives $l!(l-2)(l-3)/(48n)$. If therefore the subscript $l$ exceeds $m$ by 4, the second term involved in (9) is $-l!/(24n)$; and if $l$ exceeds $m$ by 2, it is $-l!(l-2)/(6n)$. With these exceptions, the cross-summation of $H_m(x)$ with $H_l(x)$ gives no term in $n^{-1}$, but begins with terms not earlier than $n^{-2}$.

## Development of a given function.

It is desired to develop a given function $f(k)$, where $\sum[f(k)]^2 \Delta x$ is finite as $n \to \infty$, in the form

$$f(k) = a_0 p_0(k) + \cdots + a_m p_m(k) + \cdots, \tag{13}$$

or briefly

$$f = a_0 p_0 + \cdots + a_m p_m + \cdots, \tag{14}$$

the coefficients (independent of $k$) being so chosen that

$$\sum_{k=-n}^{n} [f - a_0 p_0 - \cdots - a_m p_m - \cdots]^2 \Delta x \tag{15}$$

shall be a minimum. The expansion of this summation and differentiation with regard to $a_0, \cdots, a_m, \cdots$ gives the necessary conditions

$$\begin{aligned} a_0 \sum p_0^2 \Delta x + \cdots + a_m \sum p_0 p_m \Delta x + \cdots &= \sum f p_0 \Delta x, \\ \cdots\cdots\cdots\

Since by the orthogonal properties of the $p$'s the summations in the left members approximate to $m!$ or to zero as $n \to \infty$, this suggests that we try as approximating values for the $a$'s the expressions

$$a_m = \frac{\sum f p_m \Delta x}{m!}.$$

With the introduction of these values for the $a$'s in (15), it becomes, on expansion,

$$\begin{aligned}\sum f^2 \Delta x - 2a_0^2 - \cdots - 2a_m^2 m! - \cdots \\ + a_0^2 \sum p_0^2 \Delta x + \cdots + a_m^2 \sum p_m^2 \Delta x + \cdots \\ + 2a_0 a_1 \sum p_0 p_1 \Delta x + \cdots + 2a_m a_l \sum p_m p_l \Delta x + \cdots \quad (m \neq l),\end{aligned}$$

and utilizing the leading and second terms in these summations, as evaluated in the first part of the paper, this becomes

$$\begin{aligned}\sum f^2 \Delta x - a_0^2 - \cdots - a_m^2 m! - \cdots - \frac{1}{n} \sum_{m=0}^{\infty} \Big[ \frac{a_m a_{m-4} m!}{12} \\ + \frac{a_m a_{m-2} (m-2) m!}{3} + \frac{a_m^2 m(m-1) m!}{4} \Big] + \cdots.\end{aligned} \tag{17}$$

Because (17) is the equivalent of (15), it follows that

$$a_0^2 + \cdots + a_m^2 m! + \cdots$$

made up from the development coefficients of $f(k)$ must have a finite value, that the expression in brackets in (17) must be finite, and that expression (15) must converge, as $n \to \infty$, to the same non-negative limit as $\sum f^2 \Delta x - a_0^2 - \cdots - a_m^2 m! - \cdots$. We may also remark that, because $\sum (f - p_m)^2 \Delta x \geqq 0$, we have

$$2 \sum f p_m \Delta x \leqq \sum f^2 \Delta x + \sum p_m^2 \Delta x,$$

and that therefore each $\sum f p_m \Delta x$ or $a_m m!$ is finite.

If it is true, as seems probable from analogous theorems in Hilbert's theory of integral equations, in the theory of approximations, etc., that we can approximate to $f(k)$ by the functions $p_0, p_1, \cdots$ by selecting a finite $m$ so that

$$\sum (f - c_0 p_0 - \cdots - c_m p_m)^2 \Delta x < \varepsilon,$$

it will follow that the limit mentioned in the preceding paragraph must be zero. For suppose it were not, that is, suppose that for $n$ sufficiently large

$$\sum f^2 \Delta x - a_0^2 - \cdots - a_m^2 m! - \cdots = \varepsilon > 0.$$

We see, however, that the left member of the second preceding inequality may be developed into

$$\begin{aligned}
&\sum f^2\,\Delta x - 2c_0 a_0 - \cdots - 2c_m a_m m! \\
&\quad + c_0^2 \sum p_0^2\,\Delta x + \cdots + c_m^2 \sum p_m^2\,\Delta x \\
&\quad + 2c_0 c_1 \sum p_0 p_1\,\Delta x + \cdots + 2c_{m-1} c_m \sum p_{m-1} p_m\,\Delta x \\
&= \varepsilon + a_0^2 + \cdots + a_m^2 m! + a_{m+1}^2 (m+1)! + \cdots \\
&\quad - 2c_0 a_0 - \cdots - 2c_m a_m m! \\
&\quad + c_0^2 + \cdots + c_m^2 \left[ m! - \frac{m(m-1)\,m!}{4n} + \cdots \right] \\
&\quad + 2c_1 c_0 \cdot 0 + \cdots + 2c_{m-1} c_m \left( \frac{a}{n} + \cdots \right) \\
&= \varepsilon + (a_0 - c_0)^2 + \cdots + (a_m - c_m)^2 m! + a_{m+1}^2 (m+1)! + \cdots \\
&\quad + g_1(m)/n + g_2(m)/n^2 + \cdots,
\end{aligned}$$

where $a$ and $g_1(m)$, $g_2(m)$, $\cdots$ are finite functions of $m$. Hence, we may take $n$ large enough so as to make

$$\sum (f - c_0 p_0 - \cdots - c_m p_m)^2\,\Delta x \geqq \varepsilon.$$

This establishes the desired contradiction and we conclude that the limit, for $n \to \infty$, of

$$\sum (f - a_0 p_0 - \cdots - a_m p_m - \cdots)^2\,\Delta x$$

and of

$$\sum f^2\,\Delta x - a_0^2 - \cdots - a_m^2 m! - \cdots$$

is zero. The only caution to be observed in this last proof, aside from the initial assumption, is that it is vitiated if $a_0 = c_0, \cdots, a_m = c_m$, $a_{m+1} = a_{m+2} = \cdots = 0$; but in the event that $a_{m+1} = a_{m+2} = \cdots = 0$, the truth of the statement is evident without proof.

The question now arises as to how closely $f(k)$ is represented by its development. Set $f - a_0 p_0 - \cdots - a_m p_m - \cdots = A + B/n + C/n^2 + \cdots$, where $A, B, C, \cdots$ depend on $k$ and not on $n$. If we multiply this equality by $p_m$ and sum as to $k$, we have

$$\begin{aligned}
(18) \qquad & a_m m! - a_m \sum p_0 p_m\,\Delta x - \cdots - a_m \sum p_m^2\,\Delta x - \cdots \\
&\qquad = \sum A p_m\,\Delta x + \sum B p_m\,\Delta x/n + \cdots.
\end{aligned}$$

Equating the coefficients of like powers of $n$, and using the orthogonal formulas already proved, including those for the terms in $n^{-1}$, we have for each value of $m$

$$(19)\quad \sum A p_m \Delta x = 0,$$

$$(20)\quad \begin{aligned}\sum B p_m \Delta x/m! = \frac{1}{24}[&a_{m-4} + 4(m-2)a_{m-2} + 6m(m-1)a_m \\ &+ 4(m+2)(m+1)a_{m+2} + (m+4)(m+3)(m+2)(m+1)a_{m+4}].\end{aligned}$$

From (19) it follows that $A = 0$ since (19) gives the development coefficients for $A$. The summation in (20) is precisely the development coefficient for $B$; hence this "error coefficient" $B$ may be expressed as a development in the $p$'s for which the coefficient of each $p_m$ is the right member of (20), giving a cumbersome but entirely definite expression for the error term, which shows also that the error is of order $n^{-1}$. The present paragraph outlines the analysis to be employed here, without going fully into the rigorous proof.

*Example* 1. Since for $f(k) \equiv x^m F(k, n)$, $\sum f^2 \Delta x$ is finite as $n \to \infty$, we can develop it in a form which we shall assume as

$$x^m F(k, n) = a_m p_m + a_{m-1} p_{m-1} + \cdots.$$

$a_{m-2k+1}$, involving the summation from $-n$ to $n$ of an odd function of $k$, is zero. By our theory,

$$a_{m-2k} = 1/(m-2k)! \cdot \sum x^m F(k, n) p_{m-2k} \Delta x.$$

By the author's earlier paper already referred to,

$$2^{-2n} \sum \binom{2n}{n+k} k^{2p}$$

$$= 1 \cdot 3 \cdots (2p-1)\left(\frac{n}{2}\right)^p - 1 \cdot 3 \cdots (2p-1)\frac{p(p-1)}{6n}\left(\frac{n}{2}\right)^p + \cdots,$$

hence

$$2^{-2n} \sum \binom{2n}{n+k} x^{2p} = 1 \cdot 3 \cdots (2p-1) - 1 \cdot 3 \cdots (2p-1)\frac{p(p-1)}{6n} + \cdots$$

and hence further

$$\sum x^m F(k, n) p_{m-2k} \Delta x \equiv 2^{-2n} \sum x^m \binom{2n}{n+k} H_{m-2}(x),$$

becomes by fuller development

$$1\cdot 3\cdots(2m-2k-1)\left[1-\frac{(m-2k)(m-2k-1)}{2(2m-2k-1)}+\cdots\right]$$
$$-1\cdot 3\cdots(2m-2k-1)\frac{(m-k)(m-k-1)}{6n}$$
$$\times\left[1-\frac{(m-2k)(m-2k-1)}{2(2m-2k-1)}\,\frac{m-k-2}{m-k}+\cdots\right].$$

An analysis like that connected with equation (6) in this paper, replacing $2n-k$ by $m-2k$ and $2n$ by $2m-2k$, gives the result that the first term of this expression vanishes for negative $k$, that is, for the coefficient of every power higher than $x^m$, and that it equals $\frac{m!}{2^k k!}$ for every other value of $k$. Hence it follows that

$$a_{m-2k} = \frac{m!/(m-2k)!}{2\cdot 4\cdots 2k}$$

whence

$$x^m F(k,n) = p_m + \frac{m(m-1)}{2}p_{m-2} + \frac{m(m-1)(m-2)(m-3)}{2\cdot 4}p_{m-4} + \cdots$$

to within terms in $n^{-1}$. In corroboration of this, it is seen by actual evaluation that $x^m F(k,n)$ equals this development exactly.

*Example* 2. Since for $f(k) \equiv \cos x F(k,n)$, $\sum f^2 \Delta x$ is finite as $n\to\infty$, as is seen from the fact that it approaches uniformly the convergent integral

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\cos^2 x\, e^{-x^2/2}\,dx,$$

we can develop it in terms of the $p$'s. It is found that $a_{2m+1} = 0$, and that $a_{2m}$ differs from $(-1)^m e^{-1/2}/(2m)!$ by a term in $n^{-1}$ for which the coefficient is finite; hence within this error at most

$$\cos x F(k,n) = e^{-1/2}[p_0 - p_2/2! + p_4/4! - \cdots].$$

As an evaluation will show, the function actually equals this series for every finite $n$. A similar result holds for $\sin x F(k,n)$. In further corroboration we may prove for this example that $\sum f^2\Delta x - a_0^2 - \cdots - a_m^2 m! - \cdots$ approaches zero when $n\to\infty$, for $a_0^2 + \cdots + a_m^2 m! + \cdots$ will approximate to

$$e^{-1}(1+1/2!+1/4!+\cdots) = \frac{1}{2}(e^{-2}+1),$$

which is the value of the above definite integral.

*Example* 3. Since for $f(k) \equiv \sin x/x$, $\sum f^2 \Delta x$ is finite as $n \to \infty$, as is seen by the convergent integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx,$$

this function can be developed in terms of the $p$'s, and for the general coefficient

$$a_{2m} = \frac{1}{(2m)!}\sum \frac{\sin x}{x} p_{2m}\, \Delta x = \frac{e^{-1/2}}{(2m)!}\sum (p_1 - p_3/3! + \cdots)\frac{p_{2m}}{x}\, \Delta x.$$

Now by formula (2) and by the repeated use of the recurrence formula for the Hermite polynomials, which may be written

$$H_{2l+1}/x = H_{2l} - 2l \cdot H_{2l-1}/x,$$

we have

$$\sum \frac{p_{2l+1}}{x} p_{2m}\, \Delta x = \sum [p_{2l} - 2l p_{2l-2} + 2l(2l-2)p_{2l-4} - \cdots] p_{2m}\, \Delta x.$$

If $2l < 2m$, the leading term of this summation is zero by the orthogonal property. If $2l = 2m + 2n$, the last equation and the orthogonal property give

$$\sum \frac{p_{2m+2n+1}}{x} p_{2m}\, \Delta x = (-1)^n (2m+2n)(2m+2n-2)\cdots(2m+2)(2m)! \qquad (n = 0, 1, 2, \cdots)$$

whence

$$a_{2m} = \frac{e^{-1/2}}{(2m+1)!}\left[\frac{(2m)!}{(2m+1)!} + \frac{(2m+2)(2m)!}{(2m+3)!} + \cdots\right]$$

$$= \frac{e^{-1/2}}{(2m+1)!}\left[1 + \frac{1}{2m+3} + \frac{1}{(2m+3)(2m+5)} + \cdots\right].$$

This last involves an absolutely convergent series, and for large values of $m$, $a_{2m}$ approximates to

$$\frac{e^{-1/2}}{(2m)!\, 2m}.$$

## Change of scale.

If from the beginning we introduce the scale-parameter $\sigma = (n/2)^{1/2}\, \Delta x$, instead of setting it equal to one as was done, there will be the following obvious changes:

$[F(k, n)]^2$ approaches uniformly $e^{-x^2/(2\sigma^2)}/(2\pi)^{1/2}$,

$$p_1(k) = \frac{k}{\sigma(n/2)^{1/2}} F(k, n), \qquad p_2(k) = \frac{k^2 - n/2}{\sigma^2 \cdot n/2} F(k, n), \cdots,$$

$$p_m(k) = H_m\left(\frac{x}{\sigma}\right) F(k, n)/\sigma^m.$$

The two results of the initial theorem on orthogonal properties are to be multiplied by $\sigma^m$ and $\sigma^{(m+1)/2}$, respectively.

The development formula is now, as an approximation,

$$f = a_0 p_0 + \cdots + a_m p_m + \cdots$$

where

$$a_m = \sigma^{2m} \sum f p_m \Delta x/m!$$

approximately; and $a_0^2 + \cdots + a_m^2 m!/\sigma^{2m} + \cdots$ approximates to the same limit as $\sum f^2 \Delta x$, when $n \to \infty$.

In example 1 the two expressions for the general coefficients will read

$$a_{m-2k} = \frac{\sigma^{2m-4k-1}}{(m-2k)!} \sum x^m F(k, n) p_{m-2k} \Delta x = \frac{m!/(m-2k)!}{2 \cdot 4 \cdots 2k} \sigma^{m-2k}$$

and the development will be

$$x^m F(k, n) = \sigma^{2m} p_m + \frac{m(m-1)}{2} \sigma^{2m-2} p_{m-2} + \cdots.$$

# CONCERNING THE SUMMABILITY OF DOUBLE SERIES OF A CERTAIN TYPE.*

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The ultimate purpose of the present paper is to discuss the Cesàro summation indices of the series $\sum\sum a_{m,n}\, m^{-h} n^{-k}$, $h$ and $k \geqq 0$, provided those of the series $\sum\sum a_{m,n}$ are known, with particular attention to the case when either $h$ or $k$ is zero, In the case of single series, the corresponding problem has been solved by Hardy and M. Riesz‡; therefore the obvious method of attacking the above problem is an extension of their methods and results to the two-dimensional case.

As their discussion is carried on by the use of "Rieszian summation means",§ the equivalence of certain ones of which to Cesàro means had been demonstrated,|| it is first necessary to define double means analogous to them, and to show the equivalence of particular cases of these double means, as a method of summation, to the double Cesàro means; this is done in §§ 1 and 4. The answer to the leading question of the paper, contained in Theorem IV of § 6, follows from auxiliary results obtained in §§ 2, 3 and 5. One of these results, embodied in Theorems I and I′, is worthy of mention, being a consistency theorem for the summation of double series; it shows that the summability, Rieszian or Cesàro¶, of a double series, indices $(r, s)$, implies summability for every pair of indices $(r', s')$, $r' > r$, $s' > s$.

1. **Definitions.** What have been termed "auxiliary results" (Lemmas 1–4 and Theorems I and II) are derived for a more general series than the one indicated in the statement of the leading problem of the paper, i. e. for the series

$$\sum_1^\infty \sum_1^\infty a_{m,n}\, e^{-(\lambda_m h + \lambda_n k)}, \tag{1}$$

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* Presented to the American Mathematical Society, May 7, 1927; received May 13, 1927.

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‡ Hardy-Riesz, "The general theory of Dirichlet's series", Cambridge (1915), Theorem 48. This tract will be referred to in the rest of the present paper by the initials H. R.

§ H. R., pp. 21–22.

|| M. Riesz, "Une méthode de sommation équivalente à la méthode des moyennes arithmétiques", Comptes Rendus, vol. 152 (1911), pp. 1651–1654.

¶ The result relating to Cesàro summability has been proved for integral values of $r$ and $s$, $r'$ and $s'$, by C. N. Moore, "On convergence factors in double series and the double Fourier's series", Trans. Amer. Math. Soc., vol. 14, pp. 73–104.

where $(\lambda_m)$ and $(\lambda_n)$ are sequences of real numbers, increasing without bound; and where $h = \eta + \theta i$ and $k = \varkappa + \theta_1 i$, $i = \sqrt{-1}$. If $\lambda_m = \log m$, $\lambda_n = \log n$, then (1) becomes

$$\sum_1^\infty \sum_1^\infty a_{m,n}\, m^{-h}\, n^{-k}. \tag{2}$$

Finally, if $h = 0$, or $k = 0$, we arrive at the two series in which we are primarily interested; in obtaining our final results, § 6, we shall consider only the former,

$$\sum_1^\infty \sum_1^\infty a_{m,n}\, n^{-k}, \tag{3}$$

the results for the other being wholly analogous to those herein obtained for (3).

We shall write*

$$\begin{gathered}
\sum_1^m \sum_1^n a_{m,n} = A_{m,n}, \\
\sum_1^m a_{m,\beta} = A_{m,\beta}\,(\beta \text{ fixed}), \qquad \sum_1^n a_{\alpha,n} = A_{\alpha,n}\,(\alpha \text{ fixed}), \\
\sum_1^x \sum_1^y a_{m,n} = A(x,y), \\
\sum_1^x a_{m,\beta} = A(x)\,(\beta \text{ fixed}), \qquad \sum_1^y a_{\alpha,n} = A(y)\,(\alpha \text{ fixed}).
\end{gathered} \tag{4}$$

$$\begin{gathered}
e^{\lambda_m} = l_m, \qquad e^{\lambda_n} = l_n, \\
a_{m,n}\, e^{(-\lambda_m h + \lambda_n k)} = a_{m,n}\, l_m^{-h}\, l_n^{-k} = c_{m,n}, \\
C_\lambda(\chi,\omega) = \sum_{\lambda_m < \chi} \sum_{\lambda_n < \omega} c_{m,n}, \\
C_l(v,w) = \sum_{l_m < v} \sum_{l_n < w} c_{m,n}.
\end{gathered} \tag{5}$$

We are now ready to define, for the double series (1), the analogues of the single Rieszian means: we write

$$\begin{aligned}
C_\lambda^{r,s}(\chi,\omega) &= \sum_{\lambda_m < \chi} \sum_{\lambda_n < \omega} (\chi - \lambda_m)^r\, (\omega - \lambda_n)^s\, c_{m,n} \\
&= rs \int_0^\chi \int_0^\omega C_\lambda(\sigma,\tau)\,(\chi - \sigma)^{r-1}\,(\omega - \tau)^{s-1}\, d\tau\, d\sigma, \\
C_l^{r,s}(v,w) &= \sum_{l_m < v} \sum_{l_n < w} (v - l_m)^r\, (w - l_n)^s\, c_{m,n} \\
&= rs \int_1^v \int_1^w C_l(t,u)\,(v - t)^{r-1}\,(w - u)^{s-1}\, du\, dt,
\end{aligned} \tag{6}$$

* For the notation in the last two of these equations, cf. H. R., p. 2.

where $r$ and $s$, $(\chi, \omega)$ and $(v, w)$ are any positive numbers.* We shall call the functions

$$C_\lambda^{r,s}(\chi, \omega)/\chi^r \omega^s \tag{7}$$

$$C_\lambda^{r,s}(v, w)/v^r w^s \tag{8}$$

the "typical Rieszian means" of order $(r, s)$ of the first and second kinds respectively, associated with the series (1).

If

$$\chi^{-r} \omega^{-s} C_\lambda^{r,s}(\chi, \omega) \to C$$

as $\chi$ and $\omega$ become infinite, we say that (1) is summable $(\lambda, r, s)$ to sum $C$. A similar criterion obtains when we use (8). For the single series formed by a row or a column of (1), we at all times use the definitions and formulas as given in the Hardy-Riesz monograph.†

The results in §§ 2–3 will be couched in terms of the first mean; but it should be understood that precisely similar reasoning and results hold for the second mean. The procedure in §§ 4, 5 and 6 requires further explanation: When $\lambda_m = m$, $\lambda_n = n$, the first mean is what is called an "arithmetic" mean, while the second is of little importance; when $\lambda_m = \log m$, $\lambda_n = \log n$, the second mean is "arithmetic" and the first is called "logarithmic".‡

The results in § 4, concerning the equivalence of double Cesàro and certain double Rieszian means, will obtain of course only for the arithmetic Rieszian means, and will, for convenience, be given in terms of the first one. Finally, our treatment of (3) in § 6 will be undertaken, naturally, through the second mean, as it is the "arithmetic" one in regard to (3).

2. **Four lemmas.** LEMMA 1: *If* $r>0, s>0, \mu>0, \nu>0$, *then*§

$$C^{r+\mu,s+\nu}(\chi, \omega) = \frac{\Gamma(r+\mu+1)\Gamma(s+\nu+1)}{\Gamma(r+1)\Gamma(s+1)\Gamma(\mu)\Gamma(\nu)} \int_0^\chi \int_0^\omega C^{r,s}(\sigma, \tau)(\chi-\sigma)^{\mu-1} \times (\omega-\tau)^{\nu-1} d\tau\, d\sigma,$$

$$C^{r+\mu,s}(\chi, \omega) = \frac{\Gamma(r+\mu+1)}{\Gamma(r+1)\Gamma(\mu)} \int_0^\chi C^{r,s}(\sigma, \omega)(\chi-\sigma)^{\mu-1} d\sigma,$$

$$C^{r,s+\nu}(\chi, \omega) = \frac{\Gamma(s+\nu+1)}{\Gamma(s+1)\Gamma(\nu)} \int_0^\omega C^{r,s}(\chi, \tau)(\omega-\tau)^{\nu-1} d\tau.$$

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* For the limits of integration in (6), cf. H. R., p. 21, footnotes § and ||.

† H. R., pp. 21–23. We shall indicate the sum of a row of (1) by $C_R$, of a column by $C_C$.

‡ H. R., pp. 22–23.

§ $\mu$ and $\nu$ are the "order functions" of the function developed in series (1). Cf. H. R., p. 14, for detailed explanations.

*If* $r>0$, $s>0$, $\mu \leqq r$, $\nu \leqq s$, $\mu, \nu<1$, *then*

$$C^{r-\mu, s-\nu}(\chi, \omega) = \frac{\Gamma(r-\mu+1)\,\Gamma(s-\nu+1)}{\Gamma(r+1)\,\Gamma(s+1)\,\Gamma(1-\mu)\,\Gamma(1-\nu)} \times \int_0^{\chi}\int_0^{\omega} \frac{\partial^2 C^{r,s}(\sigma,\tau)}{\partial\sigma\,\partial\tau}(\chi-\sigma)^{-\mu}(\omega-\tau)^{-\nu}\,d\tau\,d\sigma.$$

The proofs of the first three formulas are entirely analogous to that of the first one in Lemma 6 of H. R.; their details will be omitted. For the fourth, we first observe that

$$\begin{aligned} C^{r-\mu, s-\nu}(\chi, \omega) &= \frac{1}{(r-\mu+1)(s-\nu+1)}\,\frac{\partial^2}{\partial\chi\,\partial\omega}\,C^{r-\mu+1, s-\nu+1}(\chi, \omega) \\ &= \frac{\Gamma(r-\mu+1)\,\Gamma(s-\nu+1)}{\Gamma(r+1)\,\Gamma(s+1)\,\Gamma(1-\mu)\,\Gamma(1-\nu)} \\ &\quad \times \frac{\partial^2}{\partial\chi\,\partial\omega}\int_0^{\chi}\int_0^{\omega} C^{r,s}(\sigma,\tau)(\chi-\sigma)^{-\mu}(\omega-\tau)^{-\nu}\,d\tau\,d\sigma \end{aligned} \tag{9}$$

by the first formula. But we have also, integrating by parts,

$$\begin{aligned} &\int_0^{\chi}\int_0^{\omega} C^{r,s}(\sigma,\tau)(\chi-\sigma)^{-\mu}(\omega-\tau)^{-\nu}\,d\sigma\,d\tau \\ &= \frac{1}{(1-\mu)(1-\nu)}\int_0^{\chi}\int_0^{\omega}\frac{\partial^2 C^{r,s}(\sigma,\tau)}{\partial\sigma\,\partial\tau}(\chi-\sigma)^{1-\mu}(\omega-\tau)^{1-\nu}\,d\tau\,d\sigma. \end{aligned} \tag{10}$$

Substituting (10) in (9), and carrying out the required differentiation, we obtain the fourth formula of Lemma 1.

LEMMA 2: *If* $c_{m,n}$ *is real,* $0 \leqq \xi \leqq \chi$, $0 \leqq \eta \leqq \omega$, and $0<r<1$, $0<s<1$, *then*

$$\begin{aligned} |g(\xi, \eta, \chi, \omega)| &= \left| rs\int_0^{\xi}\int_0^{\eta} C(\sigma,\tau)(\chi-\sigma)^{r-1}(\omega-\tau)^{s-1}\,d\tau\,d\sigma\right| \\ &\leqq \operatorname*{Max}_{\substack{0\leqq\sigma\leqq\xi \\ 0\leqq\tau\leqq\eta}} |C^{r,s}(\sigma,\tau)|. \end{aligned}$$

We have that

$$\begin{aligned} g(\xi, \eta, \chi, \omega) &= \frac{\sin r\pi}{\pi}\,\frac{\sin s\pi}{\pi}\int_0^{\xi}\int_0^{\eta}(\chi-\sigma)^{r-1}(\omega-\tau)^{s-1}\,d\tau\,d\sigma \\ &\quad \times \int_0^{\sigma}\int_0^{\tau}\frac{\partial^2 C^{r,s}(t,u)}{\partial t\,\partial u}(\sigma-t)^{-r}(\tau-u)^{-s}\,du\,dt \end{aligned}$$

by using the fourth formula of Lemma 1, with $\mu = r$, $\nu = s$. Thus

$$g(\xi, \eta, \chi, \omega) = \int_0^{\xi}\int_0^{\eta} \frac{\partial^2 C^{r,s}(t,u)}{\partial t\, \partial u} h(t,u)\, du\, dt, \tag{11}$$

where

$$h(t,u) = \frac{\sin r\pi}{\pi}\frac{\sin s\pi}{\pi}\int_t^{\xi}\int_u^{\eta} (\chi-\sigma)^{r-1}(\omega-\tau)^{s-1}(\sigma-t)^{-r}(\tau-u)^{-s}\, d\tau\, d\sigma.$$

But

$$\begin{aligned} h(t,u) = &\left[1 - \int_{\xi}^{\chi} (\chi-\sigma)^{r-1}(\sigma-\tau)^{-r}\, d\sigma\right] \\ &\times \left[1 - \int_{\eta}^{\omega} (\omega-\tau)^{s-1}(\tau-u)^{-s}\, d\tau\right]. \end{aligned} \tag{12}$$

Now the two integrals in (12) are, for $0 \leqq t \leqq \xi$, $0 \leqq u \leqq \eta$, always positive and less than unity; for $\sigma$ fixed in the interval $\xi \leqq \sigma \leqq \chi$, and $\tau$ fixed in the interval $\eta \leqq \tau \leqq \omega$, they are both steadily increasing functions of $t$ and $u$. Hence, each factor of (12), and consequently $h(t,u)$ itself, is a steadily decreasing bounded function of $t$ and $u$ in the given region. Applying the second mean value theorem for integrals* to (11), we have

$$g(\xi, \eta, \chi, \omega) = h(0,0)\, C^{r,s}(\bar{\sigma}, \bar{\tau}),$$

$(0 \leqq \bar{\sigma} \leqq \xi,\ 0 \leqq \bar{\tau} \leqq \eta)$, which proves the lemma.

LEMMA 3: *If* $c_{m,n}$ *is real,* $0 \leqq \xi \leqq \chi$, $0 \leqq \eta \leqq \omega$, $\mu > 0$, $\nu > 0$, $0 < r < 1$, $0 < s < 1$, *then*

$$\frac{\Gamma(r+\mu+1)\Gamma(s+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(r)\Gamma(s)} \left| \int_0^{\xi}\int_0^{\eta} C^{\mu,\nu}(\sigma,\tau)(\chi-\sigma)^{r-1}(\omega-\tau)^{s-1}\, d\tau\, d\sigma \right| \leqq \max_{\substack{0 \leqq \sigma \leqq \xi \\ 0 \leqq \tau \leqq \eta}} |C^{r+\mu, s+\nu}(\sigma,\tau)|.$$

The proof of Lemma 3 is carried out along the lines of that of Lemma 2.

LEMMA 4. *Let* $\varphi(x,y)$ *and* $\psi(x,y)$ *be functions of* $x$ *and* $y$ *satisfying the following properties*:

(*a*) $\varphi(x,y)$ *and* $\psi(x,y)$ *are continuous functions of* $x$ *and* $y$, *and also of either variable alone*;

(*b*) *considered as functions of both variables,*

$$\varphi(x;y) \approx Px^{\alpha}y^{\beta}, \qquad \psi(x,y) \approx Qx^{\gamma}y^{\delta} \qquad (\alpha, \beta, \gamma, \delta \geqq 0)$$

*as* $x$ *and* $y$ *become infinite*;†

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* H. Geiringer, "Trigonometrische Doppelreihen", Monatshefte für Math. und Phys., vol. 29, § 39.

† This notation means that $\lim_{x,y\to\infty} \varphi/x^{\alpha}y^{\beta} \to P$, etc.

(c) *considered as functions of single variables, $x_1$, $y_1$ being specific values,*

$$\varphi(x_1, y) \approx P_1 y^\beta, \qquad \varphi(x, y_1) \approx P_2 x^\alpha,$$
$$\psi(x_1, y) \approx Q_1 y^\delta, \qquad \psi(x, y_1) \approx Q_2 x^\gamma;$$

*then*

$$\Psi(x, y) = \int_0^x \int_0^y \varphi(t, u)\, \psi(x-t, y-u)\, du\, dt$$
$$\approx PQ \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\beta+1)\Gamma(\delta+1)}{\Gamma(\alpha+\gamma+2)\Gamma(\beta+\delta+2)} x^{\alpha+\gamma+1} y^{\beta+\delta+1}.$$

Let us put

$$\varphi(t, u) = P t^\alpha u^\beta + \varphi_1, \qquad \psi(t, u) = Q t^\gamma u^\delta + \varphi_2,$$

where

$$\varphi_1 = o(t^\alpha u^\beta), \qquad \psi_1 = o(t^\gamma u^\delta).$$

If we put these forms in $\Psi(x, y)$, we obtain a sum of four integrals. The first one is

$$PQ \int_0^x \int_0^y t^\alpha (x-t)^\gamma u^\beta (y-u)^\delta\, du\, dt$$
$$= PQ \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\beta+1)\Gamma(\delta+1)}{\Gamma(\alpha+\gamma+2)\Gamma(\beta+\delta+2)} x^{\alpha+\gamma+1} y^{\beta+\delta+1}.$$

Hence we must show, in order to prove our lemma, that each of the other three integrals is $o(x^{\alpha+\gamma+1} y^{\beta+\delta+1})$.

For example, consider the integral

$$P \int_0^x \int_0^y t^\alpha u^\beta \psi_1(x-t, y-u)\, du\, dt$$
$$= P \int_0^x \int_0^y (x-t)^\alpha (y-u)^\beta \psi_1(t, u)\, du\, dt.$$

Now, given $\varepsilon$, $\varepsilon_1$ and $\varepsilon_2$, we can choose $\xi$ and $\eta$ so that

$$|\psi_1(t, u)| < \varepsilon t^\gamma u^\delta, \qquad (\xi \leqq t \leqq x,\ \eta \leqq u \leqq y),$$
$$|\psi_1(t, u_1)| < \varepsilon_1 t^\gamma, \qquad (\xi \leqq t \leqq x),$$
$$|\psi_1(t_1, u)| < \varepsilon_2 u^\delta, \qquad (\eta \leqq u \leqq y).$$

Also, for $0 \leqq t \leqq \xi$, $0 \leqq u \leqq \eta$, $|\psi_1(t, u)| < M(\xi, \eta)$, $|\psi_1(t, u_1)| < M_1(\xi)$, and $|\psi_1(t_1, u)| < M_2(\eta)$, the $M$'s being numbers depending only on $\xi$ and $\eta$.

We have then that

$$P \int_0^x \int_0^y (x-t)^\alpha (y-u)^\beta \psi_1(t, u)\, du\, dt$$
$$= P \left\{ \int_0^\xi \int_0^\eta + \int_0^\xi \int_\eta^y + \int_\xi^x \int_0^\eta + \int_\xi^x \int_\eta^y \right\} (x-t)^\alpha (y-u)^\beta \psi_1(t, u)\, du\, dt. \tag{13}$$

The first integral of (13) is, in absolute value, less than $|P|\xi\eta M(\xi,\eta)\,x^{\alpha}\,y^{\beta}$; the fourth is in absolute value less than $|P|\,\varepsilon\int_0^x\int_0^y (x-t)^{\alpha}\,(y-u)^{\beta}\,t^{\gamma}\,u^{\delta}\,du\,dt$; the second, regarded as an iterated integral, is in absolute value less than $|P|\,M_1(\xi)\,\xi\varepsilon_2 x^{\alpha}\int_0^y (y-u)^{\beta}\,u^{\delta}\,du$; the third is less in absolute value than $|P|\,M_2(\eta)\,\eta\varepsilon_1 y^{\beta}\int_0^x (x-t)^{\alpha}\,t^{\gamma}\,dt$. Hence, by choice, first of $\xi$ and $\eta$, then of $x$ and $y$, the integral (13) is $o(x^{\alpha+\gamma+1}\,y^{\beta+\delta+1})$.

Similar considerations of the remaining two integrals serve to complete the proof of Lemma 4.

**3. Two subsidiary theorems.** We prove in this section, two theorems relating to (1), which, though subsidiary to our main results, are yet very important in the theory of summability.

THEOREM I: *If the series* (1) *is summable* $(\lambda, r, s)$, *then the series is summable* $(\lambda, r', s')$, $r'>r$, $s'>s$, *to the same sum as formerly.*

We put $r'=r+\mu$, $s'=s+\nu$, and apply Lemma 1:

$$C^{r+\mu,s+\nu}(\chi,\omega)=\frac{\Gamma(r+\mu+1)\,\Gamma(s+\nu+1)}{\Gamma(r+1)\,\Gamma(s+1)\,\Gamma(\mu)\,\Gamma(\nu)}\int_0^{\chi}\int_0^{\omega} C^{r,s}(\sigma,\tau)\,(\chi-o)^{\mu-1}\times(\omega-\tau)^{\nu-1}\,\delta\tau\,\delta\sigma.$$

But

$$C^{r,s}(\sigma,\tau)\approx C\sigma^{r}\,\tau^{s},\qquad C^{r,s}(\sigma,\tau_1)\approx C_1\sigma^{r},\qquad C^{r,s}(\sigma_1,\tau)\approx C_2\tau^{s},$$

so that the theorem follows at once from an application of Lemma 4.

THEOREM II: *If* (1) *is summable* $(\lambda, r, s)$, *and each row and column is summable* $(\lambda, s)$ *and* $(\lambda, r)$ *respectively, all to zero,* then, if* $0<r'\leqq r$, $0<s'\leqq s$, $\lambda_m\leqq\chi\leqq\lambda_{m+1}$, $\lambda_n\leqq\omega\leqq\lambda_{n+1}$,

(i) *if* $r'$ *and* $s'$ *are both integral,*†

$$C^{r',s'}(\chi,\omega)=o\left\{\frac{\lambda_{m+1}^{r}\,\lambda_{n+1}^{s}}{(\lambda_{m+1}-\lambda_m)^{r-r'}\,(\lambda_{n+1}-\lambda_n)^{s-s'}}\right\};$$

* That we can take all these sums as zero without loss of generality, follows thus: in the given series, we can first make the sum of each row zero by altering the elements of the first column to $\overline{c}_{m,1}$; then the sum of each column can be made zero by changing the elements of the first row to $\overline{c}_{1,n}$, and $\overline{c}_{1,1}$ to $\overline{\overline{c}}_{1,1}$; although this last change might seem to affect the summability to zero of the first row, we infer that such is not the case because the sum of the double series and those of all the other rows are zero.

† Moreover, this result is true for $r'$ or $s'$, or both, equal to zero, provided $\lambda_m<\chi$ or $\lambda_n<\omega$ or both.

(ii) *if $r'$ and $s'$ are both non-integral,*

$$C^{r',s'}(\chi,\omega)=o\left\{\frac{\lambda_m^r\lambda_n^s}{(\lambda_m-\lambda_{m-1})^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}}+\frac{\lambda_m^r\lambda_{n+1}^s}{(\lambda_m-\lambda_{m-1})^{r-r'}(\lambda_{n+1}-\lambda_n)^{s-s'}}\right.$$
$$\left.+\frac{\lambda_{m+1}^r\lambda_n^s}{(\lambda_{m+1}-\lambda_m)^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}}+\frac{\lambda_{m+1}^r\lambda_{n+1}^s}{(\lambda_{m+1}-\lambda_m)^{r-r'}(\lambda_{n+1}-\lambda_n)^{s-s'}}\right\};$$

(iii) *if $r'$ is integral and $s'$ is non-integral,*

$$C^{r',s'}(\chi,\omega)=o\left\{\frac{\lambda_{m+1}^r\lambda_{n+1}^s}{(\lambda_{m+1}-\lambda_m)^{r-r'}(\lambda_{n+1}-\lambda_n)^{s-s'}}+\frac{\lambda_{m+1}^r\lambda_n^s}{(\lambda_{m+1}-\lambda_m)^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}}\right\};$$

(iiii) *if $r'$ is non-integral, and $s'$ is integral, the result is analogous to that in* (iii).

(i) To prove (i), we suppose that $c_{m,n}$ is real; and we put $R=[r]$, $S=[s]$. Let us also write

$$\begin{aligned} X_i &= \lambda_m+ih_1, && (i=1,2,\cdots,R),\\ X_{R+1} &= \lambda_{m+1},\\ \Omega_j &= \lambda_n+jh_2, && (j=1,2,\cdots,S),\\ \Omega_{S+1} &= \lambda_{n+1}, \end{aligned}$$

where

$$h_1=\frac{\lambda_{m+1}-\lambda_m}{R+1},\quad h_2=\frac{\lambda_{n+1}-\lambda_n}{S+1};$$

let $(X,\Omega)$ denote any pair of these numbers.

Now, by Lemma 1,

$$C^{r,s}(X,\Omega)=\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(R+1)\Gamma(S+1)\Gamma(r-R)\Gamma(s-S)}\int_0^X\int_0^\Omega C^{R,S}(\sigma,\tau)(X-\sigma)^{r-R-1}\times(\Omega-\tau)^{s-S-1}\,d\tau\,d\sigma.$$

If we use Lemma 3 and the fact that

$$C^{r,s}(X,\Omega)=o(X^r\Omega^s)=o(\lambda_{m+1}^r\lambda_{n+1}^s),$$

we obtain

$$(14)\qquad \int_{\lambda_m}^X\int_{\lambda_n}^\Omega C^{R,S}(\sigma,\tau)(X-\sigma)^{r-R-1}(\Omega-\tau)^{s-S-1}\,d\tau\,d\sigma=o(\lambda_{m+1}^r\lambda_{n+1}^s).$$

Regarding (14) as an iterated integral, we integrate it $R$ times by parts with respect to $\sigma$, and then $S$ times by parts with respect to $\tau$. Observing that $C(\sigma,\tau)$ is constant in the region of integration, we find that (14) can be expressed as the sum of constant multiples of the $(R+1)(S+1)$ functions

$$
\begin{array}{ll}
(\lambda_{m+1}-\lambda_m)^{r-R} & (\lambda_{n+1}-\lambda_n)^{s-S} \quad C^{R,S}\ (\lambda_m, \lambda_n), \\
(\lambda_{m+1}-\lambda_m)^{r-R+1} & (\lambda_{n+1}-\lambda_n)^{s-S} \quad C^{R-1,S}(\lambda_m, \lambda_n), \\
(\lambda_{m+1}-\lambda_m)^{r-R} & (\lambda_{n+1}-\lambda_n)^{s-S+1} C^{R,S-1}(\lambda_m, \lambda_n), \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
(\lambda_{m+1}-\lambda_m)^{r} & (\lambda_{n+1}-\lambda_n)^{s} \qquad C(\lambda_m+0,\ \lambda_n+0).
\end{array} \tag{15}
$$

Now, if we give $(X, \Omega)$ the $(R+1)(S+1)$ different pairs of values exhibited above, we obtain $(R+1)(S+1)$ linear relations involving the quantities (15), each relation being $o(\lambda_{m+1}^r \lambda_{n+1}^s)$. Since the determinant of this system is different from zero,* we conclude that each function of (15) is of that form.

The last function is

$$(\lambda_{m+1}-\lambda_m)^r (\lambda_{n+1}-\lambda_n)^s C(\chi, \omega) = o(\lambda_{m+1}^r \lambda_{n+1}^s),$$

whence

$$C(\chi, \omega) = o\left\{\frac{\lambda_{m+1}^r \lambda_{n+1}^s}{(\lambda_{m+1}-\lambda_m)^r (\lambda_{n+1}-\lambda_n)^s}\right\}.$$

The next-to-last function is†

$$(\lambda_{m+1}-\lambda_m)^r (\lambda_{n+1}-\lambda_n)^{s-1} C^{0,1}(\chi, \lambda_n) = o(\lambda_{m+1}^r \lambda_{n+1}^s);$$

therefore $C^{0,1}(\chi, \lambda_n)$ is of the prescribed form. Hence

$$
\begin{aligned}
C^{0,1}(\chi, \omega) &= C^{0,1}(\chi, \lambda_n) + \int_{\lambda_n}^{\omega} C(\chi, \tau)\, d\tau \\
&= C^{0,1}(\chi, \lambda_n) + \int_{\lambda_n}^{\omega} o\left\{\frac{\lambda_{m+1}^r \lambda_{n+1}^s}{(\lambda_{m+1}-\lambda_m)^r (\lambda_{n+1}-\lambda_n)^s}\right\} d\tau \\
&= o\left\{\frac{\lambda_{m+1}^r \lambda_{n+1}^s}{(\lambda_{m+1}-\lambda_m)^r (\lambda_{n+1}-\lambda_n)^{s-1}}\right\},
\end{aligned}
$$

which is as desired. Similarly, we find that

$$C^{1,0}(\chi, \omega) = o\left\{\frac{\lambda_{m+1}^r \lambda_{n+1}^s}{(\lambda_{m+1}-\lambda_m)^{r-1} (\lambda_{m+1}-\lambda_m)^s}\right\}.$$

Next, we have that

$$
\begin{aligned}
C^{1,1}(\chi, \omega) &= C^{1,1}(\lambda_m, \lambda_n) + \int_{\lambda_m}^{\chi} C^{0,1}(\sigma, \lambda_n)\, d\sigma \\
&\quad + \int_{\lambda_n}^{\omega} C^{1,0}(\lambda_m, \tau)\, d\tau + \int_{\lambda_m}^{\chi}\int_{\lambda_n}^{\omega} C(\sigma, \tau)\, d\tau\, d\sigma.
\end{aligned} \tag{16}
$$

* It is, in fact, a Vandermondian; cf. H. R., p. 37, for the same argument.

† Although $C_\lambda^{0,1}(\chi, \omega)$ is not defined by the integral form in (6), it is clear that we mean by it

$$C_\lambda^{0,s}(\chi, \omega) = \sum\sum (\omega - \lambda_n)^s c_{m,n} = s \int_0^{\omega} C_\lambda(\chi, \tau)(\omega - \tau)^{s-1}\, d\tau.$$

Upon evaluation of the integrals in (16), the result follows. Proceeding in this way, we finally prove (i) for any pair of values of $r' = 0, 1, \cdots, R$, $s' = 0, 1, \cdots, S$.

(ii) We turn now to the proof of (ii). Put $R' = [r']$, $S' = [s']$, and then, by Lemma 1,

$$C^{r',s'}(\chi, \omega) = \frac{\Gamma(r'+1)\Gamma(s'+1)}{\Gamma(R'+1)\Gamma(S'+1)\Gamma(r'-R')\Gamma(s'-S')} \times \int_0^{\chi}\int_0^{\omega} C^{R',S'}(\sigma, \tau)(\chi-\sigma)^{r'-R'-1}(\omega-\tau)^{s'-S'-1}\, d\tau\, d\sigma. \tag{17a}$$

We rewrite (17a) as

$$\begin{aligned} C^{r',s'}(\chi, \omega) &= K\left\{\int_{\lambda_{m-1}}^{\chi}\int_{\lambda_{n-1}}^{\omega} + \int_0^{\lambda_{m-1}}\int_0^{\lambda_{n-1}} + \int_0^{\lambda_{m-1}}\int_{\lambda_{n-1}}^{\omega} + \int_{\lambda_{m-1}}^{\chi}\int_0^{\lambda_{n-1}}\right\} \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{17b}$$

each integral of (17b) having the integrand of (17a); $K$ is a constant.

First consider $I_1$. We break it put into four integrals:

$$\int_{\lambda_{m-1}}^{\chi}\int_{\lambda_{n-1}}^{\omega} = \int_{\lambda_{m-1}}^{\lambda_m}\int_{\lambda_{n-1}}^{\lambda_n} + \int_{\lambda_m}^{\chi}\int_{\lambda_n}^{\omega} + \int_{\lambda_{m-1}}^{\lambda_m}\int_{\lambda_n}^{\omega} + \int_{\lambda_m}^{\chi}\int_{\lambda_{n-1}}^{\lambda_n}.$$

If, from this point on, we argue analogously to the discussion in Theorem 22 of the Hardy-Riesz monograph,* we find easily that

$$\begin{aligned} I_1 = o\Bigg\{ &\frac{\lambda_m^r \lambda_n^s}{(\lambda_m-\lambda_{m-1})^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}} + \frac{\lambda_m^r \lambda_{n+1}^s}{(\lambda_m-\lambda_{m-1})^{r-r'}(\lambda_{n+1}-\lambda_n)^{s-s'}} \\ &+ \frac{\lambda_{m+1}^r \lambda_n^s}{(\lambda_{m+1}-\lambda_m)^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}} + \frac{\lambda_{m+1}^r \lambda_{n+1}^s}{(\lambda_{m+1}-\lambda_m)^{r-r'}(\lambda_{n+1}-\lambda_n)^{s-s'}}\Bigg\}. \end{aligned} \tag{18}$$

To obtain an upper limit for $I_2$, we follow a procedure so analogous to that in the corresponding place of H. R., that it is unnecessary to record it; we find that

$$I_2 = o\left\{\frac{\lambda_m^r \lambda_n^s}{(\lambda_m-\lambda_{m-1})^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}}\right\}. \tag{19}$$

For $I_3$, we proceed as follows: We integrate $(R-R')$ times by parts with respect to $\sigma$, obtaining a double integral and a set of $(R-R')$ terms made up of single integrals each multiplied be a factor consisting of powers of $(\chi-\lambda_{m-1})$. We treat the single integrals in a manner analogous to the discussion of $J_2$ in H. R., using result (i) for the functions $C^{R'+\mu,S'}(\sigma,\tau)$ in the integrands; the factors before the integrals may be discussed as in

* That is, the argument leading to equation (5), p. 38 of their tract.

the treatment of the terms of $J_1$ which do not involve an integral. In the double integral, we apply the law of the mean, and use Lemma 3, and proceed analogously to the discussion of the integral term in $J_1$ of the H. R. tract. In this way, we find that

$$(20)\quad I_3 = o\left\{\frac{\lambda_m^r\,\lambda_n^s}{(\lambda_m-\lambda_{m-1})^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}} + \frac{\lambda_m^r\,\lambda_{n+1}^s}{(\lambda_m-\lambda_{m-1})^{r-r'}(\lambda_{n+1}-\lambda_n)^{s-s'}}\right\}.$$

Similarly,

$$(21)\quad I_4 = o\left\{\frac{\lambda_m^r\,\lambda_n^s}{(\lambda_m-\lambda_{m-1})^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}} + \frac{\lambda_{m+1}^r\,\lambda_n^s}{(\lambda_{m+1}-\lambda_m)^{r-r'}(\lambda_n-\lambda_{n-1})^{s-s'}}\right\}.$$

Combining (18)–(21), we complete (ii).

In treating the oblique case (iii), we supplant the use of the first formula in Lemma 1 by the second. Hence, using result (i) for the function $C^{R',s'}(\sigma,\omega)$, and dealing with the single integrals involved as in the relevant part of Theorem 22 of H. R., we prove (iii); (iiii) follows analogously.

Theorem II is thus complete.

**4. Equivalence of the "double Cesàro" and the "arithmetic double Rieszian" means.** We have next to establish the equivalence of (7) with $\lambda_m = m$, $\lambda_n = n$, and the Cesàro means for double series. This results from a generalization to double series of the corresponding result for single series, obtained by M. Riesz.*

Let us consider a double series $\sum\sum_{0,0}^{\infty,\infty} c_{m,n}$, and form

$$(22)\quad S_{m,n}^{r,s} = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+r-i}{r}\binom{n+s-j}{s}s_{i,j},$$

where

$$(23)\quad s_{i,j} = \sum_{p=0}^{i}\sum_{q=0}^{j} c_{p,q},$$

Let us put

$$(24)\quad A_{m,n}^{r,s} = \binom{m+r}{m}\binom{n+s}{n}.$$

Then we say that $\sum\sum c_{m,n}$ is summable $(C, r, s)$ to the limit

$$(25)\quad \lim_{m,n\to\infty} S_{m,n}^{r,s}/A_{m,n}^{r,s} = L,$$

if this limit exists.

Let us also put, in accordance with the definition of the Rieszian mean (6),

$$(26)\quad C^{r,s}(\chi,\omega) = \sum_{p<\chi}\sum_{q<\omega} c_{p,q}(\chi-p)^r(\omega-q)^s.$$

---

* M. Riesz, loc. cit., pp. 1652–1655. The proof has been expanded by E. W. Hobson, "Theory of Functions of a real Variable", vol. 2, ed. 2, p. 90ff.

We wish to prove the following result:

THEOREM III: *The limit* (25) *and the limit*

$$\lim_{\chi,\omega\to\infty} C^{r,s}(\chi,\omega)/\chi^r\,\omega^s \tag{27}$$

*exist at the same time, and are equal, provided that the rows and columns of* $\sum\sum c_{m,n}$ *are summable as single series in either the Cesàro or the Rieszian manners, indices* $s$ *and* $r$.

The proof of Theorem III is contained in the combination of the following three lemmas.

LEMMA 5: *If* $\sum\sum c_{m,n}$ *is summable* $(C, r, s)$, *then*

$$\lim_{m,n\to\infty} S^{r',s'}_{m,n}/m^r\,n^s = 0$$

*where* $r'<r$, $s'<s$, *and* $r$ *and* $s$ *are greater than* $-1$.

To prove this lemma, we employ the following relation, which is readily established from the relations (22)–(24):

$$S^{r',s'}_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^i(-1)^j\binom{r-r'}{i}\binom{s-s'}{j}S^{r,s}_{m-i,n-j}. \tag{28}$$

Since $\sum\sum c_{m,n}$ is summable $(C, r, s)$, we can write

$$\frac{S^{r,s}_{m,n}}{A^{r,s}_{m,n}} = L+\varrho_{m,n}$$

where

$$\lim_{m,n\to\infty}\varrho_{m-i,n-j} = 0 \text{ for every } i,j.$$

Hence

$$\begin{aligned}\frac{S^{r',s'}_{m,n}}{m^r\,n^s} &= \frac{L}{m^r\,n^s}\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{r-r'}{i}\binom{m+r-i}{m-i}\binom{s-s'}{j}\binom{n+s-j}{n-j}\\ &\quad+(-1)^{i+j}\frac{a_{0,0}}{m^r\,n^s}\binom{r-r'}{m}\binom{s-s'}{n}\\ &\quad+\frac{1}{\Gamma(r+1)\Gamma(s+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(-1)^{i+j}\binom{r-r'}{i}\left(1-\frac{i}{m}\right)^r\\ &\qquad\times\binom{s-s'}{j}\left(1-\frac{j}{n}\right)^s\varrho_{m-i,n-j}(1+\delta_{m-i,n-j})\\ &\quad+\frac{1}{\Gamma(r+1)n^s}\sum_{i=0}^{m-1}(-1)^i\binom{r-r'}{i}\left(1-\frac{i}{m}\right)^r\varrho_{m-i,0}(1+\delta_{m-i,0})\\ &\quad+\frac{1}{\Gamma(s+1)m^r}\sum_{j=0}^{n-1}(-1)^j\binom{s-s'}{j}\left(1-\frac{j}{n}\right)^s\varrho_{0,n-j}(1+\delta_{0,n-j}),\end{aligned} \tag{29}$$

where, in the sums involving the $\varrho_{m,n}$, we have used the formula*

$$\binom{r+m}{m}\binom{s+n}{n} = \frac{m^r n^s}{\Gamma(r+1)\Gamma(s+1)}(1+\delta_{m,n}), \tag{30}$$

where $\lim_{m,n\to\infty} \delta_{m,n} \to 0$.

The first term of (29) is seen easily to converge to zero; and the second term also need not be considered hereafter, for if $r-r'$ and $s-s'$ are integers less than $m$ and $n$, the series stops before the last term appears, and if they are not both integers, it converges to zero. We consider, then, only the last three terms of (29).

If $r-r'$ and $s-s'$ are integers less than $m$ and $n$ respectively, it follows readily that, since $\lim_{m,n\to\infty} \varrho_{m-i,n-j} = 0$ for each pair of values $(i, j)$, and since the series is finite, the three terms under consideration converge to zero.

If $r-r'$ and $s-s'$ are both non-integral, we have four cases: $r>0$, $s>0$; $r>0$, $0>s>-1$; $0>r>-1$, $s>0$, $0>r>-1$, $0>s>-1$. It is readily established by standard convergence methods that in each of these cases the expression for $S^{r',s'}_{m,n}/m^r n^s$ converges to zero for proper choices of $m$ and $n$. The case when $r-r'$ is integral and $s-s'$ is not (and vice versa) is just as easily handled, and will be omitted.

Hence the lemma is established.

LEMMA 6: *If the series $\sum\sum c_{m,n}$ is summable as a double series, and its rows and columns as single series, in the Rieszian manner, indices $(r, s)$, then for every pair of integers $E<r$ and $G<s$,*

$$\lim_{m,n\to\infty} S^{E,G}_{m,n}/m^r n^s = 0. \tag{31}$$

*Case* 1: *$r$ and $s$ both integral.* In this case, the $(r, s)$th difference† of the quantities $C^{r,s}(m, n)$, $C^{r,s}(m+1/r, n)$, $C^{r,s}(m, n+1/s)$, $C^{r,s}(m+1/r, n+1/s)$, $\cdots$, $C^{r,s}(m+1, n+1)$, leads to $S^{0,0}_{m,n}$, modified by a constant factor, so that (31) is verified for $E=G=0$. Similar treatment of the $(r, s-1)$th, $(r-1, s)$th, $(r-1, s-1)$th, $\cdots$, $(1, 1)$st differences, proves (31) for all permutations of $E=0, 1, \cdots, r-1$, $G=0, 1, \cdots, s-1$.

*Case* 2: *$r$ and $s$ non-integral.* (a) $0<r<1$, $0<s<1$. In this case, we have only to prove that

$$\lim_{m,n\to\infty} \frac{S^{0,0}_{m,n}}{A^{r,s}_{m,n}} = \lim_{m,n\to\infty} \frac{S^{0,0}_{m,n}}{m^r n^s} = 0.$$

* Hobson, loc. cit., p. 71.

† C. N. Moore, loc. cit., p. 77.

We suppose that

$$\lim_{\chi,\omega\to\infty} C^{r,s}(\chi,\omega)/\chi^r\,\omega^s = 0,$$

$$\lim_{\chi\to\infty} C^r(\chi)/\chi^r = 0, \qquad \text{(for every column)},$$

$$\lim_{\omega\to\infty} C^s(\omega)/\omega^s = 0. \qquad \text{(for every row)}. \tag{32}$$

Let us also put

$$\begin{aligned} S(\sigma,\tau) &= S^{0,0}_{p,q}, \\ S_1(\sigma) &= S^0_{p,\beta}, \qquad (\beta \text{ fixed}), \\ S_2(\tau) &= S^0_{\alpha,q}, \qquad (\alpha \text{ fixed}), \end{aligned} \qquad \begin{pmatrix} p<\sigma\leqq p+1 \\ q<\tau\leqq q+1 \end{pmatrix}. \tag{33}$$

Hence we obtain

$$S(\sigma,\tau) = g\int_0^{\sigma}\int_0^{\tau} \frac{\partial^2}{\partial\chi\,\partial\omega} C^{r,s}(\chi,\omega)\,(\sigma-\chi)^{-r}\,(\tau-\omega)^{-s}\,d\omega\,d\chi,$$

$$S_1(\sigma) = g_1\int_0^{\sigma} \frac{d}{d\chi} C^r(\chi)\,(\sigma-\chi)^{-r}\,d\chi, \tag{34}$$

$$S_2(\tau) = g_2\int_0^{\tau} \frac{d}{d\omega} C^s(\omega)\,(\tau-\omega)^{-s}\,d\omega,$$

where $g$, $g_1$ and $g_2$ are constants.

On the other hand, we have

$$\begin{aligned} C^{r,s}(m,n) &= \sum_1^{m-1}\sum_1^{n-1} c_{p,q}\,(m-p)^r\,(n-q)^s \\ &= S^{0,0}_{m-1,n-1} + \sum_{p=1}^{m-2} S^0_{p,n-1}\,[(m-p)^r-(m-p-1)^r] \\ &\quad + \sum_{q=1}^{n-2} S^0_{m-1,q}\,[(n-q)^s-(n-q-1)^s] \\ &\quad + \sum_1^{m-2}\sum_1^{n-2} S^{0,0}_{p,q}[(m-p)^r(n-q)^s-(m-p-1)^r(n-q)^s \\ &\qquad -(m-p)^r(n-q-1)^s+(m-p-1)^r(n-q-1)^s], \end{aligned}$$

whence, by (33) and (34),

$$\begin{aligned} C^{r,s}(m,n) &= S^{0,0}_{m-1,n-1} + r\int_0^{m-1} S_1(\sigma)\,(m-\sigma)^{r-1}\,d\sigma \\ &\quad + s\int_0^{n-1} S_2(\tau)\,(n-\tau)^{s-1}\,d\tau \\ &\quad + rs\int_0^{m-1}\int_0^{n-1} S(\sigma,\tau)\,(m-\sigma)^{r-1}\,(n-\tau)^{s-1}\,d\tau\,d\sigma \end{aligned} \tag{35a}$$

$$= S^{0,0}_{m-1,n-1} + rg_1 \int_0^{m-1} \frac{d}{d\chi} C^r(\chi)\, M_1(\chi)\, d\chi$$

(35b)
$$+ sg_2 \int_0^{n-1} \frac{d}{d\omega} C^s(\omega)\, M_2(\omega)\, d\omega$$

$$+ rsg \int_0^{m-1} \int_0^{n-1} \frac{\partial^2}{\partial\chi\, \partial\omega} C^{r,s}(\chi, \omega)\, M(\chi, \omega)\, d\omega\, d\chi,$$

where

$$M(\chi, \omega) = \int_\chi^{m-1} \int_\omega^{n-1} (m-\sigma)^{r-1} (n-\tau)^{s-1} (\sigma-\chi)^{-r} (\tau-\omega)^{-s}\, d\tau\, d\sigma,$$

$$M_1(\chi) = \int_\chi^{m-1} (m-\sigma)^{r-1} (\sigma-\chi)^{-r}\, d\sigma,$$

$$M_2(\omega) = \int_\omega^{n-1} (n-\tau)^{s-1} (\tau-\omega)^{-s}\, d\tau.$$

But we have that

$$M(\chi, \omega) < \int_\chi^{m} \int_\omega^{n} (m-\sigma)^{r-1} (n-\tau)^{s-1} (\sigma-\chi)^{-r} (\tau-\omega)^{-s}\, d\tau\, d\sigma = M,$$

$$M_1(\chi) < \int_\chi^{m} (m-\sigma)^{r-1} (\sigma-\chi)^{-r}\, d\sigma = M_1,$$

$$M_2(\omega) < \int_\omega^{n} (n-\tau)^{s-1} (\tau-\omega)^{-s}\, d\tau = M_2,$$

where $M$, $M_1$ and $M_2$ are constants independent of $\sigma$, $\tau$, $\chi$ and $\omega$. Moreover, $M(\chi, \omega)$, $M_1(\chi)$ and $M_2(\omega)$ are increasing monotonic functions, $M(\chi, \omega)$ in the region $0 \leqq \chi \leqq m-1$, $0 \leqq \omega \leqq n-1$, the others for $0 \leqq \chi \leqq m-1$ and $0 \leqq \omega \leqq n-1$ respectively. Hence, applying the second mean value theorem for double and single integrals, we obtain from (35)

$$|C^{r,s}(m, n) - S^{0,0}_{m-1,n-1}|$$
$$= |rsg\, M(0, 0)\, C^{r,s}(\bar\sigma, \bar\tau) + rg_1 M_1(0) C^r(\bar\sigma) + sg_2 M_2(0) C^s(\bar\tau)|,$$

whence

$$|C^{r,s}(m,n) - S^{0,0}_{m-1,n-1}| < |rsg\, M C^{r,s}(\bar\sigma, \bar\tau)| + |rg_1 M_1 C^r(\bar\sigma)| + |sg_2 M_2 C^s(\bar\tau)|,$$

where $0 < \bar\sigma < m-1$, $0 < \bar\tau < n-1$. An application of hypothesis (32) gives the desired result.

*Case* 2 (b): $r > 1$, $s > 1$. The discussion for this case is carried out on the same lines as that of Theorem II (i).

The oblique cases (e. g. $r$ integral, $s$ not), can be handled by suitable combinations of the methods at hand. Lemma 6 is therefore completely outlined.

LEMMA 7: *If equation* (31) *holds, then*

$$\lim_{m,n\to\infty}\left[\frac{C^{r,s}(\chi,\omega)}{\chi^r\omega^s}-\frac{S^{r,s}_{m,n}}{A^{r,s}_{m,n}}\right]=0$$

*for* $m\leqq\chi\leqq m+1$, $n\leqq\omega\leqq n+1$.

The proof of this lemma offers no inherent difficulties, and its details will be omitted for considerations of space.*

5. **Corollaries of Theorems I, II and III.** Theorem III allows us to translate Theorems I and II into terms of Cesàro summability in the special cases when the Rieszian mean becomes "arithmetic". Thus we obtain

THEOREM I′: *If the series* (1) *is summable* $(C, r, s)$ *to sum* $C$, *and its rows and columns are summable* $(C, s)$ *and* $(C, r)$ *respectively to sums* $C_R$ *and* $C_C$, *then the series is summable* $(C, r', s')$, $r'>r$, $s'>s$, *and its rows and columns are summable* $(C, s')$ *and* $(C, r')$, *all to the same sums as before.*

THEOREM II′: *If the series* (1) *is summable* $(C, r, s)$, *and its rows and columns are summable* $(C, s)$ *and* $(C, r)$ *respectively, all to zero, then if* $0\leqq r'\leqq r$, $0\leqq s'\leqq s$,

$$C^{r',s'}(\chi,\omega)=o(\chi^r\omega^s).$$

6. **Leading results.** We now consider series (3), and deduce our main theorem:

THEOREM IV: *If, for* $k=k_0$, (3) *is summable* $(C, r, s)$ *as a double series, and if each row and each column of* (3) *is summable* $(C, s)$ *and* $(C, r)$ *respectively, as a single series, then, in the domain* $\varkappa\geqq\varkappa_0+\varrho$, $|\theta'|\leqq\Theta$, (3) *is uniformly summable* $(C, r, s')$ *as a double series, and its rows and columns are uniformly summable* $(C, s')$ *and* $(C, r)$ *respectively,* $s'$ *being the greater of the numbers* $s-\varrho$ *and* 0; *moreover, the sums of corresponding series are the same.*

We shall suppose $k_0=0$, and that the sums of (3) and each of its rows and colums are zero. We shall prove Theorem IV only (i) when $r$ and $s-\varrho$ are integral, and (ii), when $r$ and $s$ are both between 0 and 1; the other cases are treated by combinatory methods.†

(i) $r$ *and* $s-\varrho$ *integral*: We choose a value of $\varrho$ such that $s-\varrho\geqq 0$, and we consider the arithmetic mean (8) of order $(r, s-\varrho)$. The numerator of this mean is verified to be‡

$$(36)\quad C^{r,s-\varrho}(v,w)=\int_1^v\int_1^w A(t,u)\frac{\partial^2}{\partial t\,\partial u}\{(v-t)^r u^{-k}(w-u)^{s-\varrho}\}\,du\,dt.$$

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* Cf. Hobson, loc. cit., p. 96 for the one-dimensional analogue.

† H. R., V § 7, p. 33.

‡ In the case of the series (2), we have that ($p$ and $q$ being the indices of the last $l$'s less than $v$ and $w$ respectively)

If we integrate (36) $s-\varrho$ times by parts with respect to $u$, all the integrated terms vanish but one, which is $w^{-k} A^{0,s-\varrho}(t, w) \frac{d}{dt}(v-t)^r$. Besides this term, we have

$$\frac{(-1)^{s-\varrho+1}}{(s-\varrho)!}\int_1^w A^{0,s-\varrho}(t,u)\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u}\right)^{s-\varrho+1}\{(v-t)^r u^{-k}(w-u)^{s-\varrho}\}\,du.$$

Now integrate this result $r$ times by parts with respect to $t$; we obtain, after division by $v^r w^{s-\varrho}$,

$$\begin{aligned}(37)\quad v^{-r}w^{-s+\varrho}C^{r,s-\varrho}(v,w) &= v^{-r}w^{-s+\varrho}\Big[w^{-k}A^{r,s-\varrho}(v,w)\\ &+\frac{(-1)^{r+1}}{r!}w^{-k}\int_1^v A^{r,s-\varrho}(t,w)\left(\frac{d}{dt}\right)^{r+1}(v-t)^r\,dt\\ &+\frac{(-1)^{r+s-\varrho+2}}{r!\,(s-\varrho)!}\int_1^w A^{r,s-\varrho}(v,u)\left(\frac{d}{du}\right)^{s-\varrho+1}\{u^{-k}(w-u)^{s-\varrho}\}\,du\\ &+\frac{(-1)^{r+s-\varrho+2}}{r!\,(s-\varrho)!}\int_1^v\int_1^w A^{r,s-\varrho}(t,u)\left(\frac{\partial}{\partial t}\right)^{r+1}\left(\frac{\partial}{\partial u}\right)^{s-\varrho+1}\\ &\qquad\times\{(v-t)^r u^{-k}(w-u)^{s-\varrho}\}\,du\,dt\Big].\end{aligned}$$

The first of (37) is $o(1)$ by an application of Theorem II′. The second and fourth terms vanish because of the presence of the $(r+1)$st derivative of an $r$th power. The third term, being a single integral, can be shown to approach a limit uniformly as $w$ becomes infinite, by the methods used in proving Theorem 23 (a) of the Hardy-Riesz monograph.

(ii) *$r$ and $s$ both between* 0 *and* 1. We can rewrite (36) as

$$\begin{aligned}(38)\quad &v^{-r}w^{-s+\varrho}\int_1^v\int_1^w A(t,u)\frac{\partial^2}{\partial t\,\partial u}\{(v-t)^r u^{-k}(w-u)^{s-\varrho}\}\,du\,dt\\ &= v^{-r}w^{-s+\varrho}\int_1^v\int_1^w A(t,u)\frac{\partial^2}{\partial t\,\partial u}\{(v-t)^r(u^{-k}-w^{-k})(w-u)^{s-\varrho}\}\,du\,dt\\ &+r(s-\varrho)v^{-r}w^{-s+\varrho}\int_1^v\int_1^w A(t,u)(v-t)^{r-1}(w-u)^{s-\varrho-1}\,du\,dt.\end{aligned}$$

---

$$\begin{aligned}&\sum_{l_m<v}\sum_{l_n<w}c_{m,n}(v-l_m)^r(w-l_n)^s\\ &=\sum_1^{p-1}\sum_1^{q-1}A_{m,n}\Delta_1^1\{l_m^{-k}(v-l_m)^r l_n^{-k}(w-l_n)^s\}\\ &+\sum_1^{p-1}A_{m,q}\Delta_1\{l_m^{-k}(v-l_m)^r l_q^{-k}(w-l_q)^s\}\\ &+\sum_1^{q-1}A_{p,n}\Delta^1\{l_p^{-k}(v-l_p)^r l_n^{-k}(w-l_n)^s\}+A_{p,q}l_p^{-k}(v-l_p)^r l_q^{-k}(w-l_q)^s\\ &=\int_1^v\int_1^w A(t,u)\frac{\partial^2}{\partial t\,\partial u}\{v^{-k}(v-t)^r u^{-k}(w-u)^s\}\,du\,dt\end{aligned}$$

as $m$ and $n$ become infinite. In the case of the series (3), this last expression becomes (36). The $A$ must not be confused with that in § 4.

The second term of (38) is

$$v^{-r}w^{-s+\varrho-k}A^{r,s-\varrho}(v,w),$$

which is, using Theorem II′, o(1) uniformly, when $k \geqq \varrho$.

The first term of (38) we integrate by parts, obtaining

$$(39)\quad v^{-r}w^{-s+\varrho}\int_1^v\int_1^w A^{1,1}(t,u)\left(\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial u^2}\right)\{(v-t)^r(u^{-k}-w^{-k})(w-u)^{s-\varrho}\,du\,dt$$

plus single integrals which, by the methods used in the corresponding circumstances in H. R.* approach uniformly a limit as $v$ and $w$ become infinite.

We divide (39) into three integrals, $J_1$, $J_2$, and $J_3$, their integrands being $A^{1,1}(t,u)$ multiplied respectively by

(a) $\quad 2(s-\varrho)r(r-1)k(v-t)^{r-2}u^{-k-1}(w-u)^{s-\varrho-1}$,

(b) $\quad r(r-1)(s-\varrho)(s-\varrho-1)(v-t)^{r-2}(u^{-k}-w^{-k})(w-u)^{s-\varrho-2}$,

(c) $\quad r(r-1)k(k+1)(v-t)^{r-2}u^{-k-2}(w-u)^{s-\varrho}$.

To consider $J_1$, we write

$$\begin{aligned} J_1 = \int_1^v\int_1^w &= \int_1^\xi\int_1^\eta + \int_\xi^v\int_\eta^w + \int_1^\xi\int_\eta^w + \int_\xi^v\int_1^\eta \\ &= J_1' + J_1'' + J_1''' + J_1''''. \end{aligned}$$

Now, since $\sum\sum a_{m,n}$ is summable $(C,1)$, by Theorem I′, $A^{1,1}(t,u) = o(t,u)$. Therefore, we can choose $\xi$ and $\eta$ so that $|A^{1,1}(t,u)| < \varepsilon tu$ for $t > \xi$, $u > \eta$; and we also suppose that $v-\xi > 1$, $w-\eta > 1$. Again, there is a constant $M$ such that $|A^{1,1}(t,u)| < Mtu$ for all values of $t$ and $u$ greater than zero. Finally, we put $|-k|/\varkappa \leqq M$ throughout the region considered, and we write $K = 2(s-\varrho)r(r-1)$.

We now have

$$\begin{aligned} (40)\quad |J_1'| &< MM_1Kv^{-r}w^{-s+\varrho}\int_1^\xi\int_1^\eta \varkappa t(v-t)^{r-2}u^{-k}(w-u)^{s-\varrho-1}\,du\,dt \\ &< \frac{MM_1Kv^{-r}w^{-s+\varrho}(\xi^2-1)(\eta^{-k+1}-1)}{2(1-k)}, \end{aligned}$$

since $r-2$ and $s-\varrho-1$ are negative. Next,

$$\begin{aligned} (41)\quad |J_1''| &< M_1K\varepsilon v^{-r}w^{-s+\varrho}\int_\xi^v\int_\eta^w \varkappa t(v-t)^{r-2}u^{-k}(w-u)^{s-\varrho-1}\,du\,dt \\ &< \frac{M_1K}{(r-1)(s-\varrho)}\left[\varepsilon\xi v^{-1} + \frac{\varepsilon}{r-2}\right]. \end{aligned}$$

* Theorem 23 (b), pp. 42–43.

Similarly.

$$(42)\qquad |J_1'''| < \frac{M_1 M K(\xi^2-1)v^{-r}}{2(s-\varrho)},$$
$$|J_1''''| < \frac{M_1 M K}{r-1}\eta w^{-s+\varrho}\left[\xi v^{-1}+\frac{1}{r-2}\right].$$

From (40), (41) and (42), it follows that, by choosing first $\xi$ and $\eta$, then $v$ and $w$, large enough, we can make $J_1$ in absolute value as small as we please.

A similar discussion enables us to infer the same with regard to $J_2$, if we first mark the observation recorded in the corresponding treatment in H. R.*

As for $J_3$, we find that the $u$-integration tends uniformly to a limit by the process carried out in the discussion of $J_1$ in the corresponding place in H. R. (p. 42). Thence, by a process similar to that outlined above, we conclude that $J_3 \to 0$ as $v$ and $w$ become infinite.

Theorem IV is then complete.

It should, perhaps, be mentioned that the proof of the corresponding result for the series (2) does not differ materially from that outlined in Theorem IV; and all the material needed for such a discussion is included in the present paper.

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* Cf. the last paragraph in H. R., VI, § 3, (b).

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# DOUBLE ELLIPTIC GEOMETRY IN TERMS OF POINT, ORDER AND CONGRUENCE.*

By Donald A. Flanders.

**1. Introduction.** The set of axioms presented in this paper possesses two properties of particular interest, in addition to the usual ones of consistency, independence and categoricity. The first of these may be termed "dimensional categoricity", by which is meant that all propositions whose hypotheses require the existence of at least, but not more than, $n$ dimensions are demonstrable in terms of axioms which require the existence of at most $n$ dimensions. The second property may be described roughly by saying that the space is characterized by its one dimensional axioms: that is to say that a set of axioms sufficient to establish fully a space of $n(>1)$ dimensions may be obtained from the one dimensional set by removing the one dimensional closure axiom and adding successive existence axioms and a new closure axiom.

The axioms of this set are stated in terms of subclasses of a single class of undefined elements, called *points*. The notation for a point is a single capital letter, as $A$, $B$, etc. Certain triads of points are assumed to be in an undefined relation, called *order*, the notation $ABC$ (to be interpreted, "the points $A$, $B$, $C$ are in the order $ABC$") being used to indicate this relation. Certain segments (subclasses of points to be defined later) are assumed to be in the undefined relation of *congruence*, one segment being said to be congruent to another segment. The symbol $\equiv$ placed between the symbols for the two segments indicates the presence of this relation.

These are the only essentially geometrical notions postulated. The writer believes that the whole of the geometry may be constructed from these elements, using in addition the concepts and laws of pure logic. No satisfactory test of consistency and categoricity have yet been devised, however, which do not involve the assumption of the existence, consistency and categoricity of the number system of algebra. The question whether these latter ideas are themselves consequences of pure logic, without additional assumptions, is not raised in this paper, and free use will be made of algebraic concepts.

For the purposes of this article double elliptic geometry of $n$ dimensions will be taken to be isomorphic with the set of all sets of $n+1$ real numbers

* Received April 13, 1927.

$(x_1, x_2, \cdots, x_{n+1})$, subject to the condition that $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$, together with all propositions derivable from them by the operations of logic and algebra. In one and two dimensions this definition gives the geometry of the circle and of the sphere, respectively.

The axioms of the set are separated into four groups: (1) the order axioms, O, stated in terms of point and order only, and primarily concerned with the properties of the order relation; (2) the congruence axioms, C, stated in terms of point, order and congruence, and primarily concerned with the properties of the congruence relation; (3) the existence axioms, E, a succession of axioms postulating in turn the existence of the first, second and third dimensions; (4) the closure axioms, S, one for each dimension, each limiting the space to a particular dimensionality.

The segregation of the group $E$ may seem rather artificial at first glance, since points are introduced into the geometry by others of the axioms, e. g. O 2 and O 6. It will be observed, however, that the axioms E are the only ones which, in the absence of all but those axioms which are necessary to satisfy the hypothesis, demand the existence of additional points.

References to supporting theorems, axioms etc. are enclosed in square brackets [ ] and are given by letter and number. The letter T is used for theoroms, D for definitions, O, C, E and S, respectively, for order, congruence, existence and closure axioms. Frequent references to the axioms and theorems of J. R. Kline's article, "Double Elliptic Geometry in terms of Point and Order alone"*, have made it desirable to assign the symbols K-I, K-II etc. to his axioms, and K 1, K 2 etc. to his theorems.

The distinctness of points, or other entities, is not to be assumed unless explicitly stated, or unless implicit in the definition of a class to which the entities belong. Symbolic statements are to be read in the grammatical construction to which the context assigns them. Thus the statement "if $ABC$" may be interpreted "if $A$ is a point, and $B$ is a point, and $C$ is a point, and if $A, B, C$ are in the order $ABC$". The symbol $\{P\}$ is used to denote a set of points $P$, particular members of the set being differentiated by subscripts. Except in algebraic equations, the symbol $=$ is used to indicate identity. A vertical line through a symbol of relation is to mean the negation of the relation.

2. **Axioms and definitions.** O 1. If $A$ is any point, and if there exists a point $B$, distinct from $A$, then at least one such point $B$ is such that $ABX$ is false for every $X$.†

D 1. If $A$ and $B$ are such that $A \neq B$, and if $ABX$ is false for every $X$, then $B$ is said to be an *opposite* of $A$ $(B = A')$. Wherever $A'$ has

* Annals of Math., (2) vol. 18, pp. 31–44.

† With reference to axioms O, E and S, cf. Kline, loc. cit.

not been specified to be a particular opposite of $A$, the symbols, $B \neq A'$, are to mean that $B$ is distinct from *every* opposite of $A$. The points of a set $\{P\}$ will be said to be *non-opposite* if no one of them is an opposite of another.

O 2. If there exist two distinct points, $A$ and $B$, then at least two such points are such that $B$ is not an opposite of $A$.

O 3. If $ABC$, then $CBA$.

O 4. If $ABC$, and if $A'$ is an opposite of $A$, then $A'CB$.

D 2. If $A$ and $B$, and if $B \neq A'$, then the set of all points $X$ such that $X = A$, or $X = B$, or $AXB$, is called the *segment* $AB$, having $A$ and $B$ as *endpoints*, and the set of points $X$ such that $AXB$ as *interior*. The symbol $AB$ will mean the segment $AB$ whenever it occurs without immediate qualification. In the presence of O 3 and O 4 (as will be seen later [T 2]) $AB = BA$. If $B = A$, then $AB$ is called a *null* segment; otherwise, a *proper* segment.

O 5. If $A, B, C, D, E$ are points such that $ABD$, $AEC$, $C \neq B'$, $E \neq D'$, then either a point $F$ on $DE$ is such that $BFC$ or a point $G$ on $BC$ is such that $DGE$.*

D 3. Let $AB$ be any segment. Let $\{S\}$ and $\{T\}$ be any two subsets of $AB$ such that $\{S\} \neq 0$, $\{T\} \neq 0$, $\{S\} \cdot \{T\} = 0$, $\{S\} + \{T\} = AB$, and such that $S_1 T_0 S_2$ and $T_1 S_0 T_2$ are false for every $S_0, S_1, S_2, T_0, T_1, T_2$. If, for every two such sets, $\{S\}$ and $\{T\}$, there exists a point $H$ such that $SHT$ for every $S(\neq H)$ and for every $T(\neq H)$, then the segment $AB$ is said to be *continuous*.

O 6. If $AB$ is any segment, then $AB$ is continuous.

D 4. If $A$ and $B$ are such that $A \neq B \neq A'$, then the set of all points $X$ such that $X = A$, or $X = B$, or $AXB$, or $ABX$, is called the *ray* $AB$, *starting from* $A$. The ray $AB$ plus the point $A'$ is called the *halfline* $ABA'$.

C $1^{a}_{b}$. If $AB$ is any segment and $CD$ is any ray, then on ray $CD$ there exists $\{^{\text{at least}}_{\text{not more than}}\}$ one point $E$ such that $AB \equiv CE$.†

C 2. If $A, B, C, D, E$ are points such that $A \neq C \neq A'$, $D \neq E \neq D'$, $B$ lies on $AC$, $B \neq C$, $AC \equiv DE$, and if there exists a point $F$ on ray $DE$ such that $AB \equiv DF$, then at least one such point $F$ is on $DE$ and distinct from $E$.

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* Cf. G. H. Hallet, Jr., Linear order in three dimensional Euclidean and double elliptic spaces, Annals of Math. (2) vol. 21 (1921) p. 200.

† See R. L. Moore, Sets of metrical hypotheses for geometry, Trans. Amer. Math. Soc. vol. 9 (1908), pp. 487—512. Making due allowance for the different types of order in the two geometries, the following relations between his congruence axioms and ours may be observed: our C 1 is analogous to his Theorem 1, which is a consequence of his $C_1$; our C 2 contains another implication of his $C_1$; our C 3 is necessitated by our different definition of segment and has no analogue in his set; our C 4 is his $C_2$; our C 5 combines his $C_3$ and $C_4$ (Cf. his note on $C_4$).

C3. If $A \neq B \neq A'$, then $AB \equiv AA$ is false.

C4. If $AB \equiv CD$ and $CD \equiv EF$, then $AB \equiv EF$.

C5. If $A, B, C, D, E, F, G, H$ are points such that $ACD$, $D \neq A'$, $EGH$, $H \neq E'$, $B \neq A'$, $B \neq C'$, $D \neq B \neq D'$, $F \neq E'$, $F \neq G'$, $H \neq F \neq H'$, $AB \equiv EF$, $AC \equiv EG$, $AD \equiv EH$, $CB \equiv GF$, then $DB \equiv HF$.

E1. There exist at least two distinct points.

D5. If $A$ and $B$ are points such that $A \neq B \neq A'$, then the set of all points $X$ such that $X = A$, or $X = B$, or $X = A'$, or $X = B'$, or $XAB$, or $AXB$, or $ABX$, or $AX'B$, is called the *line* $AB$. In the presence of O3 and O4 (as in D2), line $AB$ = line $BA$. The points of a set $\{P\}$ will be said to be *collinear* if there exists a line containing all of them.

E2. If there exist three distinct, non-opposite points, then there exist at least three such points $A, B, C$, such that $C$ does not lie on line $AB$.

D6. If $A, B, C$ are distinct and non-collinear, and if $P$ and $Q$ are distinct and non-opposite points of distinct segments $AB, BC, CA$, then the set of all points $X$ such that $X$ lies on some segment $PQ$ is called the *triangle* $ABC$ (or $ACB$ etc.), having $A, B, C$ as *vertices*, the interiors of the segments $AB, BC, CA$ as *sides*, and the sum of the interiors of the segments $PQ$ as *interior*. The set of all points $X$ such that $X$ lies on some line $PQ$, but not on the triangle $ABC$, is called the *exterior* of the triangle.

D7. If $ABC$ is any triangle, then the triangle $ABC$ plus its exterior is called the *plane* $ABC$. The points of a set $\{P\}$ will be said to be *complanar* if there exists a plane containing all of them.

E3. If there exist three distinct, non-collinear points, then there exist at least one plane $ABC$ and one point $D$ such that $D$ does not lie on the plane $ABC$.

D8. If $A, B, C, D$ are distinct and non-complanar, and if $P$ and $Q$ are distinct and non-opposite points of distinct triangles $ABC$, $ABD$, $ACD$, $BCD$, then the set of all points $X$ such that $X$ lies on some segment $PQ$ is called the *tetrahedron* $ABCD$ (or $ABDC$ etc.), having $A, B, C, D$ as *vertices*, the interiors of the segments $AB, AC, AD, BC, BD, CD$ as *edges*, the interiors of the triangles $ABC, ABD, ACD, BCD$ as *faces*, and the sum of the interiors of the segments $PQ$ as *interior*. The set of all points $X$ such that $X$ is on some line $PQ$, but not on the tetrahedron $ABCD$, is called the *exterior* of the tetrahedron.

D9. If $ABCD$ is any tetrahedron, then the tetrahedron, $ABCD$ plus its exterior is called the *three-space* $ABCD$.

S1. If there exist points $A$ and $B$ such that $A \neq B \neq A'$, then at least two such points are such that every point lies on the line $AB$.

S2. If there exist three distinct, non-collinear points, then there exist three such points, $A$, $B$, $C$, such that every point lies on the plane $ABC$.

S3. If there exist four distinct, non-complanar points, then there exist four such points, $A$, $B$, $C$, $D$, such that every point lies on the three-space $ABCD$.

3. **One dimension.** T1. If $A$ and $B$ are points such that $A \neq B \neq A'$, then there exists a point $C$ such that $ABC$ [K1].

T2. If $B$ is an opposite of $A$, then $A$ is an opposite of $B$ [K2].

T3. If $ABC$, then $C \neq A$ and $C \neq B'$ [Part of K3].

T4. If $ABC$, $ACD$, $D \neq B'$, then $BCD$ [Weak form of K12].

*Proof.* $C \neq C'$ [D1]. $ABC$, $ACD$, $D \neq B'$, $C \neq C'$ imply that some point $F$ on $CC$ is such that $DFB$ [O5] since $G$ on $DB$ and such that $CGC$ is impossible [$T$3]. $F$ on $CC$ implies $CFC$ or $F = C$ [D2]. $CFC$ is impossible [T3]. Hence $F = C$, i. e. $BCD$.

T5. If $ABC$, then $C \neq B$ and $C \neq A'$ [Remainder of K3].

*Proof.* (1) Suppose $ABB$. Then $B \neq B'$ [D1]; and $ABB$, $ABB$, $B \neq B'$, imply $BBB$ [T4], which is impossible [T3]. (2) Suppose $ABA'$. Then $A'A'B$ [O4], and $BA'A'$ [O3], which is impossible by case 1.

T6. If $ABC$, then not $ACB$.

*Proof.* $B \neq B'$ [D1]. $ABC$, $ACB$, $B \neq B'$ imply $BCB$ [T4], which is impossible [T3].

T7. If $ABC$ and $ACD$, then $BCD$ [K12 in full].

*Proof.* The theorem is immediate if it can be shown that $D = B'$ is impossible [T4]. Now $D = B'$ implies $ACB'$ [hyp.], which implies $B'CA$ [O3], which implies $BAC$ [T2, O4], which implies $CAB$ [O3]. $CAB$ implies not $CBA$ [T6], which implies not $ABC$ [O3], contradicting the hypothesis.

T8. If $ABC$ and $ACD$, then $ABD$ [K–V].

*Proof.* $ABC$ implies $CBA$ [O3], and $ACD$ implies $DCA$ [O3], $D'AC$ [O4], $CAD'$ [O3]. Then $CBA$, $CAD'$ imply $BAD'$ [T7], which implies $ABD$ [O3, T2, O4, O3].

It is easily seen that Kline's first two axioms are equivalent to O1, O2 and E1. His third and fourth axioms are identical with O3 and O4, respectively. His fifth axiom has been shown by T8 to be a consequence of certain axioms of our set. Since his first sixteen theorems are consequences of his first five axioms, we shall assume without further proof such of these theorems as have not already been proved explicitly.

The implications of $ABC$ with respect to the points $A, B, C, A', B', C'$ and the possible orders connecting these points are of frequent occurrence. For the sake of brevity we collect these statements in the next theorem and we shall use the theorem without explicit reference to it. The proof of the theorem will be omitted, since some parts of it have already been

demonstrated and the remainder are, in each case, almost immediate consequences of axioms and theorems already stated, or of simple combinations of them.

T9. $ABC$ implies the distinctness of every two of the points $A$, $A'$, $B$, $B'$, $C$, $C'$; $ABC$ implies the truth of the orders $ABC$, $CBA$, $A'CB$, $BCA'$, $BAC'$, $C'AB$, $B'A'C$, $CA'B'$, $AC'B'$, $B'C'A$, $A'B'C'$, $C'B'A'$; $ABC$ implies the falsity of every other order connecting points selected from $A$, $A'$, $B$, $B'$, $C$, $C'$.

T10. If $AB$ is any ray and $C(\neq A)$ is any point on ray $AB$, then $B$ is on ray $AC$.

*Proof.* This is evidently an immediate consequence of D4.

T11. If $C$ is on ray $AB$, then $C'$ is not on ray $AB$.

*Proof.* $C$ on ray $AB$ implies $C = A$, or $C = B$, or $ACB$, or $ABC$; $C'$ on ray $AB$ implies $C' = A$, or $C' = B$, or $AC'B$, or $ABC'$ [D4]. Comparison with D4 and T9 shows that no one of the first four statements is compatible with any one of the second four.

T12. If $AB$ is any segment, then $AB \equiv AB$.*

*Proof.* (1) Suppose $AB$ is a null segment, $AA$. Let $K$ be any point such that $A \neq K \neq A'$ [K 6]. Then on ray $AK$ there exists $C$ such that $AA \equiv AC$ [C1a]. Suppose $C \neq A$. Then $ACK$, or $C = K$, or $AKC$, any one of which implies $C \neq A'$. Hence there exists ray $AC$, and on it a point $D$ such that $AC \equiv AD$ [C1a] and such that $D \neq A$ [C3]. Hence $AA \equiv AD$ [C4] and $C = D$ [C1b], i. e. $AC \equiv AC$. If we now identify the points $A$, $B$, $C$, $D$, $E$ of C2 with the points $A$, $A$, $C$, $A$, $C$, respectively of this theorem, we see that the hypothesis of this axiom is satisfied in such a way that only the point $C$ on ray $AC$ is such that $AA \equiv AC$, and therefore $C \neq C$, which is impossible. Hence $C = A$, i. e. $AA \equiv AA$.

(2) Suppose $AB$ is a proper segment. Then there exists $C(\neq A)$ on ray $AB$ such that $AB \equiv AC$ [C1a, C3]. As in case 1, it can be shown that $AC \equiv AC$. Then $C(\neq A)$ on ray $AB$ implies $ACB$, or $ABC$, or $C = B$. Either $ACB$ or $ABC$ implies $C \neq B$. If $ACB$, then by identifying the points $A$, $B$, $C$, $D$, $E$ of C2 with the points $A$, $C$, $B$, $A$, $C$ of this theorem, it results (as in case 1) that $C \neq C$. The order $ABC$ gives rise in a similar manner to the same contradiction. Hence $C = B$, i. e $AB \equiv AB$.

T13. If $AB \equiv CD$, then $CD \equiv AB$.

*Proof.* Let $K$ be a point such that $A \neq K \neq A'$ and such that $B$ lies on ray $AK$. (If $AB$ is a proper segment, let $K = B$). Then there exists $E$ on ray $AK$ such that $CD \equiv AE$ [C1a]. $AB \equiv AE$ [C4] and $AB \equiv AB$ [T12]. Hence $B = B$ [C1b], i. e. $CD \equiv AB$.

* Several of the next few theorems, which deal with the properties of the congruence relation, are the counterparts of theorems of Moore, loc. cit.

*T*14. If $AB \equiv CD$ and $AB \equiv EF$, then $CD \equiv EF$.

*Proof.* This follows at once from T13 and C4.

T15. If $AB$ is any proper segment and $CC$ is any null segment, then $AB \nmid CC$ and $CC \nmid AB$.

*Proof.* Suppose $AB \equiv CC$. Let $K$ be any point such that $C \nmid K \nmid C'$, and let $D$ be any point on the interior of $AB$. Then there exists exactly one point $E$ on ray $CK$ such that $AD \equiv CE$. Now $E \nmid C$, since $E = C$ implies $AD \equiv AB$ [T13, C4] and (since $AD \equiv AD$) $D = B$, contrary to the assumption that $D$ was interior to $AB$. Then $CE \equiv AD$ and $CC \equiv AB$; hence $B$ lies on the segment $AD$ and is distinct from $D$ [C2]. This implies either $B = A$, contrary to hypothesis, or $ABD$, which contradicts the assumption that $D$ was on $AB$. Hence $AB \nmid CC$.

$CC \nmid AB$ is an immediate consequence of this and T13.

T16. If $AA$ and $BB$ are null segments, then $AA \equiv BB$.

*Proof.* Let $K$ be any point such that $B \nmid K \nmid B'$, and let $C$ be the point on ray $BK$ such that $AA \equiv BC$. $C \nmid B$ is impossible [T15]. Hence $AA \equiv BB$.

D10. If $AB$ and $CD$ are segments, and if there exists $E(\nmid D)$ on $CD$ such that $AB \equiv CE$, then $AB$ is said to be *less than* $CD$ ($AB < CD$), and $CD$ is said to be *greater than* $AB$ ($CD > AB$).

T17. If $AB$ and $CD$ are any segments, then exactly one of the statements, $AB < CD$, $AB \equiv CD$, $AB > CD$, is true.

*Proof.* (1) Suppose that both $AB$ and $CD$ are null segments. Then $AB \equiv CD$ [T16]. Neither $AB < CD$ nor $AB > CD$ is possible [D10]. Hence only the statement $AB \equiv CD$ is true.

(2) Suppose $AB$ is a null segment and $CD$ is a proper segment. Then $C$ is the only point $E$ on ray $CD$ such that $AB \equiv CE$ [T16, C1]. Since $C$ is on $CD$ and $C \nmid D$, $AB < CD$ [D10]. $AB \equiv CD$ is impossible [T15]. $AB > CD$ is impossible [D10, T3, T15]. Hence only the statement $AB < CD$ is true.

(3) Suppose $AB$ is a proper segment and $CD$ is a null segment. By reasoning similar to that in case 2, only the statement $AB > CD$ is true.

(4) Suppose both $AB$ and $CD$ are proper segments. Let $E(\nmid C)$ be the point on ray $CD$ such that $AB \equiv CE$. Then exactly one of the statements $CED$, $E = D$, $CDE$ is true [D4]. (a) If $CED$, then $AB < CD$. $AB \equiv CD$ is impossible, since it would imply $D = E$, which contradicts $CED$. If $AB > CD$, then there exists an interior point $F$ of $AB$ such that $CD \equiv AF$ [D10, T15]; $AFB$, $AF \equiv CD$, $AB \equiv CE$, $D$ on ray $CE$ would imply $CDE$ [C2], contrary to the assumption $CED$. Hence $CED$ implies only $AB < CD$. (b) If $E = B$, then $AB \equiv CD$. $AB < CD$ and $AB > CD$ are evidently impossible [C1]. (c) If $CDE$, then there exists

exactly one point $F$ on ray $AB$ such that $CD \equiv AF$, and $F$ is interior to $AB$. As in case (a), only the statement $CD < AB$, i. e. $AB > CD$, is true.

T18. If $APB$ and $AQB$, then $APQ$, or $P = Q$, or $AQP$ [K25].

*Proof.* $A \dagger P \dagger B$ and $A \dagger Q \dagger B$. Exactly one of the statements, $AP < AQ$, $AP \equiv AQ$, $AP > AQ$, is true [T17]. If $AP < AQ$, then there exists just one point $R$ such that $ARQ$ and $AP \equiv AR$. $ARQ$ and $AQB$ imply $ARB$ [T7], i. e. $R$ on ray $AB$. Hence $P = R$ [C1], i. e. $APQ$.

If $AP \equiv AQ$, then $P = Q$.

If $AP > AQ$, then $AQ < AP$ and, by reasoning similar to above, $AQP$.

T19. If $ABC$ and $ABD$, then $ACD$, or $C = D$, or $ADC$ [K24].

*Proof.* $ABC$ implies $A'CD$, $ABD$ implies $A'DB$. Hence $A'DC$, or $C = D$, or $A'CD$ [T18]. $A'DC$ implies $ACD$, and $A'CD$ implies $ADC$.

T20. If $ABC$ and $ABD$, then $BCD$, or $C = D$, or $BDC$ [K23].

*Proof.* This follows easily from T19 and T7.

T21. If $ABC$ and $BCD$, then $ABD$, or $D = A'$, or $AB'D$ [K22].

*Proof.* Since $ABC$ implies $BCA'$, this theorem is a ready consequence of T19.

T22. If $ABC$ and $BCD$, then $D$ is on line $AB$ [K17].

*Proof.* $BCD$ implies $CDB'$, $ABC$ implies $CBA$. $CBD'$ and $CBA$ imply $BAD'$, or $BD'A$, or $D' = A$ [T29]. Hence $ABD$, or $BDA'$, or $D = A'$; i. e. $D$ lies on line $AB$ [D5].

The proofs of K18 and K19 are based on Kline's first five axioms and on his Theorem 17; they do not involve his Axiom VII explicitly. Since we have now demonstrated K17 [T22], we shall assume the truth of K18 and K19.

Kline's axiom X (a form of the Dedekind cut postulate) is a consequence of D3 and O6, since every segment in his sense is a segment in our sense.

We have now assumed or proved each of Kline's specifically one dimensional axioms and theorems which are stated in terms of point and order. However, since the existence of congruence relations in his geometry depends upon three dimensional concepts, we shall not now attempt to show separately that every theorem in one dimension in terms of point and order which is a consequence of his set of axioms is a consequence of this set. That will follow immediately when this set has been established as categorical.

T23. If $ABC$, $DEF$, $AB \equiv DE$, $AC \equiv DF$, then $BC \equiv EF$.

*Proof.* $ABC$ implies $C \dagger A'$, $B \dagger C$, $B \dagger A'$, $B \dagger C'$. Similarly $DEF$ implies $F \dagger D \dagger E$ and $F \dagger E \dagger F'$. $BB \equiv EE$ [T14]. If now the points $A, B, C, D, E, F, G, H$ of C5 are identified with the points $A, B, B, C, D, E, E, F$ of this theorem, the conditions of the hypothesis of C5 are

fulfilled and we have $CB \equiv FE$. Since $CB = BC$ and $FE = EF$ [D2], $BC \equiv EF$.

The theorem just proved constitutes a specialization of C5 which is more readily applicable to one dimensional theorems.

T24. If $ABC$, $DEF$, $AB \equiv DE$, $BC \equiv EF$, then $AC \equiv DF$.

*Proof.* The point $G$ on ray $DF$ such that $AC \equiv DG$ [C1] is such that $DEG$ [C2]. Also, $BC \equiv EG$ [T23] and $EF \equiv EG$ [T14]. $DEF$ and $DEG$ imply $G$ on ray $EF$ [T20]. Hence $F = G$, i. e. $AC \equiv DF$.

T25. If $AB \equiv CD$, then $A'B \equiv C'D$.

*Proof.* If $A'B < C'D$, there exists $E$ such that $A'B \equiv DE$ and $DEC'$, i. e. such that $CDE$. Then there exists $F$ such that $CE \equiv AF$ and $ABF$. Then $BF \equiv DE$ [T24] and $BF \equiv BA'$, i. e. $F = A'$, which contradicts $ABF$. Similarly, $A'B > C'D$ is impossible.

It is now possible to develop a theory of addition and subtraction of segments, either preserving the restrictions imposed by our type of order, or by extending order and congruence by definition. It is also possible to prove the various theorems of the type, "If $AB < CD$ and $CD < EF$, then $AB < EF$", and "If $AB < CD$ and $CD \equiv EF$, then $AB < EF$" etc.

D11. If $AB$ is any segment and $C$ is a point of $AB$ such that $AC \equiv BC$, then $C$ is called a *midpoint* of $AB$. If $ABA'$ is a half-line and $C$ is a point of $ABA'$ such that $AC \equiv A'C$, then $C$ is said to be a *midpoint* of $ABA'$, and $AC$ and $A'C$ are called *quadrants*.

T26. If $AB$ (or $ABA'$) is any segment (or half-line), then there exists exactly one midpoint of $AB$ (or of $ABA'$).

*Proof.* If $AB$ is a null segment, the point $A$ is evidently a midpoint, and the only midpoint, of $AB$. In the case of a proper segment $AB$ or a half-line $ABA'$, the proof of the existence of a midpoint involves a familiar type of argument based on O6, and differs only in detail from R. L. Moore's proof of his Theorem 1, section 5*. The uniqueness of the midpoint is evidently assured by C2.

T27. If $AB \equiv CD$, and if $E$ and $F$ are the midpoints of $AB$ and $CD$ respectively, then $AE \equiv CF$.

T28. All quadrants are congruent.

The truth of these theorems follows from C 2.

The space under consideration has now been developed to a point where it seems unnecessary to carry detailed investigation further. With the help of the continuity axiom [O 6], it is possible to assign to each point $P$ of our space a distinct real number $\alpha$ $(0 \leqq \alpha < 2\pi)$ in such a way that (1) to each real number $\alpha$ $(0 \leqq \alpha < 2\pi)$ there corresponds exactly one point $P$;

* R. L. Moore, loc. cit., p. 499.

(2) points $P_1, P_2, P_3$ are in the order $P_1 P_2 P_3$ if, and only if, there exist numbers $\beta_1, \beta_2, \beta_3$ ($\beta_i = 0, 1$) such that either (a) $\alpha_1 + 2\beta_1\pi < \alpha_2 + 2\beta_2\pi < \alpha_3 + 2\beta_3\pi$ and $\alpha_3 + 2\beta_3\pi - (\alpha_1 + 2\beta_1\pi) < \pi$ or (b) $\alpha_1 + 2\beta_1\pi > \alpha_2 + 2\beta_2\pi > \alpha_3 + 2\beta_3\pi$ and $\alpha_1 + 2\beta_1\pi - (\alpha_3 + 2\beta_3\pi) < \pi$; (3) $P_1 P_2 \equiv P_3 P_4$ if, and only if, one of the quantities, $\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 2\pi + (\alpha_1 - \alpha_2), 2\pi + (\alpha_2 - \alpha_1)$, is equal to one of the quantities $\alpha_3 - \alpha_4, \alpha_4 - \alpha_3, 2\pi + (\alpha_3 - \alpha_4), 2\pi + (\alpha_4 - \alpha_3)$. This procedure is equivalent to establishing one dimensional double elliptic geometry, as defined in the first section. This constitutes a demonstration of the consistency and categoricity of the set of axioms composed of O, C, E1, S1.

The independence of the set will be established in the usual manner by exhibiting, for each axiom Xn, a geometry GXn in which axiom Xn is false and the remaining axioms are true. Detailed proof of the validity of the examples is omitted.

G O 1. The set of all points in space is the set of all points on a Euclidean straight line; order and congruence have their usual significance.*

G O 2. There exist only the points $Q$ and $R$; every order is false; every segment is congruent to every segment.

G O 3. There exist exactly three points, $Q, R, S$; only the orders $QQS$, $QSS$, $RSQ$, $RSS$, $SSS$ are true; every null segment is congruent to every null segment, every proper segment is congruent to every proper segment, but no other congruence relations are true.

G O 4. There exist exactly three points, $Q, R, S$; only the orders $QRS$ and $SRQ$ are true; congruence is defined as in G O 3.

G O 5. There exist exactly four points, $P, P', Q, Q'$; only the orders $PPQ$, $QPP$, $P'P'Q$, $QP'P'$, $PPQ'$, $Q'PP$, $P'P'Q'$, $Q'P'P'$, $PQP'$, $P'QP$, $PQ'P'$, $P'Q'P$ are true; congruence is defined as in G O 3.

G O 6. The set of all points is the set of all points on the unit circle each of which can be obtained from the point (1,0) by a rotation about the center of the circle through an angle which is a rational multiple or sub-multiple of $2\pi$.

In the examples for C 1–C 5 the set of all points is the set of all points lying on a circle; order has its usual significance for points lying on an arc subtending a central angle less than $\pi$.

G C 1 a. $AB \equiv CD$ if, and only if, the following conditions are satisfied simultaneously: (1) the segments are congruent in the ordinary sense; (2) a particular point $O$ is not an endpoint of either segment.

G C 1 b. Every null segment is congruent to every null segment, every proper segment is congruent to every proper segment, but no other congruence relations are true.

* Many of these examples will be found to be identical with, or slight modifications of, examples to be found in the papers of R. L. Moore and Kline already referred to.

GC 2. $AB \equiv CD$ if, and only if, $CD$ is congruent in the ordinary sense to a particular segment $PQ$.

GC 3. Every segment is congruent to every null segment.

GC 4. $AB \equiv CD$ if, and only if, the central angle subtended by $CD$ is equal to $\frac{2}{3}\pi$ plus one-third of the central angle subtended by $AB$.

GC 5. In order to define congruence three points, $Y_1$, $Y_2$, $Y_3$, are selected at intervals of 120° along the circle. Let $l(AB)$ be the length of the shorter arc $AB$, measured in terms of 120° of arc as a unit. Let $m(AB)$, called the measure of $AB$, be defined as follows: (1) If $AB$ lies wholly on one of the shorter arcs $Y_1Y_2$, $Y_2Y_3$, $Y_3Y_1$, then $m$ is given as a function of $l$ by the relations

$$0 \leqq l \leqq \frac{1}{2}, \quad m = \frac{1}{2}l \quad \text{and} \quad \frac{1}{2} \leqq l \leqq 1, \quad m = \frac{1}{2}(3l-1)$$

and $l$ is given as a function of $m$ by the relations

$$0 \leqq m \leqq \frac{1}{4}, \quad l = 2m \quad \text{and} \quad \frac{1}{4} \leqq m \leqq 1, \quad l = \frac{1}{3}(2m+1);$$

(2) if $AB$ lies so that any of the points $Y_i$ separate it into distinct subsegments so that each of these subsegments lies wholly on one of the shorter arcs $Y_iY_j$, then the measure of $AB$ is taken as the sum of the measures of the subsegments.

Then $AB \equiv CD$ if, and only if, $m(AB) = m(CD)$.

GE 1. There exists only one point; every order is false; there is no congruence.

GS 1. Double elliptic geometry of two dimensions.

4. **Two dimensions.** T 29. Two distinct lines can have at most a point and its opposite in common [K 20].

*Proof.* As in the case of K 18 and K 19 (v. section 3), this theorem is a consequence of Kline's Axioms I–V and his Theorem 17, so that his proof may be accepted.

T 30. If $AB$ is any line and $C$ is any point not on line $AB$, then there exist an unique line $AB$ and an unique line $BC$, and lines $AC$ and $BC$ are distinct from each other and from line $AB$.

*Proof.* The existence and uniqueness of lines $AC$ and $BC$ are assured by $A \nmid C \nmid A'$ and $B \nmid C \nmid B'$ [D 5]. The distinctness of lines $AC$, $BC$ and $AB$ is assured by the fact that $C$ is not on line $AB$ [K 19].

T 31. If $D$ is not on line $AB$, and if $C$ and $E$ are such that $ABC$ and $BED$, and if there exists $F$ such that $DFA$, then there exists $G$ such that $CEG$ and $DGA$ [K–VI].

*Proof.* There exist distinct lines $AB$, $AD$, $BD$ [T 30]. $ABC$ implies $BAC'$. $D$ not on line $AB$ implies $D \nmid A'$ [D 5, T 29]. $BED$ implies $E$ on line $BD$

and $B \dagger E \dagger D'$ and these, together, imply $C \dagger E \dagger C'$ [T 29, T 30]. $BAC'$, $BED$, $D \dagger A'$, $E \dagger C$ imply the existence of $G$ on $C'E$ such that $AGD$ [C 5].

$G$ on $C'E$ implies $G = C'$, or $G = E$, or $C'GE$. But $G = C'$ implies $G$ on line $AB$, while $AGD$ implies $G$ on line $AD$, i. e., since lines $AB$ and $AD$ are distinct, $G = A$, or $G = A'$; either of which contradicts $AGD$. Similarly, $G = E$ implies $G = D$ or $G = D'$, either of which contradicts $AGD$. Hence $C'GE$ and $AGD$, i. e. $CEG$ and $DGA$.

We have shown in the preceeding section that Kline's first five axioms are consequences of certain of our one dimensional axioms (not including S 1); his Axiom VI has just been proved [T 31]; his Axiom VII appears in our set under different wording as E 2. Hence we shall accept his Theorems 1–36, which are based on his Axioms I–VII, as consequences of our set.

D 12. Let $AB$ be a line and $C$ a point not on the line. Then $C$ plus the set of all interior points of segments having $C$ as one endpoint and a point of line $AB$ as the other endpoint, constitute the *half-plane* $C—AB$. It is evident from D 7, K 34, K 35 and K 36, that any line of a plane separates the plane into exactly two half-planes, and that the half-plane $C—AB$ is identical with the half-plane $D—AB$, where $D$ is any point of $C—AB$. Also, by applying K 30 and D 5 to the triangles into which a point $D$ of $C—AB$ divides $C—AB$, we may conclude that a point and its opposite do not lie in the same half-plane.

T 32. If $ABC$ and $EFG$ are triangles such that $AB \equiv EF$, $AC \equiv EG$, $CB \equiv GF$, and if $D$ and $H$ are points of rays $AC$ and $DG$, respectively, and such that $AD \equiv EH$, then $BD \equiv FH$.

*Proof.* By hypothesis, $D = A$, or $D = C$, or $ADC$, or $ACD$. If $D = A$ or $D = C$, then $H = E$ or $H = G$, respectively [C 1], and the theorem is trivial. If $ACD$, then $EGH$ [C 2], and the theorem is a special case of C 5. If $ADC$, then $EGH$; $AC \equiv EG$ implies $A'C \equiv E'G$ and $AB \equiv EF$ implies $A'B \equiv E'F$ [T 25]. Also, $A'CD$ and $E'GH$. The theorem again reduces to C 5.

D 13. If $ABC$ is any triangle, then the rays $AB$ and $AC$ are said to form the *angle* $BAC(\angle BAC)$, whose *vertex* is $A$, and whose *sides* are the rays $AB$ and $AC$ without the vertex. Those points of the plane $ABC$ which are common to the half-planes $B—AC$ and $C—AB$ form the *interior* of $\angle BAC$, and those points of the plane $ABC$ which are neither on the angle $BAC$ nor in its interior, except $A'$, form its *exterior*. The following is evidently an equivalent definition of the exterior and interior of the angle: if $P$ and $Q$ are non-opposite points of distinct sides of the angle, then the set of all points interior to some $PQ$ form the interior of the angle, and the set of all points on some line $PQ$, but not on the segment $PQ$, form the exterior of the angle.

D 14. $\angle BAC$ is said to be *congruent* to $\angle EDF$ ($\angle BAC \equiv \angle EDF$) if there exist points $K$, $L$, $M$, $N$ such that $K$ and $L$ lie on sides $AB$ and $AC$, respectively, of $\angle BAC$, $M$ and $N$ lie on sides $DE$ and $DF$, respectively, of $\angle EDF$, $AK \equiv DM$, $AL \equiv DN$, and $KL \equiv MN$.

T 33. If $ABC$ and $DEF$ are triangles such that $\angle BAC \equiv \angle EDF$, $AB \equiv DE$, $AC \equiv DF$, then $BC \equiv EF$, $\angle ABC \equiv \angle DEF$ and $\angle ACB \equiv \angle DFE$.

*Proof.* Let $K$, $L$, $M$, $N$ be points (v. D 14) by virtue of which $\angle BAC \equiv \angle EDF$. By T 32, $KC \equiv MF$, and, applying the theorem again, $BC \equiv EF$. Then $\angle ABC \equiv \angle DEF$ and $\angle ACB \equiv \angle DFE$ [D 14].

T 34. If $BAC$ is any angle and $D$–$EF$ is any half-plane, then in $D$—$EF$ there exists one and only one ray $EG$, starting from $E$ and such that $\angle BAC \equiv \angle GEF$.

*Proof.* Let $P$ be the midpoint of $ABA'$ and $Q$ be the midpoint of $ACA'$. Let $R$ be the midpoint of $EDE'$ and $S$ be the midpoint of $EFE'$. $R$ and $S$ are distinct and non-opposite [D 12]. On ray $SR$ let $T$ be the point such that $QP \equiv ST$. In triangles $ESR$ and $E'SR$, $ES \equiv E'S$ and $ER \equiv E'R$ [T 25], and $SR \equiv SR$. Also, $T$ lies on ray $SR$ and $ST \equiv ST$. Hence $ET \equiv E'T$ [T 32]. In triangles $EST$ and $AQP$, $ES \equiv AQ$ and $ET \equiv AP$ [T 25], and $ST \equiv QP$. Hence $\angle BAC \equiv \angle TEF$.

Suppose there exists $H$, not on ray $ET$, in $D$—$EF$ and such that $\angle BAC \equiv \angle HEF$. Ray $EH$, except for $E$, lies in $D$—$EF$, and line $EH$ lies in plane $DEF$. Line $EH$ intersects line $RS$, which also lies in plane $ABC$ [K 35], in two opposite points, $J$ and $J'$. One of these points, say $J$, lies in $D$—$EF$. Then $EJ \equiv E'J$ (as above for $ET \equiv E'T$). Hence, in triangles $AQP$ and $ESJ$, $AQ \equiv ES$, $AP \equiv EJ$, $\angle PAQ \equiv \angle JES$, and therefore $PQ \equiv JS$. But since $J$ lies in $D$–$EF$ and on line $SR$, it lies on ray $SR$, i.e. $J = T$. Hence there is one and only one ray satisfying the conditions of the theorem.

In the chapter on pure spherics in his "Rational geometry",* Halsted developes two dimensional double elliptic geometry rather fully. His set of axioms is consistent and, except for the lack of an explicitly stated closure axiom, categorical. In order to demonstrate the consistency and categoricity of our set of axioms, O, C, E 1, E 2, S 2, we shall show that his undefined notions may be defined in terms of ours in such a way that each of his axioms is a logical consequence of our set.

He assumes as undefined the notions of point, straightest and sphere. These we define as identical with our point, line and plane, respectively. The implicit assumption is made that all points and straightests lie on a single sphere; the truth of this assumption in our geometry is assured by

* G. B. Halsted, Rational geometry, Chapter 15.

K 35 and S 2. The opposite of a point $A$ is defined as a point $A'$ ($\neq A$) which with $A$ does not determine a straightest; in our geometry the opposite of a point is a point, and the only point, having this property. All the axioms of his first group, the "assumptions of association on the sphere", now follow at once: I 1′ is equivalent to O 1; I 2′ is a part of D 5; the first part of I 3′ (which is superfluous in his set) follows from K 19; the second part of I 3′ is implicit in D 5; I 4′ is equivalent to E 1, O 2, E 2.

The undefined relation of betweenness is next introduced. We may define this relation as follows: a point $B$ is said to lie between points $A$ and $C$ if, and only if, $ABC$ or $CBA$. He defines a sect as a pair of distinct non-opposite points, the points lying between these points being said to be the points of the sect. A sect is, then, our proper segment. Under his definition a ray is equivalent to our ray minus the starting point. Of his second set of axioms, the "assumptions of betweenness on the sphere", II 1′, II 2′, II 4′, II 6′ and II 8′ are special cases of T 9; II 5′ is T 1; II 3′ is a consequence of T 1 and O 3; II 7′ is T 8; II 9′ is a consequence of K 28 and K 36; and II 10′ is a consequence of D 4 and D 5.

Halsted then introduces the undefined relation of congruence between sects. We define sect $AB$ congruent to sect $CD$ if, and only if, segment $AB$ is congruent to segment $CD$. The points which lie on one side of a line in his terminology form what we call a half-plane. His definition of an angle and of the exterior and interior of an angle are fully equivalent to ours. He also assumes the relation of congruence between angles; this we may define as in D 13. We can now show at once that the axioms of his last set, the "assumptions of congruence on the sphere", are also consequences of our set. The first part of III 1′ is a re-wording of C 1 as applied to proper segments; $AB \equiv AB$ is T 10, while $AB \equiv BA$ is superfluous. III 2′ is T 22. III 3′ contains the implicit assumption that if points $A, B, C, D, E, F$ are such that $ABC$, $D \neq E \neq D'$, $F$ on ray $ED'$, $AB \equiv DE$, $BC \equiv EF$, then $F$ is such that $DEF$. The proof of T 21 demonstrates the truth of this assumption, and the axiom then becomes a restatement of T 21. The first part of III 4′ is equivalent to T 34, the location of the inner points of the angle being assured by D 13 and C 2; under our definition of congruence of angles, the latter part of III 4′ follows from the corresponding propositions for segments. III 5′ is an immediate consequence of T 22. III 6′ is equivalent to T 36. This completes the list of Halsted's assumptions.

As in the one dimensional case independence will be demonstrated by examples. Since in nearly every case these examples are merely slight modifications of those used in the preceeding section, only the modifications will be cited wherever possible.

GO 1 is a Euclidean plane.

GO 2–GO 5 are unchanged.

GO 6 is obtained from the corresponding one dimensional example by substituting "sphere" for "circle" and "point (1, 0, 0)" for "point (1, 0)".

GC 1–GC 5 are obtained from the corresponding one dimensional examples by substituting "sphere" for "circle" and "arc of a great circle" for "arc".

GE 1 remains unchanged.

GE 2. One dimensional double elliptic geometry.

GS 2. Three dimensional double elliptic geometry.

5. **Three dimensions.** We may take the geometry resulting from Kline's full set of axioms as a representation of this geometry. We have already shown that Kline's Axioms I–VII and X are consequences of our Axioms O, C, E 1 and E 2. His Axioms VIII and IX are identical in content with E 3 and S 3, respectively. Hence we may assert that the set of Axioms O, C, E and S 3 are sufficient to determine three dimensional double elliptic geometry, i. e. this set is consistent and categorical.

The independence examples of the preceeding section for O, C, E 1, E 2 may evidently be modified to serve for three dimensions in a manner exactly analogous to that in which the examples in section 3 for O, C, E 1 were modified to suit two dimensions. We may take GE 3 and GS 3 as two and four dimensional double elliptic geometry, respectively.

It seems clear that the number of dimensions may be extended indefinitely by the removal of S 3 and the addition of suitable existence and closure axioms.

In the preparation of this paper the writer has encountered certain problems which he has not yet solved. These relate to the further subdivision of certain of the axioms here presented. For example, each of the axioms, O 5 and C 5, might be subdivided into two parts, one applying to one dimension and the other to two dimensions. The writer has constructed a consistent and categorical set based on this modification of the present set, but he has not been able to settle the independence of the two dimensional form of O 5. Axiom O 6 might also be subdivided by assuming continuity for a single segment only and adding some form of Archimedean axiom. This can be done for the one dimensional set without sacrificing independence, but the writer has not found a two dimensional geometry of the double elliptic type in which the Archimedean axiom holds but the continuity axiom does not.

In conclusion the writer wishes to thank most sincerely Professor J. R. Kline, whose advice and encouragement have been invaluable in the preparation of this paper.

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# THE REPLACEMENT THEOREM AND RELATED QUESTIONS IN THE PROJECTIVE GEOMETRY OF PATHS.†

By Tracy Yerkes Thomas.

1. **Introduction.** This paper on the projective theory of affinely connected manifolds $\mathfrak{M}$ or *geometry of paths*, as it is also called, is one of a series of papers of which I shall give the following brief account. In 1925 in a note, *On the projective and equi-projective geometries of paths* in the Proceedings of the National Academy of Sciences, vol. 11, pp. 199–203, I introduced the projective connection $\Pi^i_{\alpha\beta}$ and stated the fundamental problem of the projective geometry as the invariant theory of its equations of transformation. Simultaneously with this paper there appeared in the same issue of the Proceedings, pp. 204–207, a paper by O. Veblen and J. M. Thomas, entitled: *Projective normal coördinates for the geometry of paths*, in which these authors used the projective connection $\Pi^i_{\alpha\beta}$ to set up a type of normal coördinates for the projective geometry by means of which they defined a set of non-tensor invariants $Q$ which are further considered in the following paper. Although not directly related to the projective theory of the manifold $\mathfrak{M}$ reference should also be made to another paper of mine: *The identities of affinely connected manifolds* in the Mathematische Zeitschrift, vol. 25 (1926), pp. 714–722, where the idea of the complete set of identities of an invariant of the manifold, which is used in the present paper, is introduced. In this paper the complete sets of identities are determined with more or less detail of calculation for the affine theory of the general affinely connected manifold and for the Riemann geometry. Finally in an article under the title: *A projective theory of affinely connected manifolds* in the Mathematische Zeitschrift, vol. 25 (1926), pp. 723–733, I showed how a complete set of projective tensor invariants $^*A$ could be obtained by replacing the original $n$-dimensional manifold $\mathfrak{M}$ by an $(n+1)$-dimensional manifold $^*\mathfrak{M}$ having affine connection $^*\Gamma^i_{\alpha\beta}$ determined in a simple manner by the affine connection $\Gamma^i_{\alpha\beta}$ of the manifold $\mathfrak{M}$.

In the paper last mentioned the replacement theorem‡ is given subject to an apparent restriction, namely that the invariants considered be ex-

† Received April 20, 1927.

‡ This theorem appears to correspond to what is called the "Reduktionssatz" by German writers. However I have never come across a proof of the "Reduktionssatz" which has seemed satisfactory to me (see, for example, R. Weitzenböck, Invariantentheorie, (1923), § 19, p. 349–§ 22, p. 359 where other references are also given) and have accordingly devised a theorem in which the idea of a "replacement" enters as explained in § 4 of this paper.

pressible as conditions of integrability of the equations of transformation of the components of affine connection ${}^*\Gamma^i_{\alpha\beta}$ when the functions ${}^*\Gamma^i_{\alpha\beta}$ are assumed to be arbitrary, whereas in reality ${}^*\Gamma^i_{\alpha\beta}$ is of the particular form ${}^*C^i_{\alpha\beta}$ given by (15). In the present paper, however, the replacement theorem is proved for the projective geometry of paths in a direct manner which does not require that the invariant be expressible in its above general form, and in addition the question of the order of the components of the tensors ${}^*A$ which arise by the process of replacement, is considered. The complete sets of identities of certain of the ${}^*A$ invariants are calculated in a way which can obviously be applied to any ${}^*A$ invariant. These results which require calculation for the case of ${}^*A$ invariants can be obtained immediately for the non-tensor invariants $Q$ owing to analogy with the normal tensors $A$ of the affine theory of the manifold $\mathfrak{M}$. Lastly the number of functionally independent $Q$ and ${}^*A$ invariants of any order $p$ is determined.

2. **The non-tensor invariants of O. Veblen and J. M. Thomas.** When we subject the coördinates $(x^1, \cdots, x^n)$ of the manifold $\mathfrak{M}$ to an arbitrary analytic transformation

$$\mathfrak{S}: \quad x^i = f^i(y^1, \cdots, y^n) \tag{1}$$

the components of projective connection $\Pi^i_{\alpha\beta}(x)$ go over into the components of projective connection $P^i_{\alpha\beta}(y)$ with reference to the coördinates $(y^1, \cdots, y^n)$ in accordance with the equations:

$$P^i_{\alpha\beta} = \frac{\partial y^i}{\partial x^\sigma}\left(\frac{\partial^2 x^\sigma}{dy^\alpha dy^\beta} + \Pi^\sigma_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\right) - \delta^i_\alpha \psi_\beta - \delta^i_\beta \psi_\alpha \tag{2}$$

where

$$\psi_\alpha = \frac{1}{n+1}\cdot\frac{\partial \log|\partial x/\partial y|}{\partial y^\alpha}.$$

The projective connection $\Pi^i_{\alpha\beta}$ or $P^i_{\alpha\beta}$ is characterized among the possible affine connections of the manifold $\mathfrak{M}$ by the fact that its contracted form vanishes identically. Thus

$$P^i_{i\beta} = 0. \tag{3}$$

Hence in general we have

$$P^i_{\alpha\beta} = C^i_{\alpha\beta} - \frac{\delta^i_\alpha}{n+1}C^\sigma_{\sigma\beta} - \frac{\delta^i_\beta}{n+1}C^\sigma_{\sigma\alpha}$$

where $C^i_{\alpha\beta}$ denotes the components of any one of the affine connections relative to the coördinates $(y^1, \cdots, y^n)$.

The variables $(y^1, \cdots, y^n)$ may now be chosen so as to satisfy the identities

$$P^i_{\alpha\beta}\, y^\alpha\, y^\beta = 0 \tag{4}$$

and a set of initial conditions of the form

$$y^i = 0, \quad \frac{\partial x^i}{\partial y^\alpha} = \delta^i_\alpha, \quad \Psi_\alpha = 0 \quad \text{for } x^i = q^i. \tag{5}$$

This is easily demonstrated by eliminating $P^i_{\alpha\beta}$ from (4) by the substitution (2) and then showing that the coefficients of a transformation (1) are uniquely determined by these equations and the conditions (5). The variables $(y^1, \cdots, y^n)$ which are thus determined are called *projective normal coördinates* by O. Veblen and J. M. Thomas.† These coördinates which are analogous to affine normal coördinates are completely characterized by the equations (4) together with the initial conditions (5). Geometrically the significance of the coördinates $(y^1, \cdots, y^n)$ is that the paths through the origin have the simple parametric form

$$y^i = \eta^i t, \quad \eta = \text{const.}$$

and it is from this fact that the name of normal coördinates is derived.

The non-tensor invariants to which I have already referred in the introduction are the invariants $Q$ defined as coefficients of the power series

$$P^i_{\alpha\beta} = Q^i_{\alpha\beta\gamma}\, y^\gamma + \frac{1}{2!} Q^i_{\alpha\beta\gamma\delta}\, y^\gamma\, y^\delta + \frac{1}{3!} Q^i_{\alpha\beta\gamma\delta\varepsilon}\, y^\gamma\, y^\delta\, y^\varepsilon + \cdots \tag{6}$$

in which the constant term is lacking since

$$P^i_{\alpha\beta}(0) = 0. \tag{7}$$

It is true that the first set of coefficients $Q^i_{\alpha\beta\gamma}$ actually constitutes a tensor under transformations $\mathfrak{G}$, as may easily be shown, but this is not in general true of the remaining coefficients $Q$ and I have consequently taken the liberty to speak of them collectively as non-tens*o*r invariants.

We see immediately that the symmetry identities

$$Q^i_{\alpha\beta\gamma\delta\cdots\sigma} = Q^i_{\beta\alpha\gamma\delta\cdots\sigma}; \qquad Q^i_{\alpha\beta\gamma\delta\cdots\sigma} = Q^i_{\alpha\beta\mu\nu\cdots\zeta} \tag{8}$$

where $(\mu\nu\cdots\zeta)$ denotes any permutation of $(\gamma\delta\cdots\sigma)$, hold among the invariants $Q$. If we multiply both members of (6) by $y^\alpha$ and $y^\beta$ we find as a consequence of (4) that

† They have also obtained the result that when the $(x)$ coördinates undergo a transformation $\mathfrak{G}$, the projective normal coördinates associated with the $(x)$ coördinate system and a point $(q)$ undergo a linear fractional transformation. Further interesting results are given by these authors in the Ann. Math., vol. 27 (1926), pp. 279–296.

$$Q^i_{\alpha\beta\gamma} + Q^i_{\beta\gamma\alpha} + Q^i_{\gamma\alpha\beta} = 0; \tag{9}$$

$$Q^i_{\alpha\beta\gamma\delta} + Q^i_{\alpha\gamma\beta\delta} + Q^i_{\alpha\delta\beta\gamma} + Q^i_{\beta\gamma\alpha\delta} + Q^i_{\beta\delta\alpha\gamma} + Q^i_{\gamma\delta\alpha\beta} = 0; \tag{10}$$

and in general that

$$S(Q^i_{\alpha\beta\gamma\delta\ldots\sigma}) = 0 \tag{11}$$

where $S(\quad)$ denotes the sum of all terms obtainable by permutation of subscripts from the one in parenthesis which are not identical on account of (8). The above identities (8) and (11) are exactly similar to the identities satisfied by the affine normal tensors $A$. However in addition to these identities the invariants $Q$ satisfy the identities

$$Q^i_{i\beta\gamma\delta\ldots\sigma} = 0 \tag{12}$$

which are obtained as a consequence of (3) and which have no counterpart in the affine theory of the manifold $\mathfrak{M}$.

The identities (8), (11), and (12) constitute a complete set of identities of the invariant $Q^i_{\alpha\beta\gamma\delta\ldots\sigma}$. To prove this we select sets of numbers

$$Q^i_{\alpha\beta\gamma};\quad Q^i_{\alpha\beta\gamma\delta};\quad Q^i_{\alpha\beta\gamma\delta\varepsilon};\quad \cdots$$

chosen so as to satisfy the identities (8), (11), and (12) but otherwise arbitrary. These numbers $Q$ as coefficients of a power series (6) define a set of functions $P^i_{\alpha\beta}$ which satisfy the identities (3) and which is therefore the projective connection $P^i_{\alpha\beta}$ of an affinely connected manifold $\mathfrak{M}$. It may be shown that the sets of numbers $Q$ constitute sets of invariants $Q$ formed from the projective connection $P^i_{\alpha\beta}$, namely at the point $(0, \cdots, 0)$, and from this it follows that the identities (8), (11), and (12) are all that are in general satisfied by the invariant $Q^i_{\alpha\beta\gamma\delta\ldots\sigma}$. The details of this proof will be omitted as the analogous treatment for the affine normal tensors $A$ is given rather fully in my paper on complete sets of identities mentioned in the introduction. Here I need only emphasize that the significance of our result is that the identities (8), (11), and (12) constitute all possible functional conditions on the invariant $Q^i_{\alpha\beta\gamma\delta\ldots\sigma}$. This will have application in a following paragraph.

**3. Relation between the $Q$ and $^*A$ invariants.** The coördinates $(x^0, x^1, \cdots, x^n)$ of the $(n+1)$-dimensional manifold $^*\mathfrak{M}$ with affine connection $^*\Gamma^i_{\alpha\beta}$ are subject to transformations†

† Unless otherwise stated all indices on "starred" quantities can take on the full set of numbers $(0, 1, \cdots, n)$. Indices on starred quantities are denoted by Latin or Greek letters and an index which occurs twice in a term is to be summed over all admissible values of this index.

$$(13)\qquad {}^*\mathfrak{G}:\quad x^i = f^i(y^1, \cdots, y^n); \qquad x^0 = y^0 + \log(xy)$$

where

$$(14)\qquad x^i = f^i(y^1, \cdots, y^n)$$

is an arbitrary transformation $\mathfrak{G}$ and $(xy)$ denotes the Jacobian determinant

$$\begin{vmatrix} \dfrac{\partial x^1}{\partial y^1}, & \cdots, & \dfrac{\partial x^1}{\partial y^n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \dfrac{\partial x^n}{\partial y^1}, & \cdots, & \dfrac{\partial x^n}{\partial y^n} \end{vmatrix}.$$

Let us denote the components of affine connection by ${}^*C^i_{\alpha\beta}$ and the projective normal tensors by ${}^*V$ in the $(y^0, y^1, \cdots, y^n)$ coördinate system. As we shall need the equations of definition of ${}^*C^i_{\alpha\beta}$ and the formulas of certain of the tensors ${}^*V$ we note that

$$(15)\qquad \begin{cases} {}^*C^i_{\alpha\beta} = {}^*C^i_{\beta\alpha} = P^i_{\alpha\beta} & (\text{indices} = 1, \cdots, n), \\ {}^*C^i_{0\alpha} = {}^*C^i_{\alpha 0} = -\dfrac{\delta^i_\alpha}{n+1} & (\text{indices} = 0, 1, \cdots, n), \\ {}^*C^0_{\alpha\beta} = {}^*C^0_{\beta\alpha} = \left(\dfrac{n+1}{n-1}\right)\mathfrak{B}^\sigma_{\alpha\beta\sigma} & (\text{indices} = 1, \cdots, n) \end{cases}$$

where $\mathfrak{B}^i_{\alpha\beta\gamma}$ represents the equi-projective curvature tensor based on the projective connection $P^i_{\alpha\beta}$ defined in § 2. We have

$$(16)\qquad \mathfrak{B}^\sigma_{\alpha\beta\sigma} = \frac{\partial P^\sigma_{\alpha\beta}}{\partial y^\sigma} - P^\mu_{\nu\alpha} P^\nu_{\mu\beta}.$$

Also

$$(17)\qquad {}^*V^i_{\alpha\beta\gamma} = \frac{\partial\, {}^*C^i_{\alpha\beta}}{\partial y^\gamma} - {}^*C^i_{\alpha\beta\gamma} - {}^*C^i_{\sigma\beta}\, {}^*C^\sigma_{\alpha\gamma} - {}^*C^i_{\alpha\sigma}\, {}^*C^\sigma_{\beta\gamma}$$

where

$$(18)\qquad \begin{aligned} {}^*C^i_{\alpha\beta\gamma} = {} & \frac{1}{3}\left(\frac{\partial\, {}^*C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial\, {}^*C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial\, {}^*C^i_{\gamma\alpha}}{\partial y^\beta}\right) \\ & - \frac{2}{3}({}^*C^i_{\sigma\alpha}\, {}^*C^\sigma_{\beta\gamma} + {}^*C^i_{\sigma\beta}\, {}^*C^\sigma_{\gamma\alpha} + {}^*C^i_{\sigma\gamma}\, {}^*C^\sigma_{\alpha\beta}). \end{aligned}$$

In order to obtain the relationship between the $Q$ and ${}^*A$ invariants we must specify that (14) is a transformation from the coördinates $(x^1, \cdots, x^n)$ to a set of projective normal coördinates $(y^1, \cdots, y^n)$ defined in § 2. Then with respect to the transformation ${}^*\mathfrak{G}$ we have

$$\frac{\partial x^i}{\partial y^\alpha} = \delta^i_\alpha \text{ for } y^i = 0$$

and hence

$$^*A(q) = {}^*V(0). \tag{19}$$

It should be observed, however, that the "null" variable $x^0$ does not occur in the functions $^*A$ as this variable is lacking in the affine connection $^*\Gamma^i_{\alpha\beta}$ so that $^*A(q)$ depends only on the arguments $(q^1, \cdots, q^n)$.† Also

$$Q^i_{\alpha\beta\gamma\delta\cdots\sigma}(q) = \frac{\partial^p P^i_{\alpha\beta}(0)}{\partial y^\gamma \cdots \partial y^\sigma} \qquad (\text{indices} = 1, \cdots, n) \tag{20}$$

where $Q^i_{\alpha\beta\gamma\delta\cdots\sigma}$ is an invariant of the $p$th order in the derivatives of the projective connection $\Pi^i_{\alpha\beta}$ (see end of this paragraph).

By evaluating both members of (17) at the origin of the system of coördinates $(y^0, y^1, \cdots, y^n)$ we obtain an expression for $^*A$ in terms of the functions $Q$. We find that

$$^*C^i_{\alpha 0}(0) = {}^*C^i_{0\alpha}(0) = -\frac{\delta^i_\alpha}{n+1}; \qquad {}^*C^i_{\alpha\beta}(0) = 0 \text{ (otherwise)};$$

$$^*C^i_{\alpha 00}(0) = {}^*C^i_{0\alpha 0}(0) = {}^*C^i_{00\alpha}(0) = -\frac{2\,\delta^i_\alpha}{(n+1)^2}; \qquad {}^*C^i_{\alpha\beta\gamma}(0) = 0 \text{ (otherwise)};$$

and also that

$$\frac{\partial\, {}^*C^0_{\alpha\beta}(0)}{\partial y^\gamma} = Q^i_{\alpha\beta\gamma i}(q) \qquad (\text{indices} = 1, \cdots, n).$$

Hence the above evaluation gives

$$\begin{aligned} {}^*A^i_{\alpha\beta\gamma} &= Q^i_{\alpha\beta\gamma} && (\text{indices} = 1, \cdots, n), \\ {}^*A^0_{\alpha\beta\gamma} &= Q^i_{\alpha\beta\gamma i} && (\text{indices} = 1, \cdots, n), \\ {}^*A^i_{\alpha\beta\gamma} &= 0 && (\text{otherwise}). \end{aligned} \tag{21}$$

Conversely the first set of equations (21) gives the tensor $Q^i_{\alpha\beta\gamma}$ in terms of $^*A$ invariants, i. e.,

$$Q^i_{\alpha\beta\gamma} = {}^*A^i_{\alpha\beta\gamma} \qquad (\text{indices} = 1, \cdots, n). \tag{22}$$

If we attempt to write down the complete relationship between the $Q$ and $^*A$ invariants of the second and higher orders in the derivatives of

† This includes the statement that the variable $y^0$ does not occur in $^*C^i_{\alpha\beta}$ with the consequence that $^*V$ depends only on variables $(y^1, \cdots, y^n)$.

the projective connection $\Pi^i_{\alpha\beta}$ we are led to equations of much greater complication. It will be sufficient for our purpose, however, to show the general nature of the relationship between $Q$ and $^*A$ invariants. Suppose, then, that $^*A$ represents a projective normal tensor of order $p(\geqq 2)$. Transforming $^*A$ to the system of $(y^0, y^1, \cdots, y^n)$ coördinates and evaluating at the origin of this system we are enabled to see immediately that $^*A$ is equal to an expression in $Q$'s involving $Q$'s with $(p+2)$ subscripts and these at most, i. e., an expression in $Q$'s of the $p$th order (see end of this paragraph). To express this result by an equation we may write

(23) Any $p$th order component $^*A = f$(components $Q$ of order $r(\leqq p)$)

where $f$ denotes a function of its arguments $Q$ which actually includes $Q$'s of the $p(\geqq 1)$th order.

The converse result, namely that any $p(\geqq 1)$th order $Q$ invariant is equal to an expression in $^*A$ invariants involving $p(\geqq 1)$th order components $^*A$ and in general, also $^*A$'s of order $r(<p)$, may be shown as follows. We consider the general formula

$$(24)\qquad {}^*V^i_{\alpha\beta\gamma\cdots\sigma} = \frac{\partial^p\, {}^*C^i_{\alpha\beta}}{\partial y^\gamma \cdots \partial y^\sigma} - {}^*C^i_{\alpha\beta\gamma\cdots\sigma} + \left\{\frac{\partial\, {}^*C^i_{\mu\nu}}{\partial y^\lambda}, \cdots; {}^*C^\varepsilon_{\mu\nu}, {}^*C^\varepsilon_{\mu\nu\lambda}, \cdots; {}^*V^\varepsilon_{\mu\nu\lambda}, \cdots\right\}$$

which is obtained by repeated differentiation of equation (9.12) of my thesis† when regard is taken of the fact, which involves a change in notation only, that we are concerned with the $(n+1)$-dimensional manifold $^*\mathfrak{M}$ with coördinates $(y^0, y^1, \cdots, y^n)$ and affine connection $^*C^i_{\alpha\beta}$ instead of the original $n$-dimensional manifold $\mathfrak{M}$ with coördinates $(x^1, \cdots, x^n)$ and affine connection $\Gamma^i_{\alpha\beta}$. It is to be noted in (24) that the bracket represents a homogeneous integral rational function of its arguments; that the set of indices $(\alpha\beta\gamma\cdots\sigma)$ are taken as $(p+2)$ in number; that the derivatives of $^*C^i_{\mu\nu}$ in the bracket, all of which involve $i$ as an upper index, extend only to the $(p-1)$st order; that the functions $^*C^\varepsilon_{\mu\nu}$, $^*C^\varepsilon_{\mu\nu\lambda}$, $\cdots$ and tensors $^*V^\varepsilon_{\mu\nu\lambda}$, $\cdots$ involve at most $(p+1)$ subscripts; and finally that indices $(\varepsilon\mu\nu\lambda\cdots)$ are used to represent indices of the set $(i\alpha\beta\gamma\cdots\sigma)$ as well as indices which are summed over the entire range of values $(0, 1, \cdots, n)$. Let us now assume that the set of indices $(i\alpha\beta\gamma\cdots\sigma)$ in the equations (24) is restricted to values $(1, \cdots, n)$, and regard the general formula

† O. Veblen and T. Y. Thomas, Trans. Amer. Math. Soc., vol. 25 (1923), pp. 551–608.

$$(25)\qquad {}^*C^i_{\alpha\beta\gamma\cdots\varrho\sigma} = \frac{1}{N} P\left( \frac{\partial\, {}^*C^i_{\alpha\beta\gamma\cdots\varrho}}{\partial y^\sigma} - {}^*C^i_{\tau\beta\gamma\cdots\varrho}\, {}^*C^\tau_{\alpha\sigma} - \cdots \right.$$
$$\left. \cdots - {}^*C^i_{\alpha\beta\gamma\cdots\tau}\, {}^*C^\tau_{\varrho\sigma} \right), \qquad N = p+2$$

where $P(\ )$ denotes the sum of all the terms which can be obtained from the one in the parenthesis by cyclic permutation of the indices $(\alpha\beta\gamma\cdots\varrho\sigma)$.† From the formula (25) we can see that the highest derivatives of $P^i_{\alpha\beta}$ in ${}^*C^i_{\alpha\beta\gamma\cdots\sigma}$ are those arising from $p$th derivatives of ${}^*C^i_{\alpha\beta}(i, \alpha, \beta \neq 0) = P^i_{\alpha\beta}$ and $(p-1)$st derivatives of ${}^*C^0_{\mu\nu}$ which may involve first derivatives of $P^i_{\alpha\beta}$ at most. We shall express this result concisely by saying that the set of functions ${}^*C^i_{\alpha\beta\gamma\cdots\sigma}$(indices $= 1, \cdots, n$) is of order $p$ in the derivatives of $P^i_{\alpha\beta}$ or simply of order $p$. This same terminology will be employed throughout the following. Now consider the functions ${}^*C^0_{\mu\nu\lambda\cdots\eta}$ with $(p+1)$ indices in the set $(\mu\nu\lambda\cdots\eta = 0, 1, \cdots, n)$ which by the above statement can occur in the right member of (24). As the highest derivatives of $P^i_{\alpha\beta}$ in ${}^*C^0_{\mu\nu\lambda\cdots\eta}$ (subscripts $= 0, 1, \cdots n$) come from $(p-1)$st derivatives of ${}^*C^0_{\mu\nu}$ it follows that ${}^*C^0_{\mu\nu\lambda\cdots\eta}$ (subscripts $= 0, 1, \cdots, n$) is of order $r(\leqq p)$ in the derivatives of $P^i_{\alpha\beta}$. Hence the entire set of functions ${}^*C^\varepsilon_{\mu\nu}$, ${}^*C^\varepsilon_{\mu\nu\lambda}, \cdots$ in the bracket in the right member of (24) can involve derivatives of $P^i_{\alpha\beta}$ of order $r(\leqq p)$ at most. Finally the derivatives of ${}^*C^0_{\mu\nu}$ in the bracket in (24) involve derivatives of $P^i_{\alpha\beta}$ of an order $r(<p)$.

Let us next consider the order of the various normal tensors ${}^*V$. Beginning with the tensor ${}^*V^i_{\alpha\beta\gamma}$ we see from (17) that the components ${}^*V^i_{\alpha\beta\gamma}$ (indices $= 1, \cdots, n$) are of the first order in the derivatives of $P^i_{\alpha\beta}$ and that ${}^*V^0_{\alpha\beta\gamma}$ (subscripts $= 1, \cdots, n$) is of the second order in the derivatives of $P^i_{\alpha\beta}$. The remaining components ${}^*V^i_{\alpha\beta\gamma}$ have already been found to vanish (third set of equations (21)). Using the facts which we have established for the functions ${}^*C$ and the tensor ${}^*V^i_{\alpha\beta\gamma}$ we can see from the equations (24) that ${}^*V^i_{\alpha\beta\gamma\delta}$ (indices $= 1, \cdots, n$) is of the

† Attention is directed here to a mistake in my joint paper with Professor O. Veblen (loc. cit., p. 561) where we have hastily written

$$\frac{1}{N} P\left[ \frac{\partial\, \Gamma^i_{\beta\mu\cdots m}}{\partial x^n} - (N-1)\, \Gamma^i_{\alpha\beta\cdots l}\, \Gamma^\alpha_{mn} \right]$$

as the equivalent of the right member of (25). The same mistake also occurs in the Math. Zeit., vol. 25 (1926), p. 715 and ibid. p. 727.

second order and that ${}^*V^0_{\alpha\beta\gamma\delta}$ (subscripts $= 1, \cdots, n$) is of the third order in the derivatives of $P^i_{\alpha\beta}$. Moreover it is seen that the order of the functions ${}^*V^i_{\alpha\beta\gamma\delta}$ is not increased in case one or more of the subscripts $(\alpha\beta\gamma\delta)$ have the value zero. Considering the tensors ${}^*V^i_{\alpha\beta\gamma\delta\varepsilon}$, ${}^*V^i_{\alpha\beta\gamma\delta\varepsilon\zeta}$, etc., as we have done ${}^*V^i_{\alpha\beta\gamma\delta}$, we easily arrive at the general result that ${}^*V^i_{\alpha\beta\gamma\cdots\sigma}$ $(i \neq 0)$ with $(p+2)$ subscripts $(\alpha\beta\gamma\cdots\sigma)$ and ${}^*V^0_{\alpha\beta\gamma\cdots\varrho}$ with $(p+1)$ subsripts $(\alpha\beta\gamma\cdots\varrho)$ are of the $r(\leqq p)$th order in the derivatives of the projective connection $P^i_{\alpha\beta}$.

We must now evaluate both members of (24) at the origin of the $(y^0, y^1, \cdots, y^n)$ coördinate system. Then the functions ${}^*V(0)$ become ${}^*A(q)$ by (19) and

$$\frac{\partial^p {}^*C^i_{\alpha\beta}(0)}{\partial y^\gamma \cdots \partial y^\sigma} = Q^i_{\alpha\beta\gamma\cdots\sigma}(q) \quad \text{(indices} = 1, \cdots, n)$$

by (20). Furthermore the derivatives of the set $\partial\, {}^*C^i_{\mu\nu}(0)/\partial y^\lambda, \cdots$ in (24) give $Q$ invariants which involve less than $(p+2)$ subscripts. It can be seen, perhaps best by considering the way in which successive functions ${}^*C^i_{\alpha\beta\gamma}$, ${}^*C^i_{\alpha\beta\gamma\delta}$, etc., are formed, that the $p$th derivatives of ${}^*C^i_{\alpha\beta}(i, \alpha, \beta \neq 0) = P^i_{\alpha\beta}$ in ${}^*C^i_{\alpha\beta\gamma\cdots\varrho\sigma}$ (indices $= 1, \cdots, n$) occur in such a way that the invariants $Q^i_{\alpha\beta\gamma\cdots\sigma}$ which arise from them by evaluation at the point $(0, 0, \cdots, 0)$, vanish by (11) while the $(p-1)$st derivatives of ${}^*C^0_{\mu\nu}(\mu, \nu \neq 0)$, which are derivatives with respect to free indices of the set $(\alpha\beta\gamma\cdots\varrho\sigma)$, occur as coefficients of Kronecker's $\delta^i_\varepsilon$ $(i, \varepsilon \neq 0)$ and lead to invariants $Q$ involving $(p+2)$ subscripts, which are of the type $Q^i_{\alpha\beta\gamma\cdots\varrho i}$, and which are seen to vanish by the contracted form of the identity (11) and the identity (12) owing to the symmetry in the subscripts of the functions ${}^*C^i_{\alpha\beta\gamma\cdots\varrho\sigma}$. Hence ${}^*C^i_{\alpha\beta\gamma\cdots\sigma}(0)$ is equal to an expression in $Q$ invariants with less than $(p+2)$ subscripts. There remain the set of functions ${}^*C^0_{\mu\nu\lambda}(0), \cdots$ in (24) and of these we consider the one involving highest derivatives of $P^i_{\alpha\beta}$, namely ${}^*C^0_{\alpha\beta\gamma\cdots\varrho}$ with the $(p+1)$ subscripts having values $(1, \cdots n)$. The derivatives of $P^i_{\alpha\beta}$ of highest order in ${}^*C^0_{\alpha\beta\gamma\cdots\varrho}$ arise from $(p-1)$st derivatives of ${}^*C^0_{\alpha\beta}(\alpha\beta \neq 0)$ and, by a consideration similar to that just made, give a set of $Q$ invariants involving $(p+2)$ subscripts which vanish by the contracted form of (11) and (12). Hence the functions ${}^*C^0_{\alpha\beta\gamma\cdots\varrho}(0)$ and consequently the sets of functions ${}^*C^\varepsilon_{\mu\nu\lambda}, \cdots$ give invariants $Q$ having less than $(p+2)$ subscripts. *Thus we have expressed the invariant $Q$ with $(p+2)$ subscripts in terms of components ${}^*A$ of order $r(\leqq p)$ and $Q$'s involving less than $(p+2)$ subscripts.* Hence if we assume

that any $Q$ invariant having $(r+2)$ subscripts $(r \geqq 1)$ can be expressed in terms of components $^*A$ of order $m$ $(\leqq r)$, for $r \leqq p$, then this result is also true for $r = p+1$. The result is then always true since we have proved it in particular for $r = 1$ in the first part of the present paragraph. The expression, say $\varphi$, giving the component $Q$ with $(p+2)$ subscripts in terms of components $^*A$ of order $r$ $(\leqq p)$ must actually involve components $^*A$ of order $p$, since at the origin of the $(y^0, y^1, \cdots, y^n)$ coördinate system $\varphi$ must reduce to the derivative

$$\frac{\partial^p P^i_{\alpha\beta}}{\partial y^\gamma \cdots \partial y^\sigma}.$$

This shows, too, that the invariant $Q$ with $(p+2)$ subscripts is an invariant of the $p$th order in the derivatives of the projective connection $\Pi^i_{\alpha\beta}$. Corresponding to (23) we may therefore write

(26) Any $p$th order component $Q = \varphi$ (components $^*A$ of order $r$ $(\leqq p)$)

where $\varphi$ represents a function of the components $^*A$ which actually includes $p(\geqq 1)$th order $^*A$ components.

*It is evident that the functions $f$ and $\varphi$ are homogeneous integral rational functions of their arguments.* The equations (23) and (26) will be used in § 6 in connection with the determination of the actual number of functionally independent $Q$ and $^*A$ components of the $p(\geqq 1)$th order.

4. **The replacement theorem for $Q$ and $^*A$ invariants.** Let $^*P$ denote any projective invariant of order $p(\geqq 1)$. Transforming $^*P$ to the $(y^0, y^1, \cdots, y^n)$ coördinate system and evaluating at the origin of this system, we express $^*P$ as a function of $Q$ invariants by the substitutions (8) and (20). Eliminating the $Q$ invariants in their turn by (26) from the expressions for $^*P$ we arrive at the following replacement theorem for $Q$ and $^*A$ invariants.

*Any $p(\geqq 1)$th order projective invariant $^*P$ may be expressed in terms of $r(\leqq p)$th order components $^*A$ by replacing the projective connection $\Pi^i_{\alpha\beta}$ in $^*P$ by zero and the derivatives of $\Pi^i_{\alpha\beta}$ in $^*P$ by the corresponding component $Q$ (replacement theorem for $Q$ invariants) and then using the substitution* (26) *to eliminate the components $Q$ (replacement theorem for $^*A$ invariants).*

There are no differential invariants (either projective or affine) of the 0th order, for if such an invariant, which would depend on the affine connection alone, were to be transformed to the origin of a system of affine normal coördinates $(y)$† it would be seen to reduce to a set of absolute constants.‡

† O. Veblen, Proc. Nat. Acad. Sci., vol. 8 (1922), pp. 192–197.

‡ Concerning tensors whose components are absolute constants, see T. Y. Thomas, Ann. Math., vol. 27 (1926), pp. 548–550.

5. **Complete sets of identities of *$A$ invariants.** When we combine the relation (22) with the identities (8), (11), and (12) for the invariant $Q$ we obtain the following three sets of identities

$$
\begin{aligned}
&{}^*A^i_{\alpha\beta\gamma} = {}^*A^i_{\beta\alpha\gamma} && \text{(indices} = 1, \cdots, n),\\
&{}^*A^i_{\alpha\beta\gamma} + {}^*A^i_{\beta\gamma\alpha} + {}^*A^i_{\gamma\alpha\beta} = 0 && \text{(indices} = 1, \cdots, n),\\
&{}^*A^i_{i\beta\gamma} = 0 && \text{(indices} = 1, \cdots, n)
\end{aligned} \tag{27}
$$

for the components ${}^*A^i_{\alpha\beta\gamma}$ (indices $= 1, \cdots, n$). Conversely if the independent identities (27) are satisfied it follows from the relation (22) that the identities (8), (11), and (12) for the invariant $Q^i_{\alpha\beta\gamma}$, i. e., the complete set of identities of the invariant $Q^i_{\alpha\beta\gamma}$, are also satisfied. Hence we may say that the identities (27) constitute a complete set of identities of the components ${}^*A^i_{\alpha\beta\gamma}$ (indices $= 1, \cdots, n$).

Let us now put $i = \delta$ in the identities (10). Then when regard is taken of the identities (12), we have

$$Q^i_{\alpha\beta\gamma i} + Q^i_{\beta\gamma\alpha i} + Q^i_{\gamma\alpha\beta i} = 0$$

which by the second set of identities (21), gives the identities

$${}^*A^0_{\alpha\beta\gamma} + {}^*A^0_{\beta\gamma\alpha} + {}^*A^0_{\gamma\alpha\beta} = 0 \quad \text{(indices} = 1, \cdots, n). \tag{28}$$

By the process used to obtain the relations (21) we find that

$$Q^i_{\alpha\beta\gamma\sigma} = {}^*A^i_{\alpha\beta\gamma\sigma} + \frac{\delta^i_\sigma}{n+1} {}^*A^0_{\alpha\beta\gamma} + \frac{\delta^i_\gamma}{n+1} {}^*A^0_{\alpha\beta\sigma} \quad \text{(indices} = 1, \cdots, n). \tag{29}$$

If, now, we substitute the value of $Q^i_{\alpha\beta\gamma\delta}$ given by (29) into the identities (10), we find in consequence of (28) that

$${}^*A^i_{\alpha\beta\gamma\delta} + {}^*A^i_{\alpha\gamma\beta\delta} + {}^*A^i_{\alpha\delta\beta\gamma} + {}^*A^i_{\beta\gamma\alpha\delta} + {}^*A^i_{\beta\delta\alpha\gamma} + {}^*A^i_{\gamma\delta\alpha\beta} = 0 \quad \text{(indices} = 1, \cdots, n). \tag{30}$$

Again substituting (29) into (12) and using (28), we have

$${}^*A^i_{i\beta\gamma\delta} = \frac{{}^*A^0_{\delta\gamma\beta}}{n+1} \quad \text{(indices} = 1, \cdots, n). \tag{31}$$

Finally in addition to the identities (28), (30), and (31) which are satisfied by the components ${}^*A^i_{\alpha\beta\gamma\delta}$ (indices $= 1, \cdots, n$) and ${}^*A^0_{\alpha\beta\gamma}$ (indices $= 1, \cdots, n$) there are, of course, the symmetry identities, namely

$$(32)\qquad {}^*A^0_{\alpha\beta\gamma} = {}^*A^0_{\beta\alpha\gamma};\quad {}^*A^i_{\alpha\beta\gamma\delta} = {}^*A^i_{\beta\alpha\gamma\delta};\quad {}^*A^i_{\alpha\beta\gamma\delta} = {}^*A^i_{\alpha\beta\delta\gamma} \qquad (\text{indices} = 1, \cdots, n).$$

As a consequence of the identities (28), (30), (31), (32) and the relations (29) it follows immediately that the complete set of identities of the invariant $Q^i_{\alpha\beta\gamma\delta}$ is satisfied. This shows that the sets of identities (28), (30), (31), (32) which are evidently algebraically independent, constitute a complete set of identities of the components ${}^*A^i_{\alpha\beta\gamma\delta}$ (indices $= 1, \cdots, n$) and ${}^*A^0_{\alpha\beta\gamma}$ (indices $= 1, \cdots, n$).

On the basis of the above results the complete set of identities of the projective normal tensor ${}^*A^i_{\alpha\beta\gamma}$ is easily established without additional calculation. *The identities*

$$(33)\qquad \begin{aligned} {}^*A^i_{\alpha\beta\gamma} &= 0 && (\alpha, \beta, \text{or } \gamma = 0;\ i = 0, 1, \cdots, n),\\ {}^*A^i_{\alpha\beta\gamma} &= {}^*A^i_{\beta\alpha\gamma} && (\alpha, \beta, \gamma = 1, \cdots, n;\ i = 0, 1, \cdots, n),\\ {}^*A^i_{i\beta\gamma} &= 0 && (i,\ \beta, \gamma = 1, \cdots, n),\\ {}^*A^i_{\alpha\beta\gamma} + {}^*A^i_{\beta\gamma\alpha} + {}^*A^i_{\gamma\alpha\beta} &= 0 && (\alpha, \beta, \gamma = 1, \cdots, n;\ i = 0, 1, \cdots, n) \end{aligned}$$

*constitute a complete set of identities of the projective normal tensor* ${}^*A$ (*indices* $= 0, 1, \cdots, n$). It is to be noted that the first set of identities (33) is equivalent to the third set of identities (21) from which it is obtained.

No attempt will be made to write down the actual forms of the identities satisfied by the projective normal tensors ${}^*A$ in general, which are obtained by elimination of the $Q$'s from the identities (8), (11), and (12) and the relations (26). In fact I see no necessity to make this elimination. What appears to me to be more important is the determination of the actual number of $Q$ and ${}^*A$ components of any order $p(\geqq 1)$ which are functionally independent in virtue of their identities, and this is done in the next and last paragraph of the present paper.

6. **Numbers of independent $Q$ and ${}^*A$ invariants.** We can easily see from the equations (23) and (26) that the number of functionally independent components $Q$ ranging from the first to the $p(\geqq 1)$th order inclusive is the same as the number of functionally independent components ${}^*A$ comprised in this same range of order (1 to $p$ inclusive). It follows that the number of independent ${}^*A$ components of order $p(\geqq 1)$ is equal to the number of independent $Q$ components of the $p$th order. We may therefore avail ourselves of the simple form of the identities (8), (11), and (12) satisfied by the invariant $Q^i_{\alpha\beta\gamma\ldots\sigma}$ which is a $Q$ invariant of the $p(\geqq 1)$th order provided that there are $p$ indices in the set $(\gamma, \cdots, \sigma)$ as

is assumed, in the calculation of the number of independent $^*A$ components of the $p(\geqq 1)$th order.

Let $K(n, p)$ denote the number of combinations with repetitions of $n$ things taken $p$ at a time:

$$K(n,p) = \frac{n(n+1)(n+2)\cdots(n+p-1)}{1\cdot 2\cdot 3\cdots p}.$$

Then the number of independent quantities $Q^i_{\alpha\beta\gamma\cdots\sigma}$ when the symmetry conditions (8) alone are taken into account, is

$$n\,K(n,2)\;K(n,p).$$

However the effect of the equations (11) and (12) is to decrease the number of independent components $Q^i_{\alpha\beta\gamma\cdots\sigma}$ by an amount equal to the number of these equations themselves. The number of equations (11) is

$$n\,K(n,p+2)$$

and the number of equations (12) is

$$K(n,p+1).$$

Hence the number of functionally independent $p(\geqq 1)$th order $Q$ components in the $n$-dimensional manifold $\mathfrak{M}$ is given by

$$N(n,p) = n\,K(n,2)\,K(n,p) - n\,K(n,p+2) - K(n,p+1).$$

As we have already seen $N(n, p)$ also gives the number of functionally independent $^*A$ components of order $p(\geqq 1)$.

PRINCETON UNIVERSITY.
September, 1926.

# ON TYPES OF KNOTTED CURVES.*

By J. W. Alexander and G. B. Briggs.

1. The problem of determining the various possible types of closed, knotted curves in 3-space was originally studied by Gauss, and has since received the attention of a number of other mathematicians.† Very little progress seems to have been made, however, toward finding definite, calculable invariants with which to distinguish one type of knot from another, though classified tables of the more elementary knots have been arrived at by somewhat empirical methods.‡

A number of years ago,§ one of the authors of the present paper pointed out that if the space of a knotted curve be covered by an $n$-sheeted "Riemann 3-spread" (the three dimensional analogue of a Riemann surface) with a branch curve of order $n-1$ covering the knot itself, then, the topological invariants of the covering spread will also be invariants of the knot. He further calculated the Betti numbers and coefficients of torsion‖ of the covering spreads determined by some of the simpler knots and found these invariants sufficient, in the cases actually examined, to distinguish one type of knot from another. The torsion numbers to which we have just referred have recently been rediscovered by K. Reidemeister¶ who derives them from a study of the group of the space complementary to the knot and then identifies them with the invariants of the Riemann covering spreads.

In this paper, we propose, first of all, to obtain the torsion numbers of a knot by direct, elementary considerations, without appealing to the idea of a Riemann covering spread. Next, we shall prove, with the aid of these invariants alone, that all types of knots of eight or less crossings listed as distinct in the knot tables of Tait and Kirkman actually are

* Received April 28, 1927.

† For a list of references on the subject of knots, the reader is referred to the article on Analysis Situs, by Dehn and Heegaard, Encyclop. der Math. Wiss., III A B 3, pp. 207–213.

‡ References to the knot tables of Tait, Kirkman, and Little will be found in the Dehn-Heegaard article, loc. cit., p. 207.

§ In a paper read before the National Academy, Nov., 1920, cf. Veblen's Cambridge Colloquium Lectures on Analysis Situs (1922), last page.

‖ For the definition of these numbers cf., for example, Veblen's Colloquium Lectures, pp. 109, et. seq.

¶ K. Reidemeister, Knoten und Gruppen. Abh. aus d. Math. Semin. der Hamburgischen Universität, 1926, pp. 7–23. See also Elementare Begründung der Knotentheorie, pp. 24–32.

distinct. We shall also prove that among the eighty-two listed types of knots of nine or less crossings three cases arise where knots of presumably distinct types have the same torsion numbers, but that, in all other cases, the torsion numbers are sufficient to distinguish one type of knot from another. Finally, we shall describe, briefly, the method of obtaining the torsion numbers of a knot from its associated Riemann covering spreads, after the manner in which the invariants were originally discovered.

2. To simplify the exposition as much as possible, we shall approach the knot problem from the elementary, combinatorial point of view. Thus, for our purposes, a *knot* will be nothing more than a simple, closed, sensed* polygon in the space of three real variables $x, y, z$. The knot will be composed of a finite number of straight segments, or *edges*, together with their end points, or *vertices*.

A knot will be subject to certain simple transformations. On any edge $AB$ we may construct a triangle $ABC$, so drawn that neither the vertex $C$, the edge $AC$, the edge $CB$, nor the plane triangular region bounded by $ABC$ has a point in common with the knot. We may then transform the knot by removing the edge $AB$ and substituting in its place the edges $AC$ and $CB$, along with the vertex $C$. We may also perform the reverse operation which consists in replacing a pair of consecutive edges $AC$ and $CB$, together with their common vertex $C$, by a single edge $AB$, provided neither the edge $AB$ nor the plane triangular region bounded by $ABC$ has a point in common with the knot. Each of the transformations here described will be called an *elementary deformation*.

The operation of subdividing an edge $AB$ into two edges $AC$ and $CB$ along which a vertex $C$ may be regarded as a degenerate case of an elementary deformation; so also may the inverse operation: the amalgamation of two consecutive collinear edges and their common vertex into a single edge. By an extension of terminology, these degenerate operations will also be described as elementary deformations. Evidently, a degenerate elementary deformation is the resultant of three suitably chosen non-degenerate ones.

Two knots will be said to be of the same *type* if, and only if, the one is transformable into the other by a finite succession of elementary deformations. A knot will be said to be *unknotted* if, and only if, it is of the same type as a sensed triangle. Any two sensed triangles are, of course, of the same type,

Let $K$ be any knot. Then the knot $K^{-1}$ obtained by reserving the sense of the polygon $K$ will be called the *inverse* of the knot $K$. More-

* It is, perhaps, more customary not to impose the condition that the polygon be sensed.

oyer, the knot $K'$ obtained by reflecting the knot $K$ about the $xy$-plane will be called the *reflection* of the knot $K$. In general, no two of the three knots $K$, $K^{-1}$, and $K'$ need be of the same type.

3. If we project a knot orthogonally upon the $xy$-plane, we shall, in general, obtain, a singular polygon $\Pi$ with at most a finite number of crossing points, where each crossing point corresponds to two distinct points on different edges of the knot. In certain exceptional cases, the polygon $\Pi$ will have singularities of a more complicated sort, but we shall agree to leave such cases aside, once and for all. It will evidently be legitimate to do this, because a knot with an exceptional projection may always be transformed into a neighboring one with a projection of the ordinary sort by making a slight shift in the position of one or more vertices. Moreover, the transformed knot will be of the same type as the original one, since the shifting of a vertex may always be accomplished by means of two properly chosen elementary deformations.

The projection of a knot upon the $xy$-plane will serve as a schematical picture, or *diagram*, of the knot itself, provided we indicate by some suitable notation which of the two segments at each crossing point is to represent the "upper" branch of the knot and which the "lower". The device ordinarily used is to indicate the upper branch by a solid line and the lower one by a line from which a small segment has been removed in the neighborhood of the crossing point (cf. Fig. 1a). A better notation for our present purposes will be the following. We imagine an observer standing on the $xy$-plane and describing the polygon of the diagram once in the positive sense (as determined by the positive sense on the knot), thereby passing twice through each crossing point. Then, as the observer passes through a crossing point on the segment representing an upper branch of the knot, we mark with dots the two corners on his right; as he passes through a crossing point on the segment representing a lower branch, we make no notation of any sort. The resulting system of dots will enable us, at a glance, to distinguish between the segments corresponding to upper and to lower branches of the knot respectively; (cf. Fig. 1b which represents the same knot as Fig. 1a, but in the new notation).

If a figure like a diagram be formed without reference to any particular knot in space, by merely drawing a sensed polygon $\Pi$ in the $xy$-plane, deciding on the segments to represent the upper and lower branches at each crossing point, and dotting the corners accordingly, there will always be a family of knots in space having the figure in question as their common diagram, provided any point of the polygon $\Pi$ except a crossing point is allowed to represent a knot vertex. If only the vertices of the polygon $\Pi$ are allowed to represent knot vertices, it may happen, in certain cases,

that there exists no knot with sufficient flexibility to weave over and under itself in the manner precribed by the diagram. To make matters definite, we shall agree that only the vertices of a diagram are to correspond to the vertices of a knot, but we shall not prohibit diagrams with three or more collinear vertices. Thus, every figure of the type described above may be made into a true diagram, if it is not one already, by intoducing a certain number of new vertices at suitable points on the edges of the polygon $\Pi$, thereby subdividing the edges in question into sub-edges.

Since the knots corresponding to any giving diagram are all of the same type, as may readily be proved, the problem of classifying knots reduces to that of classifying diagrams. We shall say that two diagrams are of the same *type* if they correspond to knots of the same type; that a diagram $D_1$ is transformable into a diagram $D_2$ by an *elementary deformation* if some knot $K_1$ with the diagram $D_1$ is transformable into some knot $K_2$ with the diagram $D_2$ by an elementary deformation. A necessary and sufficient condition that two diagrams $D_1$ and $D_2$ be of the same type is, thus, that one be transformable into the other by a finite succession of elementary deformations.

If a knot $K$ is the inverse of a knot $L$, the diagram of $K$ will be called the *inverse* of the diagram of $L$. Similarly, if a knot $K$ is the reflection of a knot $L$, the diagram of $K$ will be called the *reflection* of the diagram of $L$.

4. An elementary deformation $\delta$ transforms a diagram $D$ into a diagram $D'$ which may be very different in appearance from $D$. We shall prove, however, that it is always possible to perform the deformation $\delta$ in a stepwise manner, in such a way, that, at each step, the change undergone by the diagram on which we operate is comparatively slight. Since the diagram $D$ is transformable into the diagram $D'$ by an elementary deformation, the polygons of the two diagrams must be relate to one another in the following simple manner: A certain edge $AB$ of one polygon corresponds to a vertex $C$ and a pair of edges $AC$ and $CB$ of the other, but the remaining parts of the two polygons are identical. To fix matters, we shall so assign our notation that $D$ is the diagram with the edge $AB$ and $D'$ the diagram with the edges $AC$ and $CB$. The triangle $ABC$ will be called the *characteristic triangle* both of the deformation $\delta$ and of the inverse deformation $\delta^{-1}$ transforming the diagram $D'$ into the diagram $D$. In certain trivial cases which will not require further analysis, the vertex $C$ will lie on the edge $AB$ and the characteristic triangle will be degenerate. We shall measure the degree of simplicity of an elementary deformation $\delta$ by the degree of simplicity of the figure formed by that part of the diagram $D$ which is enclosed by the characteristic triangle $ABC$ of $\delta$.

Now, suppose we divide up the triangle $ABC$ into a pair of triangles $\Delta_1$ and $\Delta_2$ by joining one of the vertices $A$, $B$, or $C$ to a point $K$ of the

edge opposite the vertex chosen. Suppose, moreover, that we select the point $K$ in general position, so that $K$ is not at a crossing point of either of the diagrams $D$ or $D'$, and so that the new edge connecting the point $K$ to the opposite vertex of the triangle $ABC$ passes through no vertex or crossing point of the diagram $D$. Then, it is obvious that the deformation $\delta$ with the characteristic triangle $ABC$ may be thought of as the resultant of two smaller deformations $\delta_1$ and $\delta_2$ with the characteristic triangles $\Delta_1$ and $\Delta_2$ respectively. Moreover, each of these last may, in turn, be thought of as the resultant of two still smaller deformations obtained by a similar process of factorization, and so on, as often as we please. In other words, the deformation $\delta$ may always be regarded as the resultant of a finite succession of deformations with characteristic triangles of arbitrarily small linear dimensions. It is, therefore, possible to carry the factorization of the deformation $\delta$ to such a point that each factor $\delta^*$ of $\delta$ has the following degree of simplicity: If any portion of the diagram $D^*$ on which the deformation $\delta^*$ operates is interior to the characteristic triangle $\Delta^*$ of the deformation, that portion is connected and includes not more than one point from among the vertices and crossing points of the diagram $D^*$; (here, we make use of the fact that since the triangles $\Delta^*$ corresponding to the different factors $\delta^*$ of $\delta$ are non-overlapping any portion of a diagram $D^*$ interior to a triangle $\Delta^*$ must also be a portion of the original diagram $D$). When the above state of affairs is realized, we see, by inspection, that each of the factors $\delta^*$ must be an elementary deformation of one of four simple categories determined resectively by the following conditions:

(i) No part of the diagram $D^*$ is interior to the triangle $\Delta^*$.

(ii) The part of $D^*$ interior to $\Delta^*$ consists of one sub-edge of $D^*$.

(iii) The part of $D^*$ interior to $\Delta^*$ consists of one vertex and two sub-edges meeting at the vertex.

(iv) The part of $D^*$ interior to $\Delta^*$ consists of one crossing point and two sub-edges passing through the crossing point.

The analysis may be carried somewhat further in the last three cases. We recall that either the deformation $\delta^*$ or its inverse—for our present purposes it makes no difference which—carries an edge $A^*B^*$ of the characteristic triangle $\Delta^*$ into the figure consisting of the vertex $C^*$ opposite $A^*B^*$ and the other two edges $A^*C^*$ and $B^*C^*$. Then, under cases (ii), and (iv), there are several essentially distinct sub-cases to be distinguished according to the location of the points in which the polygon of the diagram $D^*$ crosses the triangle $\Delta^*$. Let us first examine case (ii), where the portion of the polygon $D^*$ interior to the triangle $\Delta^*$ consists of a single sub-edge $x_0' x_0''$. There are then three sub-cases to be considered, to which all others may be reduced by a mere change of notation:

(iia) One end of the edge $x_0' x_0''$ is at the vertex $A$, the other on the side $BC$. (Fig. 2 II.)

(iib) One end of the edge $x_0' x_0''$ is on the side $AB$, the other on the side $BC$.

(iic) One end of the edge $x_0' x_0''$ is on the side $AC$, the other on the side $BC$. (Fig. 2 III a.)

However, sub-case (iib) may be eliminated from the discussion since it reduces by factorization to sub-case (iic). To prove this, we join the vertex $C$ of the triangle $\Delta^*$ to a point $D$ of the edge $AB$ in such a manner that the new edge $CD$ crosses the segment $x_0' x_0''$. Then, we regard the deformation $\delta^*$ with the triangle $ABC$ as the resultant of two elementary deformations with the triangles $DBC$ and $ACD$ respectively. Each of the latter is of the sub-category (iic).

Under case (iii), where the portion of the polygon of $D^*$ interior to the triangle $\Delta^*$ consists of two sub-edges $x_0' z$ and $z x_0''$ meeting in a common vertex $z$, we may dispose of all sub-cases, except the one where the point $x_0'$ lies on the side $AC$ and the point $x_0''$ on the side $BC$ (Fig. 2 III b). Any other deformation of the third category reduces, by an obvious process of factorization, to one of the types just described together with a certain number of deformations of the first and second categories.

Under case (iv), when the portion of the polygon $D^*$ interior to the triangle $\Delta^*$ consists of two sub-edges $x_1 x_1'$ and $x_2 x_2'$ crossing in a point $x_0$ we may dispose of all sub-cases except the one where the points $x_1$ and $x_2$ both lie on the side $AB$ while the points $x_1'$ and $x_2'$ lie on the sides $AC$ and $BC$ respectively (Fig. 2 IV). Any other deformation of the fourth category may be factored into one of the types just described together with a certain number of deformations of the first and second categories.

To sum up, the most general type preserving transformation of one diagram into another may be factored into a finite succession of elementary deformations each of which is one of the four types pictured schematically in Fig. 2. Of course, the dotting of the corners in Fig. 2 merely corresponds to one possible type of dotting that may actually arise. The letters $R$ in the figure refer to a matter that we have not yet taken up, hence, they have no significance at the present stage of the discussion. Deformations III a and III b, in Fig. 2, are going to play essentially the same role throughout the discussion; this is why we have classified them together.

5. At this point, we shall make a slight digression to remind the reader of certain arithmetical theorems about linear systems. A set of $m$ linearly independent marks

$$x_i, \qquad (i = 1, 2, \cdots, m), \tag{5.1}$$

is the *base* of a linear system $X$ consisting of all linear combinations of the marks $x_i$ with integer coefficients, where the ordering of the terms in a linear combination is immaterial. Moreover, a set of $k$ arbitrary marks

$$y_s = \sum_{i=1}^{m} e_{si} x_i, \qquad (s = 1, 2, \cdots, k), \tag{5.2}$$

belonging to the linear system $X$ is the base of a linear sub-system $Y$ of $X$. If a mark $x$ of the system $X$ is also a mark of the system $Y$, we shall indicate the fact by writing the homology

$$x \sim 0 \pmod{Y}.$$

Furthermore, we shall say that two marks $x$ and $x'$ of the system $X$ are homologous, and write

$$x \sim x' \pmod{Y},$$

if their difference $x - x'$ is a mark of the system $Y$. From the way in which homologies are defined it follows that any linear combination with integer coefficients of a given set of homologies is itself an homology; also, that the most general homology that can be written is a linear combination with integer coefficients of the fundamental homologies

$$y_s \sim 0 \qquad (s = 1, 2, \cdots, k), \tag{5.3}$$

determined by the base (5.2) of the system $Y$. In operating with homologies, it must be borne in mind that if the coefficients of an homology possess a common factor it is not, in general, permissible to simplify the homology by cancelling out the common factor, for if a multiple of a mark $x$ of the system $X$ belongs to the system $Y$ it does not necessarily follow that the mark $x$ itself belongs to $Y$.

Now, let the marks of the system $X$ be arranged in sets $z_i$ such that two marks belong to the same set if, and only if, they are homologous. Then, if we define the *negative* of a set $z_i$ containing a mark $x_s$ as the set containing the mark $-x_s$ and the *sum* of the sets $z_i$ and $z_j$ containing the marks $x_s$ and $x_t$ respectively as the set containing the mark $x_s + x_t$, we may regard the sets $z_i$ as the elements of a linear domain $Z$, in which the ordinary linear operations with integer coefficients may be performed. We shall designate an element $z_i$ of the domain $Z$ by means of any one of the marks $x_s$ in the set $z_i$.

Now, by a well known theorem, if the bases (5.1) and (5.2) of the linear systems $Y$ and $X$ are properly chosen, the fundamental homologies (5.3) will assume the form of monomials in the $x$'s,

$$\alpha_s x_s \sim 0, \qquad (s = 1, 2, \cdots, k' \leqq k), \tag{5.4}$$

where the coefficients $\alpha_1$, $\alpha_2$, $\cdots$, are positive integers with the property that each one after the first is exactly divisible by its immediate predecessor. The coefficients $\alpha_s$ are, in fact, the elementary divisors of the matrix of coefficients $\epsilon_{si}$ in (5.2), which divisors remain invariant when the bases of the systems $X$ and $Y$ undergo a transformation. Moreover, when the fundamental homologies are in the normal form (5.4) the linear domain $Z$ is also in a normal form. For it is easy to see that the elements of $Z$ are now representable, respectively, by the symbols

$$\sum_{s=1}^{k'} a_s x_s + \sum_{s=k'+1}^{m} a_s x_s, \tag{5.5}$$

where each coefficient $a_s$ of the first sum is a non-negative integer less than the corresponding elementary divisor $\alpha_s$ in (5.4), and where each coefficient $a_s$ of the second sum is wholly arbitrary integer. Thus, the internal structure of the domain $Z$ is completely characterized by the number of terms in the second sum of (5.5) together with the values of all the elementary divisors $\alpha_s$ that are greater than unity. If a divisor $\alpha_s$ is equal to unity, the corresponding term $a_s x_s$ of the first sum of (5.5) must have the coefficient zero; therefore, the presence or absence of this term has no effect upon the internal structure of the domain $Z$. It will be convenient to characterize the domain $Z$ by a sequence of integers consisting of as many zeros as there are terms in the second sum in (5.5) followed by as many of the elementary divisors $\alpha_s$ as are different from unity. The terms of this sequence will be called the *characteristic invariants* of the system $X$ with reference to its sub-system $Y$. Among the characteristic invariants, repetitions are, of course, to be expected, since there may be more than one term in the second sum in (5.5), and since two or more of the elementary divisors $\alpha_s$, $(\alpha_s > 1)$, may be equal to one another.

6. We are now ready to derive the torsion invariants of a knot. These invariants occur in sets, such that there is one set corresponding to each choice of an integer $n$ greater than unity. We shall determine the set corresponding to a definite, though arbitrary choice of the integer $n$.

Let $\nu$ be the number of crossing points of the knot diagram. Then, by Euler's theorem on polyhedra, the number of connected regions into which the polygon $\Pi$ of the diagram subdivides the $xy$-plane must be $\nu + 2$. We shall denote the $\nu$ crossing points of the diagram by the symbols

$$x_\alpha, \qquad (\alpha = 1, 2, \cdots, \nu),$$

respectively, the $\nu+2$ connected regions of the plane of the diagram by the symbols

$$R_\sigma, \qquad (\sigma = 1, 2, \cdots, \nu+2),$$

respectively. A crossing point $x_\alpha$ and a region $R_\sigma$ will be said to be *incident* to one another if, and only if, the point $x_\alpha$ is on the boundary of the region $R_\sigma$. If the point $x_\alpha$ and region $R_\sigma$ are incident, they will be said to be incident *with*, or *without a dot* according as the corner of the region $R_\sigma$ contiguous to the crossing point $x_\alpha$ is or is not one of the dotted corners of the diagram. In certain exceptional cases, it may happen that a point $x_\alpha$ and a region $R_\sigma$ are doubly incident in the sense that two different corners of the region $R_\sigma$ are contiguous to the point $x_\alpha$. Whenever this occurs, the two corners of $R_\sigma$ at $x_\alpha$ must, of course, be opposite corners, so that one of them is dotted, the other one not.

To each crossing point of the diagram we shall now assign $n$ marks, thereby obtaining, in all, a set of $n\nu$ marks

$$x_{\alpha a}, \qquad \begin{pmatrix} \alpha = 1, 2, \cdots, \nu \\ a = 1, 2, \cdots, n \end{pmatrix}.$$

These marks will be regarded as the base marks of a linear system $x$ similar to the one described in the last section. For reasons of symmetry, we shall make use of a recursion formula

$$x_{\alpha a} = x_{\alpha(a+n)},$$

and do away with the requirement that the second index $a$ of the mark $x_{\alpha a}$ lie between 0 and $n+1$. The next step will be to assign to each region $R_\sigma$ of the diagram a set of $n$ homologies of the form

$$(6.1) \qquad y_{\sigma s} = \sum_\beta x_{\beta s} + \sum_\gamma x_{\gamma(s+1)} \sim 0, \qquad (s = 1, 2, \cdots, n),$$

where the marks $x_{\beta s}$ in the first sum of the $s$th homology correspond to the crossing points $x_\beta$ that are incident without dot to the region $R_\sigma$ and the marks $x_{\gamma(s+1)}$ in the second sum correspond to the crossing points $x_\gamma$ that are incident with dot to $R_\sigma$. The marks $y_{\sigma s}$ will be the base marks of a certain sub-system $Y$ of $X$. By definition, the *torsion numbers* of the knot corresponding to the given determination of the integer $n$ will be the characteristic invariants of the system $X$ with reference to its sub-system $Y$. In § 9, we shall prove that the torsion numbers, as here defined, have a genuine topological significance.

7. We shall now illustrate by an example the actual process of computing the torsion numbers of a knot. Consider the knot pictured in Fig. 1b.

Its diagram $D$ has four crossing points $x_\alpha$ determining $4n$ marks $x_{\alpha a}$, and six regions $R_\sigma$ determining $6n$ homologies. However, by a general theorem which we shall prove at the very end of the next section, the homologies determined by the outer region $R_6$ and by the region $R_5$ adjacent to the outer region are all expressible in terms of the homologies determined by the remaining four regions. Thus, the fundamental homologies reduce to

$$\begin{aligned} x_{2a} &\sim -x_{1a}, && \text{(by } R_1\text{)}, \\ x_{3a} &\sim -x_{2a} - x_{1(a+1)}, && \text{(by } R_2\text{)}, \\ &\sim x_{1a} - x_{1(a+1)}; \\ x_{4a} &\sim -x_{1a} - x_{2(a+1)}, && \text{(by } R_3\text{)}, \\ &\sim -x_{1a} + x_{1(a+1)}; \end{aligned} \tag{7.1}$$

and

$$x_{4(a+1)} + x_{3a} + x_{2(a+1)} \sim 0, \qquad \text{(by } R_4\text{)}. \tag{7.2}$$

Moreover, by means of (7.1) we may eliminate the marks $x_{2a}$, $x_{3a}$, and $x_{4a}$ in (7.2) and thereby reduce the fundamental homologies to a set of $n$ only,

$$x_{1(a+2)} - 3x_{1(a+1)} + x_{1a} \sim 0, \qquad (a = 1, 2, \cdots, n), \tag{7.3}$$

in the $n$ marks $x_{1a}$.

For $n = 2$, $x_{1(a+2)}$ and $x_{1a}$ are equal, hence the homologies (7.3) become

$$2x_{12} - 3x_{11} \sim 0, \quad -3x_{12} + 2x_{11} \sim 0. \tag{7.4}$$

Therefore, since the matrix of the coefficients in (7.4) is of rank 2 and has the elementary divisors 1 and 5, there is a single torsion number equal to 5.

For $n = 3$, the homologies (7.3) become

$$\begin{aligned} x_{13} - 3x_{12} + x_{11} &\sim 0, \\ x_{13} + x_{12} - 3x_{11} &\sim 0, \\ -3x_{13} + x_{12} + x_{11} &\sim 0. \end{aligned}$$

This time, the rank of the matrix is 3, and the elementary divisors are 1, 4, 4. Therefore, the torsion numbers are 4, 4.

A table of knots with their invariants for $n = 2$ and $n = 3$ will be given in § 11 at the end of the paper.

8. We shall now derive a few general theorems about knot diagrams. In the notation of § 6, let $x_{\alpha a}$ be the marks determined by the crossing points $x_\alpha$ of a diagram $D$ and let

$$\sum_{\beta} x_{\beta s} + \sum_{\gamma} x_{\gamma(s+1)} \sim 0 \tag{8.1}$$

be the fundamental homologies determined by the regions $R_\sigma$ of $D$. Then, if we make the change of marks

$$x_{\alpha s} = x'_{\alpha(n-s+1)}, \qquad \begin{pmatrix} \alpha = 1, 2, \cdots, \nu \\ s = 1, 2, \cdots, n \end{pmatrix},$$

which has the effect of reversing the normal cyclical order of the second subscripts $s$, the homologies (8.1) go over into the homologies

$$\sum_{\gamma} x'_{\gamma t} + \sum_{\beta} x'_{\beta(t+1)} \sim 0, \qquad (t = n-s).$$

But these last are precisely the homologies corresponding to the inverse diagram $D^{-1}$ of $D$, for to every dotted corner of the diagram $D$ there corresponds an undotted corner of the inverse diagram $D^{-1}$ and to every undotted corner of $D$, a dotted corner of $D^{-1}$. We, therefore, have the following theorem.

*A knot and its inverse have the same torsion numbers.*

We notice, next, that to each region $R_\sigma$ of the diagram $D$ there corresponds a number, called the *index* of the region, which measures the number of times, algebraically speaking, that the sensed polygon $\Pi$ of the diagram winds around the region $R_\sigma$ in a counter clockwise manner. The index of a region is, thus, a certain integer, which may be positive, negative, or zero. Obviously, if two regions meet along an edge, their indices differ by unity. Moreover, if four regions meet at a crossing point $x_\alpha$ and if the one of lowest index is of index $k-1$, two of the others must be of index $k$ and the remaining one of index $k+1$ (cf. Fig. 3). At a crossing point such as $x_\alpha$ the corner of the region of index $k-1$ is always dotted and the corner of the region of index $k+1$ undotted. One, but not both of the corners belonging to the regions of index $k$ is dotted. We shall say that the crossing point $x_\alpha$ is of *index* $k$ (where $k$ is thus the average of the indices of the regions incident to $x_\alpha$).

Now, if at each crossing point of index $k$ ($k = 1, 2, \cdots$), we remove the dot from the corner of one region of index $k$ and place it in the corner of the other region of index $k$, the effect will be to interchange upper and lower branches at all the crossing points, and, therefore, to transform the diagram $D$ into the reflection of $D$, (§ 2). On the other hand, if at each crossing point of index $k$ ($k = 1, 2, \cdots$), we remove the dot from the corner of the region of index $k-1$ and place it in the corner of the region of index $k+1$, the effect will be not only to interchange upper and lower branches but, also, to reverse the positive sense of description of the

polygon $\pi$. Therefore, the diagram $D$ will be transformed into the inverse of the reflection of $D$. Incidentally, also, the indices of all regions and crossing points will be changed in sign owing to the change of direction on the polygon $\pi$. In view of the above remarks, it will now be easy to prove the following theorem:

*A knot and its reflection have the same torsion numbers.*

For let $k$ be the index of a region $R_\sigma$ of $D$. Then, the corners of the region $R_\sigma$ may be divided into four classes: (i) Undotted corners $x_\alpha$ at crossing points of index $k+1$; (ii) undotted corners $x_\beta$ at crossing points of index $k$; (iii) dotted corners $x_\gamma$ at crossing points of index $k$; (iv) dotted corners $x_\delta$ at crossing points of index $k-1$. The $n$ homologies determined by the region $R_\sigma$ are, therefore, of the form

$$\sum_\alpha x_{\alpha s} + \sum_\beta x_{\beta s} + \sum_\gamma x_{\gamma(s+1)} + \sum_\delta x_{\delta(s+1)} \sim 0. \tag{8.2}$$

But, suppose we make the change of marks

$$x_{\alpha i} = x'_{\alpha(i+k_\alpha)},$$

where $k_\alpha$ denotes the index of the crossing point $x_\alpha$. Then the homologies (8.2) go over into the homologies

$$\sum_\alpha x'_{\alpha(t+1)} + \sum_\beta x'_{\beta t} + \sum_\gamma x'_{\gamma(t+1)} + \sum_\delta x'_{\delta t} \sim 0, \quad (t = s+k), \tag{8.3}$$

which are precisely the ones we should obtain if we were to remove the dots from the corners of the region $R_\sigma$ at crossing points of index $k+1$ and dot the corners at crossing points of index $k-1$. That is to say, the homologies (8.3) are precisely the ones determined by the region $R_\sigma$ in the inverse of the reflection of the diagram $D$. Hence, in view of the previous theorem, the homologies determined by a diagram are equivalent to the homologies determined by the reflection of the diagram, which completes the argument.

The next theorem is intended, primarily, to shorten the labor involved in computing the torsion numbers of a knot:

*The homologies determined by any two regions with consecutive indices $k$ and $k+1$ are consequences of the homologies determined by the remaining $\nu$ regions of the diagram.*

The proof of the theorem reduces, essentially, to the determination of certain identical relations between the homologies

$$y_{\sigma s} \sim 0 \tag{8.4}$$

corresponding to the different regions $R_\sigma$ of the diagram. A first set of identities will be the following:

$$H_s = \sum_\sigma \varepsilon_\sigma y_{\sigma s} \sim 0, \tag{8.5}$$

where each coefficient $\varepsilon_\sigma$ is equal to $+1$ or $-1$ according as the index of the corresponding region $R_\sigma$ is even or odd. To verify the identical character of these relations we observe that at each crossing point $x_\alpha$ there are two dotted and two undotted corners, and that one corner of each sort belongs to a region of odd index the other to a region of even index. Because of this fact, the expressions $H_s$ will be sums of groups of terms of the form

$$x_{\alpha s} - x_{\alpha s} + x_{\alpha(s+1)} - x_{\alpha(s+1)} \sim 0,$$

and will, therefore, vanish identically.

A second set of identical relations between the homologies (8.4) will next be determined. Corresponding to each region $R_\sigma$ of non-negative index $k$, let us form the sums

$$z_{\sigma s} = y_{\sigma s} + y_{\sigma(s+1)} + \cdots + y_{\sigma(s+k-1)} \sim 0$$

and corresponding to each region $R_\sigma$ of negative index $-k$, the sums

$$z_{\sigma s} = y_{\sigma(s-k)} + y_{\sigma(s-k+1)} + \cdots + y_{\sigma(s-1)} \sim 0.$$

Then, in terms of the sums $z_{\sigma s}$ we may write the identities

$$G_s = \sum_\sigma \varepsilon_\sigma z_{\sigma s} \sim 0, \tag{8.6}$$

where the coefficients $\varepsilon_\sigma$ have the same meaning as in (8.5). To prove that the relations (8.6) are identities, let us calculate how the marks corresponding to any crossing point $x_\alpha$ enter into the expressions $G_s$. We shall suppose, to fix matters, that the index of the crossing point $x_\alpha$ is positive and odd. Then, corresponding to the corner of the region of index $k-1$ incident at $x_\alpha$ we shall have the set of marks

$$x_{\alpha(s+1)} + x_{\alpha(s+2)} + \cdots + x_{\alpha(s+k-1)},$$

corresponding to the corners of the two regions of indices $k$ the sets of marks

$$-x_{\alpha s} \quad -x_{\alpha(s+1)} - \cdots - x_{\alpha(s+k-1)},$$

and

$$-x_{\alpha(s+1)} - x_{\alpha(s+2)} - \cdots - x_{\alpha(s+k)},$$

respectively, corresponding to the corner of the region of index $k+1$ the set of marks

$$x_{\alpha s} + x_{\alpha(s+1)} + \cdots + x_{\alpha(s+k)}.$$

But the sum of all these marks vanishes, therefore the expressions $G_s$ contain no marks corresponding to crossing points with positive, odd indices. By a similar argument, we may show that the expressions $G_s$ contain no marks corresponding to crossing points with indices that are not positive and odd. Therefore, the expressions $G_s$ must vanish identically.

Now, by means of the identities (8.5) we may express the homologies determined by any region of index zero (or of index $k$, for that matter) in terms of the homologies determined by the remaining $\nu+1$ regions of the diagram. Morever, by means of the identities (8.6), we may express the homologies determined by any region of index 1 in terms of the homologies determined by the remaining regions of the diagram *of indices greater than zero.* For the identities (8.6) involve none of the marks $y_{\sigma s}$ corresponding to the regions $R_\sigma$ of index 0, and each of them involves only one mark $y_{\sigma s}$, with coefficient $-1$, corresponding to each region $R_\sigma$ of index 1.

Therefore, from the combined relations (8.5) and (8.6) we may express the homologies determined by any one region of index $D$ together with the homologies determined by any one region of index $k$ in terms of the homologies determined by the remaining $\nu$ regions of the diagram. This proves the theorem for the special case $k = 0$. To prove it for a general value of $k$, we have only to redefine indices so as to lower all their values by the same constant $k$ and proceed as before.

As a consequence of the last theorem, if we wish to compute the torsion numbers of a knot we may disregard the homologies determined by any pair of regions of the knot diagram that are contiguous along an edge. For, as we have already remarked, the indices of two such regions differ by unity. Thus, we have proved the theorem assumed in § 7.

9. Our next objective will be to prove that the torsion numbers of a knot are topological invariants. To do this, it will be sufficient for us to show that the numbers are unchanged when the knot undergoes an elementary deformation of any one of the four classes pictured in Fig. 2, because we have already proved that the most general type preserving transformation of a knot may be factored into elementary deformations of precisely the four classes in question.

A deformation of Class I obviously leaves the torsion numbers invariant, since it does not disturb the incidence relations between the regions and crossing points of the diagram. A deformation of Class II creates a new crossing point $x_0$ (Fig. 2 II), corresponding to which we shall have $n$ new

marks $x_{0i}$. It also creates a new region $R_0$ incident to the crossing point $x_0$ and to no other, corresponding to which we shall have $n$ new homologies

$$x_{0i} \sim 0. \tag{9.1}$$

If we use the homologies (9.1) to eliminate the new marks $x_{0i}$ we find that the system of marks and homologies determined by the diagram after deformation reduces to the system of marks and homologies determined by it before deformation; therefore, the torsion numbers are invariant under the deformation. A deformation of Class III creates two new crossing points $x_0'$ and $x_0''$, (Fig. 2 III). We may assume that the branch $ACB$ passes over the branch $x_0' x_0''$, otherwise we could replace the diagram by its reflection, and that the corners are dotted as indicated in the figure, otherwise we could replace the diagram by its inverse. Now, it will be observed that two new regions $R'$ and $R''$ are created, (among others), such that the first is incident to the crossing point $x_0'$ but not to the crossing point $x_0''$, whereas the second is incident to $x_0''$ but not to $x_0'$. Let

$$x_{0i}' + \psi_i' \sim 0 \tag{9.2}$$

and

$$x_{0i}'' + \psi_i'' \sim 0 \tag{9.3}$$

be the homologies determined by the regions $R'$ and $R''$ respectively, where $\psi_i'$ and $\psi_i''$ are linear combinations of marks determined by the original crossing points. Then, if we use the homologies (9.2) and (9.3) to eliminate the new marks $x_{0i}'$ and $x_{0i}''$, we may again verify that the system of marks and homologies after deformation reduces to the system of marks and homologies before deformation. Therefore, in this case also, the torsion numbers are invariant. A deformation of Class IV destroys two crossing points $x_1$ and $x_2$, and creates two new ones $x_1'$ and $x_2'$, (Fig. 2 IV). Three cases can arise according as the branch $AB$ passes over both, one, or neither of the branches $x_1 x_1'$ and $x_2 x_2'$. However, the third case reduces to the first by a reflection of the diagram, leaving only the first two to be considered. If the branch $AB$ passes over the two branches $x_1 x_1'$ and $x_2 x_2'$ the branches $AC$ and $BC$ must pass over the branches $x_1 x_1'$ and $x_2 x_2'$ respectively. We may, therefore, assume that the corners at the crossing points $x_1$, $x_1'$, $x_2$, and $x_2'$ are dotted in the manner indicated in the figure, otherwise we could replace the diagram by its inverse. There are, however, two essentially different ways in which the corners at the crossing point $x_0$ may be dotted. If they are dotted in the manner indicated in Fig. 2 IV, the change of marks

$$
\begin{aligned}
x_{0i} &= x'_{1(i+1)} + x'_{2(i+1)}, \\
x_{1i} &= x'_{0(i+1)} + x'_{2i}, \\
x_{2i} &= x'_{0i} \quad + x'_{1i}
\end{aligned}
\tag{9.4}
$$

will transform the marks and homologies before deformation into the marks and homologies after deformation, as may easily be verified. It may happen, however, that the corner of the triangle $x_0 x_1 x_2$ at the crossing point $x_0$ is not dotted but that the opposite corner at $x_0$ is. In this case, we must make the change of marks

$$
\begin{aligned}
x_{0i} &= x'_{1i} \quad + x'_{2i}, \\
x_{1i} &= x_{0(i+1)} + x'_{2i}, \\
x_{2i} &= x'_{0i} \quad + x'_{1i}
\end{aligned}
\tag{9.5}
$$

instead of the change (9.4). There are two other possible ways in which the corners at the crossing point $x_0$ may be dotted, but each reduces to one of the cases already considered by a mere change of notation.

If the branch $AB$ passes over one, but not both, of the branches $x_1 x'_1$ and $x_2 x'_2$ we may assume that it passes over the branch $x_1 x'_1$, otherwise, we could replace the diagram by its reflection. We may also assume that the positive direction along the edge $AB$ is the direction from $B$ to $A$, otherwise we could replace the diagram by its inverse. The positive direction along the branch $x_2 x'_2$ (which, in this case, passes over the branches $AB$, $x_1 x'_1$, and $AC$) may now be either the direction from $x_2$ to $x'_2$ or the direction from $x'_2$ to $x_2$. If the positive direction is from $x_2$ to $x'_2$, we make the change of marks

$$
\begin{aligned}
x_{0i} &= x'_{1(i+1)} + x'_{2i}, \\
x_{1i} &= x'_{0(i+1)} + x'_{2(i+1)}, \\
x_{2i} &= x'_i \quad + x'_{1i},
\end{aligned}
\tag{9.6}
$$

if the positive direction is from $x'_2$ to $x_2$ we make the change of marks

$$
\begin{aligned}
x_{0i} &= x'_{1i} + x'_{2i}, \\
x_{1i} &= x'_{0i} + x'_{2i}, \\
x_{2i} &= x'_{0i} + x'_{1(i-1)}.
\end{aligned}
\tag{9.7}
$$

The effect, in either case, will be to transform the system of marks and homologies before deformation into the system after deformation. Thus, a deformation of Class IV leaves the torsion numbers invariant. This completes the argument.

10. Before bringing the discussion to a close, we shall indicate how the torsion numbers of a knot $K$, for any given determination of the integer $n$, may be interpreted in terms of the Betti numbers and coefficients of torsion

of an $n$-sheeted covering space $J_n$ with a branch curve of order $n-1$ covering the knot. The covering space $J_n$ will be thought of as spread out upon the closed space $J$ obtained by adding a point at infinity to the space of the knot and treating the point at infinity as a limit point of every unbounded set of points in the space of the knot.

The first step in the argument will be to make a cellular subdivision $\Sigma$ of the carrier space $J$ with a view to determining, later on, a subdivision of the covering space $J_n$. To each crossing point $x_\alpha$ of the knot diagram there correspond two points of the knot itself, a point $P_\alpha$ on an upper branch and a point $Q_\alpha$ on a lower branch. We shall choose the combined points $P_\alpha$ and $Q_\alpha$ as the $2\nu$ vertices, or 0-cells, of the subdivision $\Sigma$. The edges, or 1-cells, of the subdivision will consist, first, of the $\nu$ linear segments $P_\alpha Q_\alpha$ joining the points of corresponding upper and lower branches of the knot, secondly, of the $2\nu$ arcs into which the points $P_\alpha$ and $Q_\alpha$ subdivide the knot. There will, thus, be $3\nu$ edges in all. The 2-cells of the subdivision will be a set of $\nu+1$ properly selected simply connected, regular surfaces $S_\sigma$, $(\sigma = 0, 1, \cdots, \nu)$, one associated with each region $R_\sigma$ of the diagram with the exception of the outer region of all. Each surface $S_\sigma$ will be so chosen that (i) it projects orthogonally in a point-for-point manner upon the region $R_\sigma$ with which it is associated and that (ii) its boundary is a simple polygon made up of the portions of the knot which project upon the boundary of the region $R_\sigma$ together with the edges $P_\alpha Q_\alpha$ which project upon the crossing points $x_\alpha$ on the boundary of $R_\sigma$. The subdivision $\Sigma$ will have a single 3-cell $C_3$ consisting of all points of the space $J$ that are not on any cell of lower dimensionality.

To the subdivision $\Sigma$ of the carrier space $J$ there corresponds a subdivision $\Sigma_n$ of the covering space $J_n$ such that each cell $C$ of the subdivision $\Sigma$ is covered by $n$ superimposed cells of the subdivision $\Sigma_n$ unless it so happens that the cell $C$ is made up of points of the knot $K$. In the latter case, the cell $C$ is covered by a single cell of the subdivision $\Sigma_n$, for the knot $K$ determines the branch curve of the space $J_n$ along which the $n$ sheets of $J_n$ merge into one. By a straightforward calculation, we find that the subdivision $\Sigma_n$ comprises $2\nu$ vertices, $\nu n + 2\nu$ edges, $n\nu + n$ 2-cells, and $n$ 3-cells.

The indices $k_\sigma$ assigned to the regions $R_\sigma$ of the diagram (§ 7) have a direct bearing on the incidence relations among the cells of the subdivision $\Sigma_n$. Let $S_\sigma$ be the 2-cell of the subdivision $\Sigma$ determined by the region $R_\sigma$, and let

$$S_{\sigma i}, \qquad (i = 1, 2, \cdots, n),$$

be the $n$ 2-cells of the subdivision $\Sigma_n$ that cover the 2-cell $S_\sigma$. Moreover, let

$$C_i, \qquad (i = 1, 2, \cdots, n),$$

be the $n$ 3-cells of the subdivision $\Sigma$. Then, we are to think of $C_i$ and $C_{i+k_\sigma}$ as the two 3-cells incident to the 2-cell $S_{\sigma i}$. (Here, of course, the index $i+k_\sigma$ is to be reduced, modulo $n$, to an integer between 0 and $n+1$.) More precisely, we are to imagine that we pass from the cell $C_i$ to the cell $C_{i+k_\sigma}$ when we travel across the cell $S_{\sigma i}$ on a path leading up to $S_{\sigma i}$ from below.

In order to calculate the topological invariants of the covering space $J_n$ with a minimum of effort, we shall replace the subdivision $\Sigma_n$ by another composed of fewer cells. Suppose two distinct 3-cells of the subdivision $\Sigma_n$ are incident to the same 2-cell. Then, we obviously have the right to combine together the 2-cell and the two 3-cells so as to form a single 3-cell. Suppose, moreover, that two distinct 0-cells are incident to the same 1-cell. Then, we also have the right to deform the elements of the subdivision in such a manner that the 1-cell and its two ends shrink up into a single 0-cell, or, what comes to the same thing, to treat the 1-cell and its two ends as a "generalized" 0-cell. The two simplifying operations described above are the duals of one another.

With these facts in mind, let $R_0$ be a region of the knot diagram which is adjacent along an edge to the very outer region of all and which, therefore, has the index $\pm 1$. Then, corresponding to the region $R_0$ there will be $n$ 2-cells,

$$S_{0s}, \qquad (s = 1, 2, \cdots, n),$$

of the subdivision $\Sigma_n$. As a first simplification, we shall merge the first $n-1$ 2-cells $S_{0s}$ with the $n$ 3-cells $C_i$ to form a single 3-cell $C'$ with a singular boundary. This amalgamation will merely involve $n-1$ repetitions of the first simplifying operation described above. On the boundary of the 3-cell $C'$, the remaining 2-cell $S_{0n}$ will appear twice, once in positive relation to the boundary and once in negative.

Now, let $K'$ denote the circuit composed of the 0-cells and 1-cells of the subdivision $\Sigma_n$ which cover the points of the knot $K$. There will then be one definite 1-cell $x$ of the circuit $K'$ which is on the boundary of the 2-cell $S_{0n}$ but not on the boundary of any of the other 2-cells $S_{\sigma s}$ $(\sigma = 1, 2, \cdots, n)$, namely, the 1-cell which projects on the arc separating the region $R_0$ from the outer region of the diagram. As a second simplification of the subdivision we shall join together all the 0-cells and 1-cells of the circuit $K'$ with the exception of the 1-cell $x$ so as to form a single generalized 0-cell. This will involve $2\nu-2$ applications of the second simplification process.

After the two simplifications described above, we shall be left with a subdivision $\Sigma_n'$ consisting of one generalized vertex, $\nu n+1$ edges, $\nu n+1$

2-cells, and one 3-cell. Moreover, each of the edges, when we include its ends, will be in the nature of a closed curve, or 1-circuit, since its two ends coincide at the one vertex present. One of the edges will be the 1-cell $x$ on the boundary of the 2-cell $S_{0n}$, the others will be the 1-cells covering the segments $P_\alpha Q_\alpha$. We shall denote the latter by the marks $x_{\alpha\alpha}$ respectively.

Now, let us write the homologies (in the sense of analysis situs) which express the fact that the boundary of each 2-cell of the subdivision $\Sigma'_n$ is a linear combination of the 1-circuits determined by the respective 1-cells of $\Sigma'_n$. Then, it is easy to verify that if the 1-cells $x_\alpha$ are suitably sensed, the homology determined by the 2-cell $S_{0n}$ is

$$x + y_{0n} \sim 0 \tag{10.1}$$

and that the homologies determined by the $\nu n$ 2-cells $S_{\sigma n}$, $(\sigma > 0)$, are

$$y_{\sigma s} \sim 0, \tag{10.2}$$

where the symbols $y_{0s}$ and $y_{\sigma s}$ have precisely the meanings attached to them in § 6. Moreover, since the relation

$$y_{0n} \sim 0$$

follows as a consequence of the relations

$$y_{\sigma n} \sim 0, \qquad (\sigma = 1, 2, \cdots, n),$$

(by a theorem in § 6), we may reduce the homology (10.1) to an homology

$$x \sim 0. \tag{10.3}$$

The geometrical significance of this last homology is that *the branch curve of the space $J_n$, is always a bounding 1-circuit of $J_n$.* For the 1-cell $x$ and its "generalized" ends represent the entire branch curve. Finally, if we eliminate the mark $x$ by means of the homology (10.3), we are left with the homologies (10.2) which determine the torsion numbers of the knot. The exact relation between the torsion numbers of the knot and the topological invariants of the covering space $J_n$ may be expressed as follows:

*There are exactly as many vanishing torsion numbers as there are 1-circuits of the space $J_n$ linearly independent with respect to homologies. Moreover, the torsion numbers that are greater than zero are the coefficients of torsion of the space $J_n$.*

11. We give below a table of invariants for all types of knots of nine or less crossings listed as distinct by Tait and Kirkman. The symbols appearing in the first column are the distinguishing symbols of the various knots pictured. In each symbol, the large numeral denotes the minimal

number of crossings that the diagram of the knot can have, while the indices $a$ or $n$ indicate whether or not a knot is alternating. The numbers in the second and third columns are the torsion numbers for the cases $n = 2$ and $n = 3$ respectively.

The two members of each of the three pairs of knots marked with the asterisks, crosses and zeros respectively have the same torsion numbers for $n = 2$ and also for $n = 3$. We have found, as a matter of fact, that they also have the same torsion numbers for each value of $n$ so that it is impossible to distinguish them by their torsion numbers. In all other cases considered, two knots were distinguished by their torsion numbers.

| Type | $N=2$ | $N=3$ | Type | $N=2$ | $N=3$ | Type | $N=2$ | $N=3$ |
|---|---|---|---|---|---|---|---|---|
| $3_{1a}$ | 3 | 2, 2 | $8_{15a}$ | 33 | 16, 16 | $9_{22a}$ | 43 | 14, 14 |
| $4_{1a}$ | 5 | 4, 4 | $8_{16a}$ | 35 | 11, 11 | $9_{23a}$ | 45 | 22, 22 |
| $5_{1a}$ | 5 | none | $8_{17a}$ | 37 | 13, 13 | $9_{24a}$ | 45 | 16, 16 |
| $5_{2a}$ | 7 | 5, 5 | $8_{18a}$ | 3, 15 | 2, 2, 8, 8 | $9_{25a}$ | 47 | 26, 26 |
| $6_{1a}$ | 9 | 7, 7 | $8_{19n}$ | 5 | 4, 4 | $9_{26a}$ | 47 | 17, 17 |
| $6_{2a}$ | 11 | 5, 5 | $8_{20n}$ | 9 | 4, 4 | $9_{27a}$ | 49 | 19, 19 |
| $6_{3a}$ | 13 | 7, 7 | $8_{21n}$ | 15 | 8, 8 | $^{\circ}9_{28a}$ | 51 | 20, 20 |
| $7_{1a}$ | 7 | none | $9_{1a}$ | 9 | none | $^{\circ}9_{29a}$ | 51 | 20, 20 |
| $7_{2a}$ | 11 | 8, 8 | $^{*}9_{2a}$ | 15 | 11, 11 | $9_{30a}$ | 53 | 22, 22 |
| $7_{3a}$ | 13 | 4, 4 | $9_{3a}$ | 19 | none | $9_{31a}$ | 55 | 23, 23 |
| $^{*}7_{4a}$ | 15 | 11, 11 | $9_{4a}$ | 21 | 7, 7 | $9_{32a}$ | 59 | 23, 23 |
| $7_{5a}$ | 17 | 7, 7 | $9_{5a}$ | 23 | 17, 17 | $9_{33a}$ | 61 | 25, 25 |
| $7_{6a}$ | 19 | 11, 11 | $9_{6a}$ | 27 | 4, 4 | $9_{34a}$ | 69 | 31, 31 |
| $7_{7a}$ | 21 | 13, 13 | $9_{7a}$ | 29 | 13, 13 | $9_{35a}$ | 3, 9 | 20, 20 |
| $8_{1a}$ | 13 | 10, 10 | $^{\times}9_{8a}$ | 31 | 17, 17 | $9_{36a}$ | 15 | 2, 2, 4, 4 |
| $8_{2a}$ | 17 | none | $9_{9a}$ | 31 | 5, 5 | $9_{37a}$ | 3, 15 | 28, 28 |
| $8_{3a}$ | 17 | 13, 13 | $9_{10a}$ | 33 | 13, 13 | $9_{38a}$ | 57 | 28, 28 |
| $8_{4a}$ | 19 | 8, 8 | $9_{11a}$ | 33 | 7, 7 | $9_{39a}$ | 55 | 32, 32 |
| $8_{5a}$ | 21 | 4, 4 | $9_{12a}$ | 35 | 20, 20 | $9_{40a}$ | 5, 15 | 4, 4, 8, 8 |
| $8_{6a}$ | 23 | 11, 11 | $9_{13a}$ | 37 | 16, 16 | $9_{41a}$ | 7, 7 | 28, 28 |
| $8_{7a}$ | 23 | 5, 5 | $9_{14a}$ | 37 | 22, 22 | $9_{42n}$ | 7 | 2, 2 |
| $8_{8a}$ | 25 | 13, 13 | $9_{15a}$ | 39 | 23, 23 | $9_{43n}$ | 13 | 2, 2 |
| $8_{9a}$ | 25 | 7, 7 | $9_{16a}$ | 39 | 8, 8 | $9_{44n}$ | 17 | 10, 10 |
| $8_{10a}$ | 27 | 8, 8 | $9_{17a}$ | 39 | 11, 11 | $9_{45n}$ | 23 | 14, 14 |
| $8_{11a}$ | 27 | 14, 14 | $9_{18a}$ | 41 | 19, 19 | $9_{46n}$ | 3, 3 | 7, 7 |
| $8_{12a}$ | 29 | 19, 19 | $9_{19a}$ | 41 | 25, 25 | $9_{47n}$ | 3, 9 | 5, 5 |
| $8_{13a}$ | 29 | 16, 16 | $9_{20a}$ | 41 | 13, 13 | $9_{48n}$ | 3, 9 | 17, 17 |
| $^{\times}8_{14a}$ | 31 | 17, 17 | $9_{21a}$ | 43 | 26, 26 | $9_{49n}$ | 5, 5 | 10, 10 |

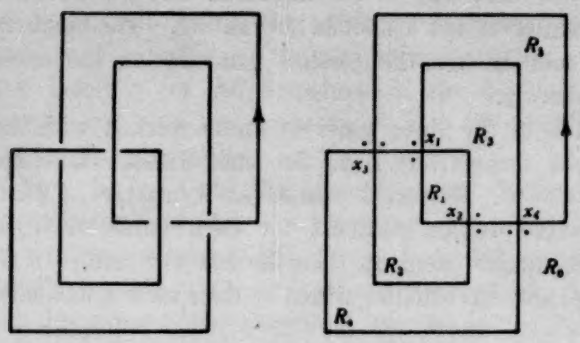
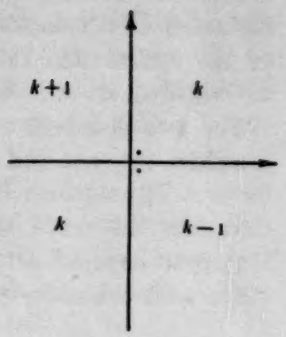



Fig. 1.

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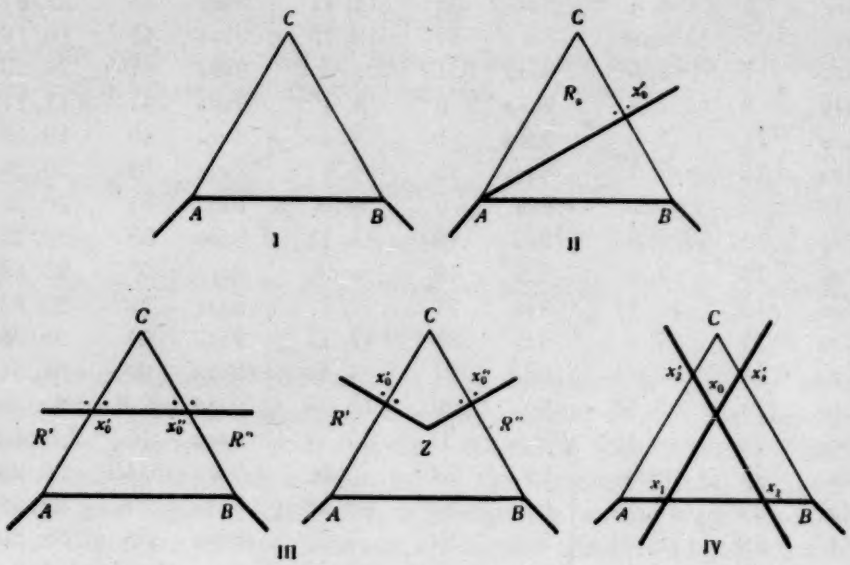
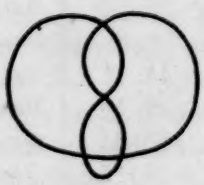



Fig. 2.

Fig. 31a.



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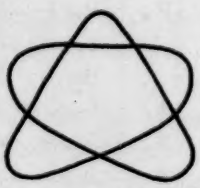
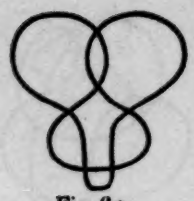


Fig. 51a.

Fig. 52a.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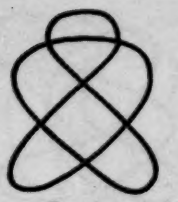
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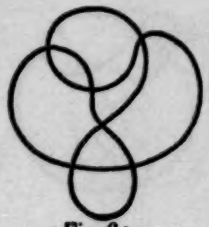
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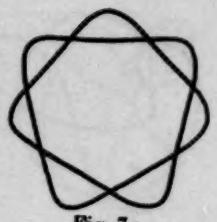
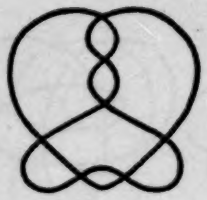


Fig. 71a.

Fig. 72a.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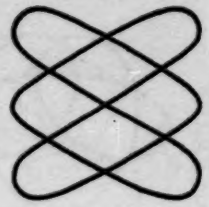
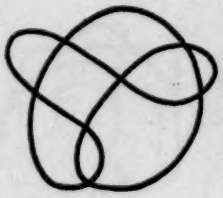


Fig. 74a.

Fig. 75a.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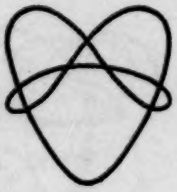
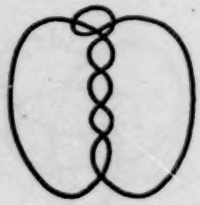


Fig. 77a.

Fig. 81a.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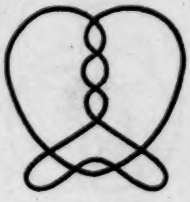


Fig. 83a.

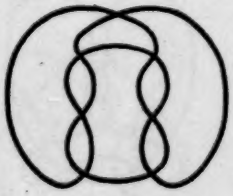


Fig. 84a.

Fig. 85a.

Fig. 86a.

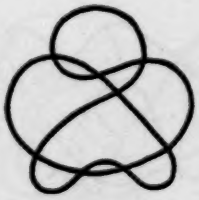


Fig. 87a.

Fig. 88a.

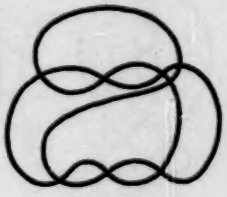


Fig. 89a.

Fig. 810a.

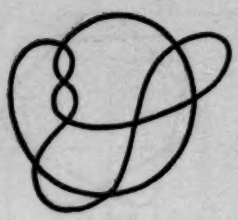
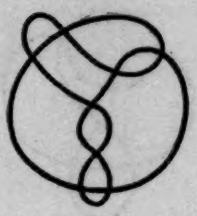


Fig. 8 11a.



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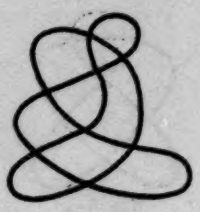
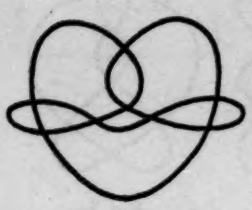


Fig. 8 13a.

Fig. 8 14a.



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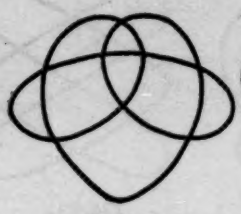

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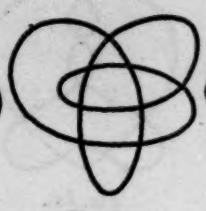


Fig. 8 17a.

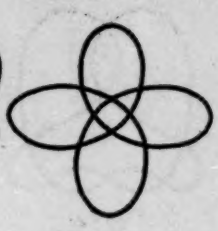


Fig. 8 18a.

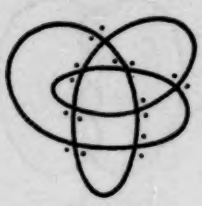


Fig. 8 19a.

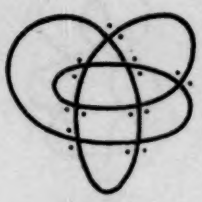


Fig. 8 20a.

Fig. 8 21a.

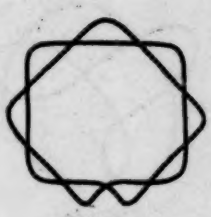


Fig. 9 1a.

Fig. 9 2a.

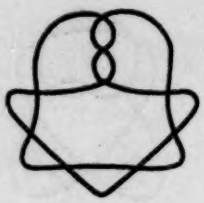


Fig. 9 3a.

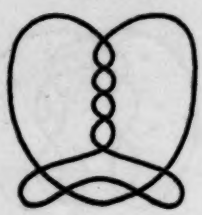


Fig. 9 4a.

Fig. 9 5a.

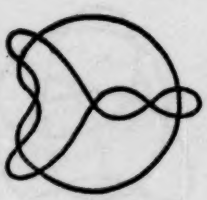


Fig. 9 6a.

Fig. 9 7a.

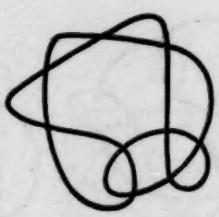


Fig. 9 8a.

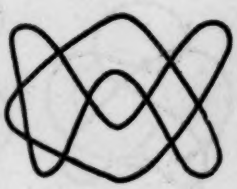


Fig. 9 9a.

Fig. 9 10a.

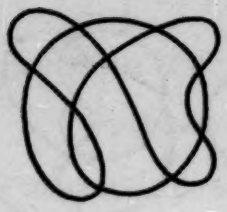


Fig. 9 11a.

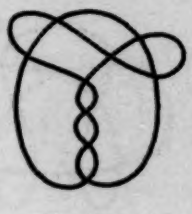


Fig. 9 12a.

Fig. 9 13a.

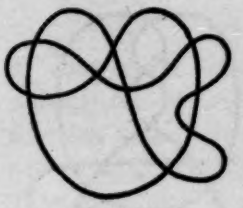


Fig. 9 14a.

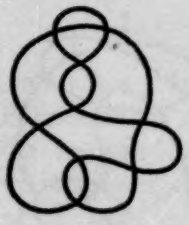


Fig. 9 15a

Fig. 9 16a.

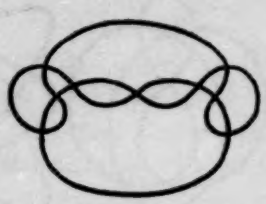
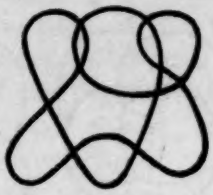


Fig. 9 17a.

Fig. 9 18a.



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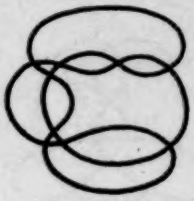


Fig. 9 20a.

Fig. 9 21a.

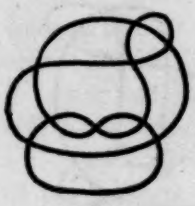
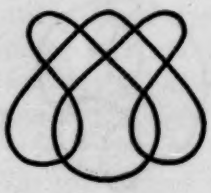


Fig. 9 22a.



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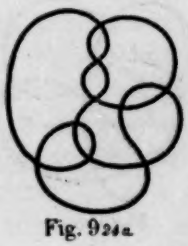

Fig. 9 24a.

Fig. 9 25a.

Fig. 9 26a.

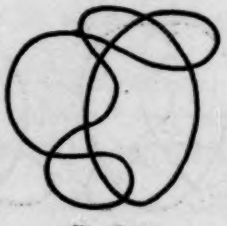

Fig. 9 27a.

Fig. 9 28a

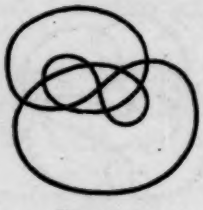

Fig. 9 29a.

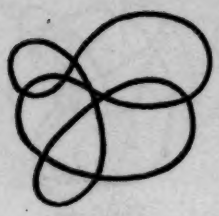

Fig. 930a.

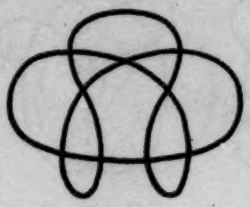

Fig. 931a.

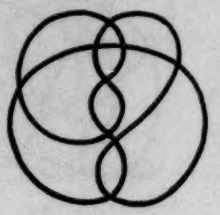

Fig. 932a.

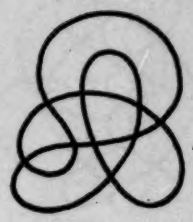

Fig. 933a.

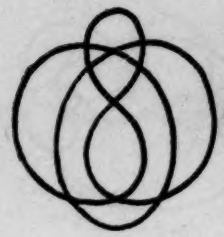

Fig. 934a.

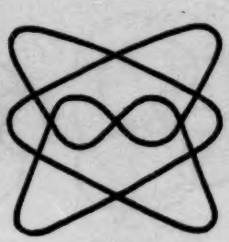

Fig. 935a.

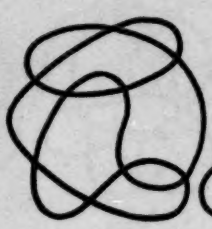

Fig. 936a.

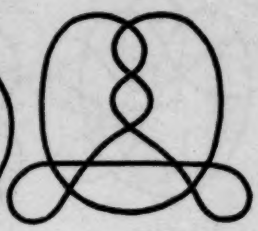

Fig. 937a.

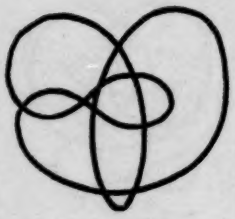

Fig. 938a.

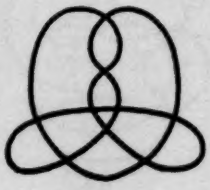

Fig. 939a.

Fig. 940a.

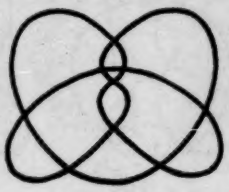

Fig. 941a.

Fig. 942n.

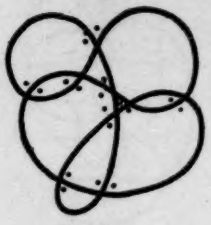

Fig. 943n.

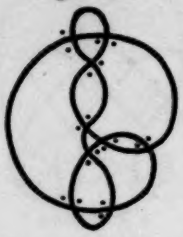

Fig. 944n.

Fig. 945n.

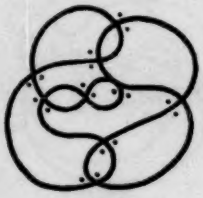

Fig. 946n.

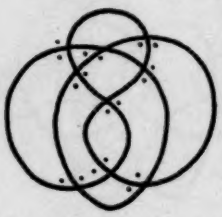

Fig. 947n.

Fig. 948n.

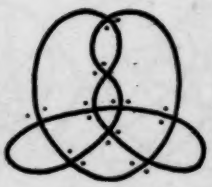

Fig. 949n.

# A PROBLEM IN MINIMA.*

By Dunham Jackson.

If $f(x)$ is a given function of period $2\pi$, the trigonometric sum $T_n(x)$, of order $n$, which minimizes the integral

$$\int_{-\pi}^{\pi}[f(x)-T_n(x)]^2\,dx,$$

in comparison with all other trigonometric sums of like order, is the partial sum of the Fourier series for $f(x)$. In terms of the geometry of function space, if $P$ is the point corresponding to $f(x)$, and $T$ the point representing $T_n(x)$, it is a question of minimizing the distance $PT$, when $P$ is fixed and $T$ ranges over the linear $(2n+1)$-dimensional space of trigonometric sums of the $n$th order. A natural step in the direction of generalization is this. Let $f(x)$ and $\varphi(x)$ be two given functions of period $2\pi$, represented by points $P$ and $Q$ in function space: to determine the point $T$ which minimizes the area of the triangle $PTQ$, or, in analytic language,† to find the trigonometric sum $T_n(x)$, of the $n$th order, which minimizes the Gramian of the functions $f(x)-T_n(x)$ and $\varphi(x)-T_n(x)$. The existence of a solution of this minimum problem will be discussed below, together with the question of the convergence of the minimizing sums $T_n(x)$ as $n$ becomes infinite.

Let it be supposed throughout, for convenience, that $f(x)$ and $\varphi(x)$ are continuous. If $f$ and $\varphi$ themselves are trigonometric sums of the $n$th order, any $T_n(x)$ of the form $cf(x)+(1-c)\varphi(x)$ will reduce the Gramian to zero; this case will be left out of consideration henceforth. There may still be a single value of $c$, though there can not be more than one, for which $cf+(1-c)\varphi$ is of the form $T_n(x)$, and so gives a trivial solution of the problem. This will happen whenever there is a linear combination $c_1f+c_2\varphi$, with $c_1+c_2\neq 0$, which is a trigonometric sum of the $n$th order, as may be seen be taking $c=c_1/(c_1+c_2)$. This special case also may be put aside.

Under all other circumstances, the points $P$, $T$, $Q$ form a triangle of positive area, whatever trigonometric sum of the $n$th order is represented by $T$. The base $PQ$ being fixed, the point $T$ has to be found so that

* Presented to the American Mathematical Society, April 16, 1927; received April 28, 1927.

† Cf., e. g., D. Jackson, The elementary geometry of function space, Amer. Math. Monthly, vol. 31 (1924), pp. 461–471; p. 463.

its perpendicular distance from the line $PQ$, constituting the altitude of the triangle, shall be as small as possible. If $\psi(x) = f(x) - \varphi(x)$, an arbitrary point of the line $PQ$ corresponds to a function of the form

$$\omega(x) = \varphi(x) + \lambda\psi(x) = \lambda f(x) + (1-\lambda)\varphi(x).$$

The foot of the perpendicular to $PQ$ from the nearest point $T$ represents the function $\omega(x)$ which can be most closely approximated by a trigonometric sum of the $n$th order, the approximation being measured by the integral of the square of the error.

One more special case remains to be considered, namely that in which $\psi(x)$ is a trigonometric sum of the $n$th order, neither $f$ nor $\varphi$ by itself being such a sum. In this case all functions of the form $\omega(x)$ can be equally well approximated, since the addition of a constant multiple of $\psi(x)$ simultaneously to the function to be approximated and to the approximating function leaves the integral of the square of the error unchanged. The minimum problem has a solution in the partial sum of the Fourier series for any function $\omega(x)$, through terms of the $n$th order.

Let it be supposed from now on that no linear combination of $f(x)$ and $\varphi(x)$ is identically equal to a trigonometric sum of the $n$th order. This is equivalent to assuming that the $2n+3$ functions $\varphi(x)$, $\psi(x)$, $1, \cos x, \cdots, \cos nx, \sin x, \cdots, \sin nx$, are linearly independent. If $T_n(x)$ is represented by the notation

$$T_n(x) = \frac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + \cdots + a_n\cos nx + b_1\sin x + b_2\sin 2x + \cdots + b_n\sin nx,$$

the problem is to determine the $2n+2$ parameters $\lambda, a_0, a_1, \cdots, a_n, b_1 \cdots, b_n$, so as to minimize the integral

$$\int_{-\pi}^{\pi}[\varphi(x) + \lambda\psi(x) - T_n(x)]^2\,dx,$$

regarded as a function of the $2n+2$ variables mentioned. To shift the emphasis slightly, a least-square approximation is sought for $\varphi(x)$, by means of a linear combination of the $2n+2$ linearly independent functions $\psi(x), 1, \cos x, \cdots, \cos nx, \sin x, \cdots, \sin nx$. *It is well known that this problem has one and just one solution.**

For each value of $n$, let $S_n(x)$ be the trigonometric sum which enters into the solution—the sum, then, which solves the problem originally

* Cf., e. g., D. Jackson, On functions of closest approximation, Trans. Amer. Math. Soc., vol. 22 (1921), pp. 117–128; Theorem I.

proposed—and let $\lambda_n$ be the corresponding value of $\lambda$. One proceeds to inquire as to the convergence of these quantities when $n$ becomes infinite. It is to be borne in mind that $S_n(x)$ is the partial sum of the Fourier series for the function $\varphi(x) + \lambda_n \psi(x)$.

Let $a_k$, $b_k$ be the coefficients of $\cos kx$ and $\sin kx$ respectively in the Fourier series for $\varphi(x)$, and let $\alpha_k$, $\beta_k$ be the corresponding coefficients in the case of $\psi(x)$. Let it be supposed that $\psi(x)$ is not a trigonometric sum of any order—in other words, that infinitely many of its Fourier coefficients are different from zero. The integral, extended over a period, of the square of the difference between $\varphi + \lambda\psi$ and the partial sum of its Fourier series through terms of the $n$th order is*

$$\pi \sum [(a_k + \lambda \alpha_k)^2 + (b_k + \lambda \beta_k)^2]$$
$$= \pi \left[ \sum (a_k^2 + b_k^2) + 2\lambda \sum (a_k \alpha_k + b_k \beta_k) + \lambda^2 \sum (\alpha_k^2 + \beta_k^2) \right],$$

the summation being extended in each case from $n+1$ to $\infty$, and $\lambda_n$ is merely the value of $\lambda$ which minimizes this expression. By simple differentiation it is found that

$$\lambda_n = -\frac{\sum (a_k \alpha_k + b_k \beta_k)}{\sum (\alpha_k^2 + \beta_k)}.$$

The next question at issue is whether $\lambda_n$ approaches a limit as $n$ becomes infinite. It is recognized by the construction of examples that no simple general answer is possible. For convenience, let $b_k = \beta_k = 0$ in each case, for all values of $k$, so that the Fourier series in question are series of cosines only. As a first example, let $a_k = 1/k^3$, $\alpha_k = 1/k^2$. The corresponding cosine series are uniformly convergent, and represent continuous functions. The quantity $\sum_{n+1}^{\infty} a_k \alpha_k$ is of the order of magnitude of $1/n^4$, while $\sum_{n+1}^{\infty} \alpha_k^2$ is of the order of $1/n^3$, and $\lim_{n=\infty} \lambda_n = 0$. On the other hand, if the definitions of $a_k$ and $\alpha_k$ are interchanged, $\lim_{n=\infty} \lambda_n = \infty$. Thirdly, if $a_k = 0$ when $k$ is even, and $\alpha_k = 0$ when $k$ is odd, then $\lambda_n = 0$ constantly, regardless of the relative orders of magnitude of the coefficients. Finally, let $\alpha_k = 1/2^{k/2}$ for all values of $k$, and let $a_k = 1/2^{k/2}$ or 0 according as $k$ is odd or even; then $\lambda_n$ is alternately equal to 2/3 and to 1/3, and remains bounded without approaching a limit.

It may be pointed out at once that if $\lambda_n$ does approach a limit $A$, and if, as will be supposed henceforth, $f(x)$ and $\varphi(x)$, and consequently $\psi(x)$, are such that their respective Fourier series are uniformly convergent,

* Cf., e. g., Bôcher, Introduction to the theory of Fourier's series, Annals of Math., (2), vol. 7 (1906), pp. 81–152; pp. 85, 107.

then $S_n(x)$ converges uniformly toward $\varphi(x)+\Lambda\psi(x)$. For the remainder in the series for $\varphi+\lambda\psi$ is merely the sum of the corresponding remainders for $\varphi$ and $\psi$ multiplied by 1 and $\lambda$ respectively, and the series for $\varphi+\lambda\psi$ converges uniformly with regard to both $\lambda$ and $x$, as long as $\lambda$ remains bounded; and the difference between $S_n(x)$ and $\varphi(x)+\Lambda\psi(x)$ is the sum of the quantity $(\Lambda-\lambda_n)\psi$ and the remainder in the series for $\varphi+\lambda_n\psi$. Similarly, if a sub-sequence of the numbers $\lambda_n$ approaches a limit, the corresponding sub-sequence of sums $S_n(x)$ uniformly approaches the combination $\varphi+\lambda\psi$ formed with the limiting value of $\lambda$ in question. So the convergence of $\lambda_n$ may be regarded as the main problem.

It may be noted also that the problem is concerned from the beginning with the family of functions $\varphi+\lambda\psi=\lambda f+(1-\lambda)\varphi$, rather than with $f$ and $\varphi$ individually. The conditions would be essentially unchanged if $f$ and $\varphi$ were replaced by two other functions of the family, and the value $\lambda=0$, which appears to have a peculiar significance in some of the examples, would be replaced by a different value.

It has been observed that in some cases the behavior of the sequence $\lambda_n$ depends on the relative orders of magnitude of the Fourier coefficients for $\varphi$ and $\psi$. More generally, by Schwarz's inequality,

$$\begin{aligned}|\sum(a_k\alpha_k+b_k\beta_k)| &\leqq |\sum a_k\alpha_k|+|\sum b_k\beta_k| \\ &\leqq (\sum a_k^2)^{1/2}(\sum\alpha_k^2)^{1/2}+(\sum b_k^2)^{1/2}(\sum\beta_k^2)^{1/2} \\ &\leqq [\sum(a_k^2+b_k^2)]^{1/2}[\sum(\alpha_k^2+\beta_k^2)]^{1/2};\end{aligned}$$

the last member is justified by noting that if $a$, $b$, $c$, $d$ are any four non-negative quantities,

$$(ab)^{1/2}+(cd)^{1/2}\leqq(a+c)^{1/2}(b+d)^{1/2}.$$

Consequently

$$|\lambda|\leqq[\textstyle\sum_{n+1}^{\infty}(a_k^2+b_k^2)]^{1/2}/[\sum_{n+1}^{\infty}(\alpha_k^2+\beta_k^2)]^{1/2},$$

and it can be stated definitely that *if the Fourier series for* $\varphi(x)$ *converges more rapidly than the series for* $\psi(x)$, *in the sense that the integral of the square of the error is an infinitesimal of higher order in the former case then in the latter, then* $\lambda_n$ *approaches* 0 *as* $n$ *becomes infinite, and* $S_n(x)$ *converges uniformly toward* $\varphi(x)$. It should be pointed out that the replacement of $f$ and $\varphi$ by two other functions of the family $\lambda f+(1-\lambda)\varphi$ does not affect the order of convergence of the Fourier seriers for their difference, since that difference is replaced merely by a constant multiple of itself. But the convergence might be more rapid for some other function

that would serve in place of $\varphi$, then it is for $\varphi$. The essential thing, then, in the formulation of a sufficient condition for the convergence of $\lambda_n$ to some definite limit, is not that there should be a peculiarity in the function that happens to have been denoted by $\varphi$, but that there should be in the family some function whose Fourier series converges with exceptional rapidity. *If the family of functions $\lambda f+(1-\lambda)\varphi$ is such that the Fourier series for an individual function of the family converges at least as rapidly as the series for the difference of two of them, the values of $\lambda_n$ will be bounded, and will converge toward one or more limits.*

For an example, not without some generality, of the conditions insuring convergence of $\lambda_n$ to zero, let $\varphi(x)$ have a continuous second derivative everywhere, and let $f(x)$ have a continuous second derivative, except that for a single point in each period, say for $x = 0$ and congruent points, the right-hand and left-hand first derivatives are different. Then $\psi(x)$ and all the functions of the family $\lambda f+(1-\lambda)\varphi$, except $\varphi$ itself, have the properties assigned to $f(x)$. It is well known, and readily verified on the basis of successive integrations by parts, like those to be performed a few lines below, that

$$\lim_{k=\infty} k^2 a_k = \lim_{k=\infty} k^2 b_k = 0.$$

Hence it follows that

$$\lim_{n=\infty} n^3 \sum_{n+1}^{\infty} (a_k^2+b_k^2) = 0.$$

Considered merely for $0 \leqq x \leqq 2\pi$, without regard to periodicity, the function $\psi(x)$ has a continuous second derivative throughout the closed interval, but $\psi'(0+) \neq \psi'(2\pi-)$. By integration by parts,

$$\pi a_k = \int_0^{2\pi} \psi(x) \cos kx\, dx = -\frac{1}{k}\int_0^{2\pi} \psi'(x) \sin kx\, dx$$

$$= \frac{1}{k^2}[\psi'(x)\cos kx]_0^{2\pi} - \frac{1}{k^2}\int_0^{2\pi} \psi''(x)\cos kx\, dx.$$

Since the last integral approaches zero,* $\pi a_k$ differs from $[\psi'(2\pi-) - \psi'(0+)]/k^2$ by an infinitesimal of higher order, $\liminf_{k=\infty} k^2 |a_k| > 0$, and

$$\liminf_{n=\infty} n^3 \sum_{n+1}^{\infty} (\alpha_k^2+\beta_k^2) > 0.$$

So it is certain that $\lim_{n=\infty} \lambda_n = 0$.

The conditions discussed in the last paragraphs, however, are merely sufficient conditions. One case has already been remarked in which the

* Cf. Bôcher, loc. cit., p. 86.

precise order of magnitude of the coefficients was irrelevant. If $\varphi(x)$ and $\psi(x)$ are so related that every non-vanishing coefficient in the Fourier series for one corresponds to a vanishing coefficient in the series for the other, all the numbers $\lambda_n$ are zero. More generally, let $\varphi_n$ and $\psi_n$ be the remainders in the Fourier series for $\varphi$ and $\psi$ respectively. In an abbreviated notation that will be readily understood, the integrals being extended over a period in each case,

$$\lambda_n = -\frac{\int \varphi_n \psi_n}{\int \psi_n^2} = -\frac{\left[\int \varphi_n^2\right]^{1/2}}{\left[\int \psi_n^2\right]^{1/2}} \cdot \frac{\int \varphi_n \psi_n}{\left[\int \varphi_n^2 \cdot \int \psi_n^2\right]^{1/2}}.$$

Since the last fraction on the right represents the cosine of the angle between the vectors $\varphi_n$ and $\psi_n$ in function space, it may be said that $\lambda_n$ will approach zero if the series for $\varphi$ and $\psi$ converge with the *same* order of rapidity, according to the interpretation that has been placed upon this mode of expression, and if at the same time $\varphi_n$ and $\psi_n$ approach orthogonality with each other, so that the angle between them approaches 90°.

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# ON THE DEVELOPMENT OF ARBITRARY FUNCTIONS IN SERIES OF HERMITE'S AND LAGUERRE'S POLYNOMIALS.*

By James V. Uspensky.

The so-called Hermite's and Laguerre's polynomials afford the simplest and the most important examples of Tchebysheff's polynomials related to an infinite interval, and it is just the necessity of taking into account the infinite values of the variable that constitutes the essential difficulty of the problem of the development of arbitrary functions in series of these polynomials. This matter has been investigated by several authors and by means of various methods.

Recently two remarkable papers dealing with the same subject have been published by G. Szegö† and Einar Hille‡ and, undoubtedly, they have both achieved a considerable progress. It seems, however, that neither of them was aware of the paper published by the author in 1916§, in which by means of a certain peculiar device far more general results have been established. This paper was written in Russian and published during the time when all the international relations were broken, and this may sufficiently account for the fact that it could pass unnoticed. However, as the method used by the author can successfully be applied whenever asymptotic expressions of a certain type exist, it seems worth while to reproduce the essential parts of this paper in a modified and simplified form.

The case of Hermite's polynomials, which is easier, will be considered first in order to have an illustration as simple as possible for the chief idea of our method.

## Part I. Hermite's Polynomials.||

### 1. The definition of Hermite's polynomials and their properties.

Hermite's polynomials $U_n(x)$ are defined by the following equation

* Received May 3, 1927.

† G. Szegö, Beiträge zur Theorie der Laguerreschen Polynome I, Math. Zeit., 25 (1926).

‡ E. Hille, On Laguerre's series, Notes 1, 2, 3. Proc. Nat. Acad. of Sci., vol. 12, 1926. From the same point of view, but in a slightly different way, the subject was treated in a paper by the author presented to the Math. Society in Kazan in 1916. Owing, however, to the political disturbances of that time, this paper never was published and was finally lost.

§ J. Uspensky, On the development of functions in series of polynomials $e^x \frac{d^n e^{-x} x^n}{dx^n}$, Bull. Russian Acad. of Sci., vol. 10 (6), 1916.

|| These polynomials were studied by Hermite in 1864, but previously to this date they occur in an important paper by Tchebysheff (1859) and even earlier in Laplace's "Théorie analytique des Probabilités".

$$U_n(x) = e^{x^2} \frac{d^n e^{-x^2}}{dx^n}$$

so that

$$U_0(x) = 1, \quad U_1(x) = -2x, \quad U_2(x) = 4x^2 - 2, \cdots$$

and in general

$$(-1)^n U_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \cdots.$$

The following are their most important properties:

1. $U_n$ satisfies the linear differential equation of the second order

$$U_n'' - 2xU_n' + 2nU_n = 0$$

whence it follows that for

$$y = e^{-\frac{x^2}{2}} U_n$$

the following equation holds good

$$y'' + (2n + 1 - x^2)y = 0.$$

2. Three consecutive polynomials $U_{n+1}$, $U_n$, $U_{n-1}$ are connected by the equation

$$U_{n+1} + 2xU_n + 2nU_{n-1} = 0$$

which is satisfied even for $n = 0$ if we put $U_{-1} = 0$.

3. For two *different* integral values $m$ and $n$ the equation of *orthogonality* holds

$$\int_{-\infty}^{\infty} e^{-x^2} U_m U_n \, dx = 0$$

and for $m = n$

$$\int_{-\infty}^{\infty} e^{-x^2} U_n^2 \, dx = 2 \cdot 4 \cdot 6 \cdots 2n \sqrt{\pi}.$$

4. Starting from the definite integral

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \cos 2xt \, dt$$

it is easy to show that

$$|U_{2n}(x)| < \frac{(2n)!}{n!} e^{x^2}.$$

The polynomial

$$W_n(x) = (-1)^n \frac{n!}{(2n)!} U_{2n}(x),$$

therefore, satisfies the following inequality

(1) $$|W_n(x)| < e^{x^2}$$

for every real $x$.

5. The functions

$$\xi = e^{-\frac{x^2}{2}} W_n(x) \text{ and } \eta = e^{-\frac{x^2}{2}} W_{n+1}(x)$$

satisfy respectively the equations

$$\xi'' + (4n+1-x^2)\xi = 0, \qquad \eta'' + (4n+5-x^2)\eta = 0$$

and for $x = 0$

$$\xi(0) = \eta(0) = 1, \qquad \xi'(0) = \eta'(0) = 0,$$

so that the difference

$$V(x) = \eta - \xi$$

is the solution of the equation

$$V'' + (4n+1)V = (x^2-1)V - 4\eta$$

satisfying the initial conditions

$$V(0) = 0, \qquad V'(0) = 0.$$

Such a solution is given by

$$V(x) = \frac{1}{\lambda}\int_0^x [(t^2-1)V(t) - 4\eta(t)] \sin\lambda(x-t)\,dt,$$
$$\lambda = \sqrt{4n+1}.$$

But for every $t$

$$|V(t)| < 2e^{\frac{t^2}{2}}, \qquad |\eta(t)| < e^{\frac{t^2}{2}},$$

so that for every $x > 0$

$$|V(x)| \leqq \frac{3}{\sqrt{n}}\int_0^x (t^2+1)e^{\frac{t^2}{2}}\,dt = \frac{3xe^{\frac{x^2}{2}}}{\sqrt{n}}$$

and this is equivalent to the inequality

$$|W_{n+1}(x) - W_n(x)| \leqq \frac{3|x|e^{x^2}}{\sqrt{n}} \tag{2}$$

which holds good for every real $x$.

**2. Asymptotic expressions of Hermite's polynomials.** For the following discussion the asymptotic expressions of Hermite's polynomials as well as those of their derivative for large values of $n$ are of fundamental importance.

These expressions were obtained first by Professor Adamoff,* and can be presented in the following form.

First, dividing $U_n(x)$ by $\sqrt{2 \cdot 4 \cdot 6 \cdots 2n \sqrt{\pi}}$ we introduce the normalized polynomials

$$V_n(x) = \frac{U_n(x)}{\sqrt{2 \cdot 4 \cdot 6 \cdots 2n \sqrt{\pi}}}$$

which satisfy the equation

$$\int_{-\infty}^{\infty} e^{-x^2} [V_n(x)]^2 \, dx = 1.$$

Now for large values of $n$ we have the following asymptotic expression

$$V_n(x) = \frac{e^{\frac{1}{2}x^2}}{\sqrt{\pi}} \sqrt[4]{\frac{2}{n}} \left[ \cos\left( x\sqrt{2n} + \frac{n\pi}{2} \right) + \frac{\varphi_n(x)}{\sqrt{n}} \right] \tag{3}$$

whence the similar expression can be derived for $V_{n+1}(x)$, viz.

$$V_{n+1}(x) = \frac{e^{\frac{1}{2}x^2}}{\sqrt{\pi}} \sqrt[4]{\frac{2}{n}} \left[ -\sin\left( x\sqrt{2n} + \frac{n\pi}{2} \right) + \frac{\theta_n(x)}{\sqrt{n}} \right]. \tag{4}$$

Again, using the relation

$$V_n'(x) = -\sqrt{2n}\, V_{n-1}(x),$$

it is easy to find the asymptotic expression for $V_n'(x)$:

$$V_n'(x) = -\sqrt{2n} \frac{e^{\frac{1}{2}x^2}}{\sqrt{\pi}} \sqrt[4]{\frac{2}{n}} \left[ \sin\left( x\sqrt{2n} + \frac{n\pi}{2} \right) + \frac{\psi_n(x)}{\sqrt{n}} \right]. \tag{5}$$

* Adamoff, On asymptotic expressions of the polynomials $U_n(x)$. Ann. Inst. Polytechnique de St. Petersbourg, 5 (1906).

In the equations (3), (4), and (5) the functions $\varphi_n(x)$, $\theta_n(x)$ and $\psi_n(x)$, independently of the values of $x$ and $n$, supposed large, satisfy the inequalities

$$|\varphi_n(x)|<Cx^3, \qquad |\theta_n(x)|<Cx^3, \qquad |\psi_n(x)|<Cx^3$$

$C$ being a suitably chosen constant. Suppose now that $x$ is confined to a given finite interval $a \leqq x \leqq b$; it is possible then to find a constant $L$ so that the inequalities

$$|\varphi_n(x)|<L, \qquad |\theta_n(x)|<L, \qquad |\psi_n(x)|<L$$

are satisfied for every $n$ whatever value $x$ may have in the interval $a \leqq x \leqq b$. Comparing the value of $V_n'(x)$ found from (5) with that given by (3) it is easy to see that $\varphi_n'(x)$ satisfies the inequality

$$|\varphi_n'(x)|<M\sqrt{n}$$

for all values of $n$, when $x$ is confined to the interval $a \leqq x \leqq b$, $M$ being another constant. In the following we shall have occasion to deal with the functions $F_n(x)$ depending on $x$ and $n$, which satisfy the inequalities

$$|F_n(x)|<L, \qquad |F_n'(x)|<M\sqrt{n}$$

when $x$ is confined to a given finite interval and $n$ has any value whatever, $L$ and $M$ being constants. For brevity we shall say that under such circumstances the function $F_n(x)$ satisfies the "condition (A)".

The asymptotic expressions (3), (4) and (5) suffice for the discussion of the development problem in case of Hermite's polynomials. But dealing with Laguerre's polynomials, we shall need to have the asymptotic expression for $V_n(x)$ correct to the terms of the order $n^{-5/4}$. Such an expression can be easily obtained by the same method which was applied to get (3), and we can give it without an elaborate proof:

$$V_n(x) = \frac{e^{\frac{1}{2}x^2}}{\sqrt{\pi}}\sqrt[4]{\frac{2}{n}}\left[\cos\left(x\sqrt{2n}+\frac{n\pi}{2}\right) + \frac{\frac{1}{6}x^3-\frac{1}{2}x}{\sqrt{2n}}\sin\left(x\sqrt{2n}+\frac{n\pi}{2}\right)+\frac{R_n(x)}{n}\right] \tag{6}$$

where $R_n(x)$ satisfies the condition (A).

3. **The fundamental relation.** The "Fourier constants" of a given function $f(x)$ with respect to the system of polynomials $V_0$, $V_1$, $V_2$, $\cdots$ being defined by

$$A_n = \int_{-\infty}^{\infty} e^{-x^2} f(x) V_n(x) dx; \qquad n = 0, 1, 2, \cdots,$$

we are led to consider the infinite series

$$A_0 V_0(x) + A_1 V_1(x) + A_2 V_2(x) + \cdots.$$

Following is the fundamental problem concerning the series of this kind: What conditions assure the convergence of this series and, in the case of convergence, what is the relation between its sum and the function $f(x)$? In order to give the answer to this question under very general conditions we first consider the sum of $n+1$ terms of the preceding series:

$$S_n = A_0 V_0(x) + A_1 V_1(x) + A_2 V_2(x) + \cdots + A_n V_n(x).$$

It is possible to give a simple expression of this sum by means of a certain definite integral. From the relation

$$\sqrt{2n+2}\, V_{n+1}(x) + 2x V_n(x) + \sqrt{2n}\, V_{n-1}(x) = 0$$

it follows easily that

$$V_n(x) V_n(y) = \frac{\sqrt{2n+2}}{2} \frac{V_{n+1}(x) V_n(y) - V_{n+1}(y) V_n(x)}{y-x} - \frac{\sqrt{2n}}{2} \frac{V_n(x) V_{n-1}(y) - V_n(y) V_{n-1}(x)}{y-x}$$

and

$$\sum_{k=0}^{n} V_k(x) V_k(y) = \frac{\sqrt{2n+2}}{2} \frac{V_{n+1}(x) V_n(y) - V_{n+1}(y) V_n(x)}{y-x}.$$

Now it is sufficient to multiply both members by $e^{-y^2} dy$ and integrate between limits $-\infty$ and $\infty$ in order to get the required expression for $S_n$:

$$S_n = \frac{\sqrt{2n+2}}{2} \int_{-\infty}^{\infty} e^{-y^2} \varphi_n(x, y) f(y) dy \tag{7}$$

where

$$\varphi_n(x, y) = \frac{V_{n+1}(x) V_n(y) - V_{n+1}(y) V_n(x)}{y-x}.$$

In case $f(x)$ is a polynomial of the degree $n$ we have the exact equation

$$S_n = f(x).$$

Such will be the case if we take

$$f(x) = \frac{V_{n+1}(\xi) V_n(x) - V_{n+1}(x) V_n(\xi)}{x-\xi}$$

which, indeed, is a polynomial of the degree $n$ containing a parameter $\xi$, and we shall have the exact equation

$$\frac{V_{n+1}(\xi)\,V_n(x)-V_{n+1}(x)\,V_n(\xi)}{x-\xi}=\frac{\sqrt{2n+2}}{2}\int_{-\infty}^{\infty}e^{-y^2}\,\varphi_n(x,y)\,\varphi_n(\xi,y)\,dy,$$

whence it follows, for $\xi=x$, that

$$(8)\qquad V_{n+1}(x)\,V_n'(x)-V_{n+1}'(x)\,V_n(x)=\frac{\sqrt{2n+2}}{2}\int_{-\infty}^{\infty}e^{-y^2}\,[\varphi_n(x,y)]^2\,dy.$$

This equation is of fundamental importance for the following discussion.

4. **Consequences derived from the equation** (8). Substituting for $V_n(x)$, $V_n'(x)$; $V_{n+1}(x)$, $V_{n+1}'(x)$ their asymptotic expressions as given by (3) and (5) it is easy to find for the left hand member of (8) the following expression

$$V_{n+1}(x)\,V_n'(x)-V_{n+1}'(x)\,V_n(x)=\frac{2e^{x^2}}{\pi}+\frac{E_n}{\sqrt{n}}$$

where $E_n$ represents, for a given $x$, a bounded function of $n$, so that in virtue of (8) we have

$$(9)\qquad \frac{\sqrt{2n+2}}{2}\int_{-\infty}^{\infty}e^{-y^2}\,[\varphi_n(x,y)]^2\,dy=\frac{2e^{x^2}}{\pi}+\frac{E_n}{\sqrt{n}}.$$

Now, let $a$ and $b$ represent any two fixed numbers satisfying the conditions $a<x<b$ and let us find the asymptotic expression of the integral

$$I=\int_a^b e^{-y^2}\,[\varphi_n(x,y)]^2\,dy.$$

For this purpose we introduce into $\varphi_n(x,y)$ instead of $V_n(x)$, $V_{n+1}(x)$; $V_n(y)$, $V_{n+1}(y)$ their asymptotic expressions, given by (3) and (4). Putting for brevity

$$\begin{aligned}G_n(x,y)=\;&\theta_n(x)\cos\left(y\sqrt{2n}+\frac{n\pi}{2}\right)-\theta_n(y)\cos\left(x\sqrt{2n}+\frac{n\pi}{2}\right)\\&+\varphi_n(y)\sin\left(x\sqrt{2n}+\frac{n\pi}{2}\right)-\varphi_n(x)\sin\left(y\sqrt{2n}+\frac{n\pi}{2}\right)\\&+\frac{\theta_n(x)\,\varphi_n(y)-\theta_n(y)\,\varphi_n(x)}{\sqrt{n}},\end{aligned}$$

we get

$$(10)\qquad \varphi_n(x,y)=\frac{1}{\pi}\sqrt{\frac{2}{n}}\,\frac{e^{\frac{x^2+y^2}{2}}}{y-x}\left[\sin\sqrt{2n}\,(y-x)+\frac{G_n(x,y)}{\sqrt{n}}\right].$$

It is important to notice that the function $G_n(x, y)$ obviously satisfies the condition (A) when $y$ is confined to the interval $(a, b)$ and that $G_n(x, x) = 0$.

Now the following lemma can be easily established:

*Lemma. Denoting by $G_n(x, y)$ and $\Gamma_n(x, y)$ two functions which with respect to $y$ confined to a given interval $a \leqq y \leqq b$ (where $a < x < b$) possess a continuous derivative, both satisfy the condition* (A) *and vanish for $y = x$, we can make the following statements:*

1. *The integral*

$$\frac{1}{\sqrt{n}} \int_a^b \frac{G_n(x, y)\, \Gamma_n(x, y)}{(y-x)^2}\, dy$$

*is bounded.*

2. $$\lim_{n=\infty} \frac{1}{\sqrt{n}} \int_a^b \frac{G_n(x, y)}{y-x}\, dy = 0.$$

Proof 1. As $G_n(x, y)$ and $\Gamma_n(x, y)$ satisfy the condition (A), a constant $M$ independent of $n$ may be assigned so that

$$|G_n(x, y)| < M, \qquad |\Gamma_n(x, y)| < M,$$
$$\left|\frac{\partial G_n(x, y)}{\partial y}\right| < M\sqrt{n}, \qquad \left|\frac{\partial \Gamma_n(x, y)}{\partial y}\right| < M\sqrt{n},$$

for every $n$ and $y$ confined to the interval $a \leqq y \leqq b$. Accordingly

$$\frac{1}{\sqrt{n}} \int_{x+\frac{1}{\sqrt{n}}}^b \frac{|G_n(x, y)\, \Gamma_n(x, y)|}{(y-x)^2}\, dy < \frac{M^2}{\sqrt{n}} \int_{x+\frac{1}{\sqrt{n}}}^b \frac{dy}{(y-x)^2} < M^2,$$

$$\frac{1}{\sqrt{n}} \int_a^{x-\frac{1}{\sqrt{n}}} \frac{|G_n(x, y)\, \Gamma_n(x, y)|}{(y-x)^2}\, dy < \frac{M^2}{\sqrt{n}} \int_a^{x-\frac{1}{\sqrt{n}}} \frac{dy}{(y-x)^2} < M^2.$$

In the interval $x - \frac{1}{\sqrt{n}} \leqq y \leqq x + \frac{1}{\sqrt{n}}$ we have

$$\frac{G_n(x, y)}{y-x} = G_n'(x, x + \vartheta(y-x)); \qquad \frac{\Gamma_n(x, y)}{y-x} = \Gamma_n'(x, x + \vartheta'(y-x))$$

where $0 < \vartheta < 1$; $0 < \vartheta' < 1$ and the derivatives are taken with respect to $y$. Therefore

$$\frac{1}{\sqrt{n}} \int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}} \frac{|G_n(x, y)\, \Gamma_n(x, y)|}{(y-x)^2}\, dy < 2M^2.$$

From all this we gather that

$$\left|\frac{1}{\sqrt{n}}\int_a^b \frac{G_n(x,y)\,\Gamma_n(x,y)}{(y-x)^2}\right| < 4M^2,$$

which proves the first part of the lemma.

2. It is obvious that

$$\left|\frac{1}{\sqrt{n}}\int_{x+\varepsilon}^b \frac{G_n(x,y)}{y-x}\,dy\right| < \frac{M}{\sqrt{n}}\log\frac{b-x}{\varepsilon},$$

$$\left|\frac{1}{\sqrt{n}}\int_a^{x-\varepsilon} \frac{G_n(x,y)}{y-x}\,dy\right| < \frac{M}{\sqrt{n}}\log\frac{x-a}{\varepsilon},$$

$$\left|\frac{1}{\sqrt{n}}\int_{x-\varepsilon}^{x+\varepsilon} \frac{G_n(x,y)}{y-x}\,dy\right| < 2M\varepsilon.$$

Taking, for instance, $\varepsilon = 1/n$ we see that all these integrals converge to 0, which proves the second part of the lemma.

Introducing the expression (10) into the integral $I$ and applying the lemma we readily get the following result

$$I = \frac{\sqrt{2n+2}}{\pi^2 n}\,e^{x^2}\int_a^b \frac{\sin^2\sqrt{2n}\,(y-x)}{(y-x)^2}\,dy + \frac{B_n}{\sqrt{n}}$$

where $B_n$ remains bounded for indefinitely increasing $n$.

Now

$$\int_a^b \frac{\sin^2\sqrt{2n}\,(y-x)}{(y-x)^2}\,dy = \sqrt{2n}\int_{-\sqrt{2n}(x-a)}^{\sqrt{2n}(b-x)} \frac{\sin^2 v}{v^2}\,dv$$

and

$$\int_{-\sqrt{2n}(x-a)}^{\sqrt{2n}(b-x)} \frac{\sin^2 v}{v^2}\,dv = \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2}\,dv + O\left(\frac{1}{\sqrt{n}}\right) = \pi + O\left(\frac{1}{\sqrt{n}}\right),$$

so that finally

$$I = \frac{2e^{x^2}}{\pi} + \frac{C_n}{\sqrt{n}}$$

where $C_n$ remains bounded. Comparing this result with the equation (9) we arrive at the most important conclusion: for given $a$ and $b$ $(a<x<b)$ a constant $\Gamma$ may be assigned so that

$$\int_b^{\infty} e^{-y^2}[\varphi_n(x,y)]^2\,dy < \frac{\Gamma}{n}, \qquad \int_{-\infty}^{a} e^{-y^2}[\varphi_n(x,y)]^2\,dy < \frac{\Gamma}{n}$$

for every $n$. For a given $x$ or for $x$ confined to a given finite interval we can choose for $a$ and $b$ two perfectly determined numbers. Denoting

then by $G$ any positive number in absolute value greater than $a$ and $b$, we shall have

$$(11)\quad \int_G^{\infty} e^{-y^2}[\varphi_n(x,y)]^2\,dy<\frac{\Gamma}{n},\qquad \int_{-\infty}^{-G} e^{-y^2}[\varphi_n(x,y)]^2\,dy<\frac{\Gamma}{n}$$

with the same constant $\Gamma$ whatever may by the value attributed to $G$.

5. **Conditions of development.** Suppose the given function $f(x)$ subject to the condition that the integrals

$$\int_a^{\infty} e^{-x^2}[f(x)]^2\,dx \text{ and } \int_{-\infty}^{-a} e^{-x^2}[f(x)]^2\,dx$$

both exist for a certain value of $a$. Taking $G$ sufficiently large and applying the known Bouniakovsky-Schwarz* inequality we have

$$\left|\frac{\sqrt{2n+2}}{2}\int_G^{\infty} e^{-y^2}f(y)\,\varphi_n(x,y)\,dy\right| < \frac{\sqrt{2n+2}}{2}\sqrt{\int_G^{\infty} e^{-y^2}[f(y)]^2\,dy}\cdot\sqrt{\int_G^{\infty} e^{-y^2}[\varphi_n(x,y)]^2\,dy},$$

$$\left|\frac{\sqrt{2n+2}}{2}\int_{-\infty}^{-G} e^{-y^2}f(y)\,\varphi_n(x,y)\,dy\right| < \frac{\sqrt{2n+2}}{2}\sqrt{\int_{-\infty}^{-G} e^{-y^2}[f(y)]^2\,dy}\cdot\sqrt{\int_{-\infty}^{-G} e^{-y^2}[\varphi_n(x,y)]^2\,dy},$$

that is, in virtue of (11)

$$\left|\frac{\sqrt{2n+2}}{2}\int_G^{\infty} e^{-y^2}f(y)\,\varphi_n(x,y)\,dy\right| < \sqrt{\Gamma}\sqrt{\int_G^{\infty} e^{-y^2}[f(y)]^2\,dy},$$

$$\left|\frac{\sqrt{2n+2}}{2}\int_{-\infty}^{-G} e^{-y^2}f(y)\,\varphi_n(x,y)\,dy\right| < \sqrt{\Gamma}\sqrt{\int_{-\infty}^{-G} e^{-y^2}[f(y)]^2\,dy}.$$

And this shows that both the integrals

$$\frac{\sqrt{2n+2}}{2}\int_G^{\infty} e^{-y^2}f(y)\,\varphi_n(x,y)\,dy,\qquad \frac{\sqrt{2n+2}}{2}\int_{-\infty}^{-G} e^{-y^2}f(y)\,\varphi_n(x,y)\,dy,$$

can be made less than any given number, however small, independently of $n$ by taking $G$ sufficiently large. This important remark removes at once

* The very important inequality generally known as "Schwarz's inequality" in reality was given by Bouniakovsky in 1861, that is long before it appeared in Schwarz's paper of 1885.

all the difficulties arising from the infinite limits. And, indeed, it remains only to find the limit of the integral

$$\frac{\sqrt{2n+2}}{2}\int_{-G}^{G} e^{-y^2} f(y)\,\varphi_n(x,y)\,dy$$

where the limits are finite constants. Substituting for $\varphi_n(x,y)$ its expression (10) we can represent the preceding integral as a sum of $K_1$ and $K_2$, where

$$K_1 = \frac{\sqrt{2n+2}}{\sqrt{2n}}\,\frac{e^{\frac{x^2}{2}}}{\pi}\int_{-G}^{G} e^{-\frac{y^2}{2}}\,\frac{\sin\sqrt{2n}\,(y-x)}{y-x}\,f(y)\,dy,$$

$$K_2 = \frac{\sqrt{2n+2}}{\sqrt{2n}}\,\frac{e^{\frac{x^2}{2}}}{\pi\sqrt{n}}\int_{-G}^{G} e^{-\frac{y^2}{2}}\,\frac{G_n(x,y)}{y-x}\,f(y)\,dy.$$

If the function $f(x)$ is supposed to be absolutely integrable in any finite interval and of limited variation in the interval $x-\delta,\ x+\delta$, then it follows from the second part of the lemma

$$\lim K_2 = 0, \qquad n \to \infty,$$

and from the known properties of Dirichlet's integrals

$$\lim K_1 = \frac{f(x+0)+f(x-0)}{2}.$$

Summing up we arrive at the following conclusion:

*If the function $f(x)$ satisfies the conditions:*

1. *$f(x)$ is absolutely integrable in any finite interval,*
2. *is of limited variation in a certain interval $x-\delta,\ x+\delta$,*
3. *and admits of the integrals*

$$\int_{\alpha}^{\infty} e^{-x^2}[f(x)]^2\,dx \quad \text{and} \quad \int_{-\infty}^{-\alpha} e^{-x^2}[f(x)]^2\,dx$$

*for certain $\alpha$,*

*then the series*

$$A_0 V_0(x) + A_1 V_1(x) + A_2 V_2(x) + \cdots$$

*converges and has for its sum*

$$\frac{f(x+0)+f(x-0)}{2},$$

These conditions are far more general than those established by G. Szegö and E. Hille, and it is even possible to substitute for 3 a certain less restrictive condition as we intend to show in another paper.

## Part II. Laguerre's polynomials.*

### 6. The definition of Laguerre's polynomials and their properties.

The following equation may serve as definition for Laguerre's polynomials

$$P_n(x) = x^{-\alpha} e^x \frac{d^n x^{n+\alpha} e^{-x}}{dx^n}$$

where, in view of the following applications, we submit $\alpha$ to the restriction $\alpha > -1$. These polynomials satisfy the linear differential equation of the second order

$$x P_n'' + (\alpha + 1 - x) P_n' + n P_n = 0 \tag{12}$$

as well as the equation in finite differences

$$P_{n+1} - (2n + 1 + \alpha - x) P_n + n(n + \alpha) P_{n-1} = 0 \tag{13}$$

remaining true even for $n = 0$ if we assume $P_{-1} = 0$.

The following two important equations could serve as complete definitions of the polynomials $P_n$:

$$\int_0^\infty x^\alpha e^{-x} P_m(x) P_n(x)\, dx = 0, \qquad (m \neq n),$$

$$\int_0^\infty x^\alpha e^{-x} P_n(x)^2\, dx = \Gamma(n+1)\Gamma(n+\alpha+1).$$

The polynomials

$$\Pi_n(x) = \frac{P_n(x)}{\sqrt{\Gamma(n+1)\Gamma(n+\alpha+1)}}$$

are normalized Laguerre's polynomials satisfying the condition

$$\int_0^\infty x^\alpha e^{-x} [\Pi_n(x)]^2\, dx = 1.$$

For $\alpha > -\frac{1}{2}$ the following important equation may be easily verified

$$\Pi_n(x) = \frac{1}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}} \int_0^\pi W_n(\sqrt{x}\cos\varphi) \sin^{2\alpha}\varphi\, d\varphi \tag{14}$$

denoting by $W_n(x)$ the same polynomial as in *I*.

* The same polynomials were studied by Tchebysheff as early as 1859.

Taking into account the inequalities (1) and (2) we see immediately that for $x$ confined to a finite interval $0 \leqq x \leqq h$ a constant $T$ may be determined such that the following inequalities hold

$$(15) \qquad |H_n(x)| < T n^{\frac{\alpha}{2}}, \qquad |H_{n+1}(x) - H_n(x)| < T n^{\frac{\alpha}{2}-\frac{1}{2}}$$

provided $\alpha > -\frac{1}{2}$.

7. **Asymptotic expressions of a certain type of integrals.** The asymptotic expressions of Laguerre's polynomials can be easily derived from (14), but first it is necessary to consider a certain type of definite integrals. Denoting by $\alpha$ a real number $> -\frac{1}{2}$ and by $m$ and $\lambda$ two positive parameters, let us consider the definite integral

$$I = \int_{-1}^{1} e^{\frac{1}{2}m^2x^2} f(mx)(1-x^2)^{\alpha-\frac{1}{2}} e^{i\lambda m x}\, dx,$$

where $f(x)$ is a given polynomial. In order to find the asymptotic expression of this integral for very large $\lambda$ we can first introduce a new path of integration which consists of three rectilinear parts joining respectively the points: $-1$ to $-1+Ti$; $-1+Ti$ to $1+Ti$; $1+Ti$ to 1. As the integral along the second part tends to 0 when $T$ increases indefinitely, we get finally

$$I = I_1 + I_2$$

where

$$I_1 = 2^{\alpha-\frac{1}{2}} e^{\frac{1}{2}m^2 + i\left(\lambda m - \frac{2\alpha+1}{4}\pi\right)} \times \int_0^{\infty} e^{im^2t - \frac{1}{2}m^2t^2} e^{-\lambda m t} f(m+imt)\, t^{\alpha-\frac{1}{2}} \left(1+\frac{it}{2}\right)^{\alpha-\frac{1}{2}} dt,$$

$$I_2 = 2^{\alpha-\frac{1}{2}} e^{\frac{1}{2}m^2 - i\left(\lambda m - \frac{2\alpha+1}{4}\pi\right)} \times \int_0^{\infty} e^{-im^2t - \frac{1}{2}m^2t^2} e^{-\lambda m t} f(-m+imt)\, t^{\alpha-\frac{1}{2}} \left(1-\frac{it}{2}\right)^{\alpha-\frac{1}{2}} dt,$$

or

$$I_1 = \frac{2^{\alpha-\frac{1}{2}}}{(\lambda m)^{\alpha+\frac{1}{2}}} e^{\frac{1}{2}m^2 + i\left(\lambda m - \frac{2\alpha+1}{4}\pi\right)} \times \int_0^{\infty} e^{\frac{imu}{\lambda} - \frac{u^2}{2\lambda^2}} e^{-u} f\left(m+\frac{iu}{\lambda}\right) u^{\alpha-\frac{1}{2}} \left(1+\frac{iu}{2\lambda m}\right)^{\alpha-\frac{1}{2}} du,$$

$$I_2 = \frac{2^{\alpha-\frac{1}{2}}}{(\lambda m)^{\alpha+\frac{1}{2}}} e^{\frac{1}{2}m^2 - i\left(\lambda m - \frac{2\alpha+1}{4}\pi\right)} \times \int_0^\infty e^{-\frac{imu}{\lambda} - \frac{u^2}{2\lambda^2}} e^{-u} f\left(-m + \frac{iu}{\lambda}\right) u^{\alpha-\frac{1}{2}} \left(1 - \frac{iu}{2\lambda m}\right)^{\alpha-\frac{1}{2}} du.$$

Before going further suppose $f(x) = 1$ and $\lambda m \geqq 1$; we have then

$$|I_1| < \frac{2^{\alpha-\frac{1}{2}} Q e^{\frac{m^2}{2}}}{(\lambda m)^{\alpha+\frac{1}{2}}}, \qquad |I_2| < \frac{2^{\alpha-\frac{1}{2}} Q e^{\frac{m^2}{2}}}{(\lambda m)^{\alpha+\frac{1}{2}}}$$

where $Q$ stands for

$$Q = \int_0^\infty e^{-u} u^{\alpha-\frac{1}{2}} du, \qquad \left(\alpha \leq \frac{1}{2}\right),$$

$$Q = \int_0^\infty e^{-u} u^{\alpha-\frac{1}{2}} \left(1 + \frac{u}{2}\right)^{\alpha-\frac{1}{2}} du, \qquad \left(\alpha > \frac{1}{2}\right).$$

Thus we have the following inequality

$$\left| \int_{-1}^{1} e^{\frac{1}{2}m^2x^2} (1-x^2)^{\alpha-\frac{1}{2}} e^{i\lambda mx} dx \right| < \frac{2^{\alpha+\frac{1}{2}} Q e^{\frac{m^2}{2}}}{(\lambda m)^{\alpha+\frac{1}{2}}} \tag{16}$$

when $\lambda m \geqq 1$, $Q$ being determined as above.

Putting

$$\varphi(u) = e^{\frac{imu}{\lambda} - \frac{u^2}{2\lambda^2}} f\left(m + \frac{iu}{\lambda}\right) \left(1 + \frac{iu}{2\lambda m}\right)^{\alpha-\frac{1}{2}}$$

we find

$$\varphi(0) = f(m), \qquad \varphi'(0) = \frac{i\left[mf'(m) + \left(m^2 + \frac{2\alpha-1}{4}\right) f(m)\right]}{\lambda m};$$

as to the second derivative $\varphi''(u)$ it is easy to see that its modulus for sufficiently large values of $\lambda$ and $m$ confined to a given interval $0 < a \leqq m \leqq b$ admits of an upper limit of the form $\omega(u)/\lambda^2$, where $\omega(u)$ is function of $u$ alone increasing less rapidly than a certain power of $u$. Applying Taylor's development we get, therefore,

$$\varphi(u) = \varphi(0) + u\varphi'(0) + \theta \frac{u^2 \omega(u)}{2\lambda^2}$$

where $|\theta| < 1$.

Substituting this development into the expression for $I$, we get

$$I_1 = \frac{2^{\alpha-\frac{1}{2}}\,\Gamma(\alpha+\frac{1}{2})}{(\lambda m)^{\alpha+\frac{1}{2}}}\, e^{\frac{1}{2}m^2+i\left(\lambda m-\frac{2\alpha+1}{4}\pi\right)}$$

$$\times\left[f(m)+\frac{i\left(mf'(m)+\left(m^2+\frac{2\alpha-1}{4}\right)f(m)\right)\left(\alpha+\frac{1}{2}\right)}{\lambda m}+\frac{R_1}{\lambda^2}\right]$$

and similarly

$$I_2 = \frac{2^{\alpha-\frac{1}{2}}\,\Gamma(\alpha+\frac{1}{2})}{(\lambda m)^{\alpha+\frac{1}{2}}}\, e^{\frac{1}{2}m^2-i\left(\lambda m-\frac{2\alpha+1}{4}\pi\right)}$$

$$\times\left[f(-m)-\frac{i\left(-mf'(-m)+\left(m^2+\frac{2\alpha-1}{4}\right)f(-m)\right)\left(\alpha+\frac{1}{2}\right)}{\lambda m}+\frac{R_2}{\lambda^2}\right]$$

where $R_1$ and $R_2$ remain both bounded for indefinitely increasing $\lambda$ when $m$ is confined to a given interval $0<a\leqq m\leqq b$. Hence we obtain the following asymptotic expression for the integral $I$:

$$I=\int_0^{\pi} e^{\frac{1}{2}m^2\cos^2\varphi}\, f(m\cos\varphi)\, e^{i\lambda m\cos\varphi}\,\sin^{2\alpha}\varphi\, d\varphi$$

$$=\frac{2^{\alpha+\frac{1}{2}}\,\Gamma(\alpha+\frac{1}{2})\, e^{\frac{1}{2}m^2}}{(\lambda m)^{\alpha+\frac{1}{2}}}\left[\frac{f(m)+f(-m)}{2}\cos\left(\lambda m-\frac{2\alpha+1}{4}\pi\right)\right.$$

$$+i\frac{f(m)-f(-m)}{2}\sin\left(\lambda m-\frac{2\alpha+1}{4}\pi\right) \tag{17}$$

$$+\frac{F(-m)-F(m)}{2\lambda}\sin\left(\lambda m-\frac{2\alpha+1}{4}\pi\right)$$

$$\left.+i\frac{F(m)+F(-m)}{2\lambda}\cos\left(\lambda m-\frac{2\alpha+1}{4}\pi\right)+\frac{R}{\lambda^2}\right].$$

Here $F(m)$ stands for

$$F(m)=\frac{\left[mf'(m)+\left(m^2+\frac{2\alpha-1}{4}\right)f(m)\right]\left(\alpha+\frac{1}{2}\right)}{m}$$

and $R$ denotes a bounded function of $\lambda$ supposing $m$ confined to the interval $0<a\leqq m\leqq b$.

8. **Asymptotic expressions of Laguerre's polynomials.** As

$$|W_n(x)| < e^{\frac{x^2}{2}}$$

we have from (14)

$$|H_n(x)| < \frac{1}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}\int_0^\pi e^{x\cos^2\varphi}\sin^{2\alpha}\varphi\,d\varphi.$$

Now, applying the inequality (16), we see that for $x$ confined to the interval $1/n \leqq x \leqq h$, where $h$ is a given number, a constant $M$ may be determined so that inequality holds

$$|H_n(x)| < \frac{Mx^{-\frac{\alpha}{2}}}{\sqrt[4]{nx}}. \tag{18}$$

Again we derive from the same equation (14)

$$H_{n+1}(x) - H_n(x)$$

$$= \frac{1}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}\int_0^\pi \{W_{n+1}(\sqrt{x}\cos\varphi) - W_n(\sqrt{x}\cos\varphi)\} \times \sin^{2\alpha}\varphi\,d\varphi$$

$$+ \frac{1}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}\left[\sqrt{\frac{n+\alpha+1}{n+1}} - 1\right] \times \int_0^\pi W_{n+1}(\sqrt{x}\cos\varphi)\sin^{2\alpha}\varphi\,d\varphi$$

and since, by the inequality (2),

$$|W_{n+1}(x) - W_n(x)| \leqq \frac{3|x|e^{\frac{x^2}{2}}}{\sqrt{n}},$$

it follows in the same way that

$$|H_{n+1}(x) - H_n(x)| < \frac{Mx^{-\frac{\alpha}{2}}}{n^{\frac{3}{4}}\,x^{\frac{1}{4}}} \tag{19}$$

where $M$ is a constant which can be assigned when $x$ is supposed to vary in a given interval $1/n \leqq x \leqq h$.

For $W_n(x)$ we had the following asymptotic expression

$$W_n(x) = e^{\frac{1}{2}x^2}\left[\cos(2x\sqrt{n}) + \frac{\frac{1}{12}x^3 - \frac{1}{4}x}{\sqrt{n}}\sin(2x\sqrt{n}) + \frac{\varrho_n(x)}{n}\right];$$

introducing it in (14) we get

$$\Pi_n(x) = \frac{1}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}\int_0^{\pi} e^{\frac{1}{2}x\cos^2\varphi}\left[\cos(2\sqrt{nx}\cos\varphi)\right.$$
$$\left.+\frac{\frac{1}{12}x^{\frac{3}{2}}\cos^3\varphi-\frac{1}{4}x^{\frac{1}{2}}\cos\varphi}{\sqrt{n}}\sin(2\sqrt{nx}\cos\varphi)+\frac{\varrho_n(\sqrt{x}\cos\varphi)}{n}\right]$$
$$\times\sin^{2\alpha}\varphi\,d\varphi.$$

Applying now to the first two terms the asymptotic expression (17), where we take $m=\sqrt{x}$, $\lambda=2\sqrt{n}$, and to the last term the inequality (16) we finally arrive at the following asymptotic expression for $\Pi_n(x)$:

$$(20)\quad \Pi_n(x) = \frac{x^{-\frac{\alpha}{2}}e^{\frac{x}{2}}}{\sqrt{\pi}\sqrt[4]{nx}}\left[\cos\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)\right.$$
$$\left.+\frac{\frac{x^2}{12}-\frac{1+\alpha}{2}x+\frac{1}{16}-\frac{\alpha^2}{4}}{\sqrt{nx}}\sin\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)+\frac{\varphi_n(x)}{n}\right]$$

where $\varphi_n(x)$ remains bounded when $x$ varies in the finite interval

$$0<a\leqq x\leqq b.$$

This expression is established only for $\alpha>-\frac{1}{2}$. Now we can find the the asymptotic expressions of $\Pi_n'(x)$ and $\Pi_n''(x)$ for every $\alpha>-1$. For, denoting temporarily by $\Pi_n(x,\alpha)$ the polynomial with the parameter $\alpha$, we have

$$\Pi_n'(x,\alpha) = -\sqrt{n}\,\Pi_{n-1}(x,\alpha+1)$$

and, since $\alpha+1>0$, we can apply to the right hand member the equation (20). Thus we get for every $\alpha>-1$ (dropping $\alpha$ again)

$$(21)\quad \Pi_n'(x) = -\frac{x^{-\frac{\alpha+1}{2}}e^{\frac{x}{2}}}{\sqrt{\pi}\sqrt[4]{nx}}\left[\sqrt{n}\sin\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)\right.$$
$$\left.-\frac{\frac{x^2}{12}-\frac{\alpha x}{2}+\frac{1}{16}-\frac{(\alpha+1)^2}{4}}{\sqrt{x}}\cos\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)+\frac{\psi_n(x)}{\sqrt{n}}\right]$$

where $\psi_n(x)$ denotes a certain function of the same nature as $\varphi_n(x)$. A similar expression may be obtained for $\Pi_n''(x)$ for every $\alpha > -1$.

Now, substituting these expressions in the equation

$$x\Pi_n''(x) + (\alpha + 1 - x)\Pi_n'(x) + n\Pi_n(x) = 0,$$

it can be proved that (20) continues to hold even for $-1 < \alpha \leqq -\frac{1}{2}$. From the equation (20) the following may be easily derived

$$(22)\quad \Pi_{n+1}(x) = \frac{x^{-\frac{\alpha}{2}} e^{\frac{x}{2}}}{\sqrt{\pi}\sqrt[4]{nx}}\left[\cos\left(2\sqrt{nx} - \frac{2\alpha+1}{4}\pi\right) + \frac{\frac{x^2}{12} - \frac{3+\alpha}{2}x + \frac{1}{16} - \frac{\alpha^2}{4}}{\sqrt{nx}}\sin\left(2\sqrt{nx} - \frac{2\alpha+1}{4}\pi\right) + \frac{\theta_n(x)}{n}\right].$$

Comparing now (20) with (21) we find easily that, for $x$ confined to a given interval $0 < a \leqq x \leqq b$, a constant $L$ may be found so that

$$|\varphi_n(x)| < L, \qquad |\varphi_n'(x)| < L\sqrt{n},$$

and the same statement can be made with respect to the function $\theta_n(x)$.

9. **The fundamental relation.** From the equation

$$\sqrt{(n+1)(n+1+\alpha)}\,\Pi_{n+1}(x) - (2n+1+\alpha-x)\Pi_n(x) + \sqrt{n(n+\alpha)}\,\Pi_{n-1}(x) = 0$$

connecting three consecutive polynomials (which holds true even for $n = 0$ if we assume $\Pi_{-1}(x) = 0$) the following relation may be easily derived

$$\sum_{k=0}^{n} \Pi_k(x)\Pi_k(y) = \sqrt{(n+1)(n+\alpha)}\,\frac{\Pi_{n+1}(x)\Pi_n(y) - \Pi_{n+1}(y)\Pi_n(x)}{y-x}.$$

Multiplying both sides by $y^\alpha e^{-y} f(y)$ and integrating between limits 0 and $\infty$, we obtain for the sum $S_n$ of the $n+1$ first terms of the development,

$$A_0\Pi_0(x) + A_1\Pi_1(x) + A_2\Pi_2(x) + \cdots,$$
$$A_k = \int_0^\infty y^\alpha e^{-y}\Pi_k(x) f(y)\,dy,$$

the following expression

$$S_n = \sqrt{(n+1)(n+1+\alpha)} \int_0^\infty y^\alpha e^{-y} f(y) \varphi_n(x, y)\, dy$$

where $\varphi_n(x, y)$ stands for

$$\varphi_n(x, y) = \frac{\Pi_{n+1}(x)\Pi_n(y) - \Pi_{n+1}(y)\Pi_n(x)}{y-x}.$$

Taking

$$f(x) = \frac{\Pi_{n+1}(\xi)\Pi_n(x) - \Pi_{n+1}(x)\Pi_n(\xi)}{x-\xi},$$

which is a polynomial in $x$ of degree $n$, we have the exact equation

$$\frac{\Pi_{n+1}(\xi)\Pi_n(x) - \Pi_{n+1}(x)\Pi_n(\xi)}{x-\xi}$$
$$= \sqrt{(n+1)(n+1+\alpha)} \int_0^\infty y^\alpha e^{-y} \varphi_n(\xi, y) \varphi_n(x, y)\, dy$$

whence for $\xi = x$ we obtain the fundamental relation

$$\Pi'_{n+1}(x)\Pi_n(x) - \Pi_{n+1}(x)\Pi'_n(x)$$
$$= \sqrt{(n+1)(n+1+\alpha)} \int_0^\infty y^\alpha e^{-y} [\varphi_n(x, y)]^2\, dy. \tag{23}$$

10. **Consequences derived from** (23). The asymptotic expression for the left hand member of (23) may be immediately obtained by introducing instead of $\Pi_n(x)$, $\Pi'_n(x)$; $\Pi_{n+1}(x)$, $\Pi'_{n+1}(x)$ their asymptotic expressions; thus we get

$$\Pi'_{n+1}(x)\Pi_n(x) - \Pi_{n+1}(x)\Pi'_n(x) = \frac{x^{-\alpha-\frac{1}{2}} e^x}{\pi\sqrt{n}} + \frac{E_n}{n} \tag{24}$$

where $E_n$ remains bounded for indefinitely increasing $n$.

Now we take two numbers $a$ and $b$ satisfying the conditions $0 < a < x < b$ and consider the integral

$$I = \sqrt{(n+1)(n+1+\alpha)} \int_a^b y^\alpha e^{-y} [\varphi_n(x, y)]^2\, dy.$$

In order to find its asymptotic expression we introduce in $\varphi_n(x, y)$ for $\Pi_n(x)$, $\Pi_n(y)$; $\Pi_{n+1}(x)$, $\Pi_{n+1}(y)$ their asymptotic expressions as given by (20) and (22); the result can be presented as follows:

$$(25)\qquad \varphi_n(x,y)=\frac{(xy)^{-\frac{\alpha}{2}-\frac{1}{4}}e^{\frac{x+y}{2}}}{\pi n(y-x)}\left[T_n(x,y)+\frac{U_n(x,y)}{\sqrt{n}}\right]$$

where

$$T_n(x,y)=\sqrt{y}\cos\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)\sin\left(2\sqrt{ny}-\frac{2\alpha+1}{4}\pi\right)$$
$$-\sqrt{x}\cos\left(2\sqrt{ny}-\frac{2\alpha+1}{4}\pi\right)\sin\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)$$

and

$$U_n(x,y)=\left[y^{\frac{1}{2}}x^{-\frac{1}{2}}g(x)-x^{\frac{1}{2}}y^{-\frac{1}{2}}g(y)\right]\sin\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)$$
$$\times\sin\left(2\sqrt{ny}-\frac{2\alpha+1}{4}\pi\right)$$
$$+\frac{x^{-\frac{1}{2}}\varphi_n(y)g_1(x)}{\sqrt{n}}\sin\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)$$
$$-\frac{y^{-\frac{1}{2}}\varphi_n(x)g_1(y)}{\sqrt{n}}\sin\left(2\sqrt{ny}-\frac{2\alpha+1}{4}\pi\right)$$
$$+\frac{y^{-\frac{1}{2}}\theta_n(x)g(y)}{\sqrt{n}}\sin\left(2\sqrt{ny}-\frac{2\alpha+1}{4}\pi\right)$$
$$-\frac{x^{-\frac{1}{2}}\theta_n(y)g(x)}{\sqrt{n}}\sin\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)$$
$$+\varphi_n(y)\cos\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)$$
$$-\varphi_n(x)\cos\left(2\sqrt{ny}-\frac{2\alpha+1}{4}\pi\right)$$
$$+\theta_n(x)\cos\left(2\sqrt{ny}-\frac{2\alpha+1}{4}\pi\right)$$
$$-\theta_n(y)\cos\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)$$
$$+\frac{\theta_n(x)\varphi_n(y)-\theta_n(y)\varphi_n(x)}{n};$$

$$g(x)=\frac{x^2}{12}-\frac{1+\alpha}{2}x+\frac{1}{16}-\frac{\alpha^2}{4};\quad g_1(x)=\frac{x^2}{12}-\frac{3+\alpha}{2}x+\frac{1}{16}-\frac{\alpha^2}{4}.$$

It is obvious that both the functions $T_n(x, y)$ and $U_n(x, y)$ satisfy the condition (A) and vanish when $y = x$. We can, therefore, apply the lemma established in § 4 and thus reach the following result

$$(26)\quad I = \frac{e^x x^{-\alpha-\frac{1}{2}}\sqrt{(n+1)(n+\alpha+1)}}{n^2\pi^2}\int_a^b y^{-\frac{1}{2}}\frac{[T_n(x,y)]^2}{(y-x)^2}\,dy + \frac{B_n}{n}$$

where $B_n$ remains bounded. Now the quotient $T_n(x, y)/(y-x)$ can obviously be presented as follows

$$\frac{T_n(x,y)}{y-x} = \frac{\cos\left(2\sqrt{nx}-\frac{2\alpha+1}{4}\pi\right)\sin\left(2\sqrt{ny}-\frac{2\alpha+1}{4}\pi\right)}{\sqrt{y}+\sqrt{x}} + \frac{\sqrt{x}\,\sin 2\sqrt{n}\,(\sqrt{y}-\sqrt{x})}{y-x}$$

$$= \Gamma_1(x, y) + \Gamma_2(x, y).$$

The integral

$$\int_a^b [\Gamma_1(x, y)]^2 y^{-\frac{1}{2}}\,dy$$

is evidently bounded. If we write $x = \xi^2$, $y = \eta^2$, $u = \eta - \xi$, the integral

$$\int_a^b y^{-\frac{1}{2}}\Gamma_1(x, y)\Gamma_2(x, y)\,dy$$

may be represented as follows

$$\int_a^b y^{-\frac{1}{2}}\Gamma_1(x, y)\Gamma_2(x, y)\,dy$$

$$= \frac{1}{8\xi}\sin\left(4\xi\sqrt{n}-\frac{2\alpha+1}{2}\pi\right)\int_{\sqrt{a}-\xi}^{\sqrt{b}-\xi}\frac{\sin 4\sqrt{n}u}{u}\,du$$

$$+\frac{1}{2\xi}\cos^2\left(2\xi\sqrt{n}-\frac{2\alpha+1}{4}\pi\right)\int_{\sqrt{a}-\xi}^{\sqrt{b}-\xi}\frac{\sin^2 2\sqrt{n}u}{u}\,du$$

$$-\frac{1}{2\xi}\cos\left(2\xi\sqrt{n}-\frac{2\alpha+1}{4}\pi\right)\int_{\sqrt{a}-\xi}^{\sqrt{b}-\xi}\frac{4\xi+u}{(u+2\xi)^2}$$

$$\times\sin\left(2\xi\sqrt{n}-\frac{2\alpha+1}{4}\pi+2u\sqrt{n}\right)\sin 2\sqrt{n}\,u\,du$$

The last term is obviously bounded, and the same can be easily proved with respect to the first two. It remains to consider the integral

$$\int_a^b y^{-\frac{1}{2}} [\Gamma_2(x,y)]^2\,dy = 2\xi^2 \int_{\sqrt{a}}^{\sqrt{b}} \frac{\sin^2 2\sqrt{n}(\eta-\xi)}{(\eta-\xi)^2} \frac{d\eta}{(\eta+\xi)^2}$$
$$= 2\xi^2 \int_{\sqrt{a}-\xi}^{\sqrt{b}-\xi} \frac{\sin^2 2\sqrt{n}u}{u^2} \frac{du}{(u+2\xi)^2}$$

which may be represented as a sum of three others

$$\frac{1}{2}\int_{\sqrt{a}-\xi}^{\sqrt{b}-\xi} \frac{\sin^2 2\sqrt{n}u}{u^2}\,du - \frac{1}{2}\xi^{-1}\int_{\sqrt{a}-\xi}^{\sqrt{b}-\xi} \frac{\sin^2 2\sqrt{n}u}{u}\,du$$
$$+\frac{1}{2}\xi^{-1}\int_{\sqrt{a}-\xi}^{\sqrt{b}-\xi} \sin^2 2\sqrt{n}u \frac{u+3\xi}{(u+2\xi)^2}\,du.$$

The last two are bounded functions of $n$, and the first can be expressed as follows

$$\sqrt{n}\int_{2\sqrt{n}(\sqrt{a}-\xi)}^{2\sqrt{n}(\sqrt{b}-\xi)} \frac{\sin^2 v}{v^2}\,dv = \pi\sqrt{n} + \varepsilon_n,$$

where $\varepsilon_n$ remains bounded. Finally we obtain

$$I = \frac{x^{-\alpha-\frac{1}{2}} e^x}{\pi\sqrt{n}} + \frac{C_n}{n}.$$

$C_n$ denoting a bounded function of $n$. The comparison of this expression for $I$ with the equation (24) leads to the following important conclusions

$$\int_b^\infty y^\alpha e^{-y} [\varphi_n(x,y)]^2\,dy < \frac{\Gamma}{n^2}, \qquad \int_0^a y^\alpha e^{-y} [\varphi_n(x,y)]^2\,dy < \frac{\Gamma}{n^2}$$

where the constant $\Gamma$ may be determined whenever $a$ and $b$ are given. Now, if $x$ remains fixed or varies in a limited interval, we can choose for $a$ and $b$ two fixed numbers. Denoting then by $H$ and $G$ any two numbers satisfying the conditions $0 < H < a$, $G > b$, we shall have

$$(27)\quad \int_0^H y^\alpha e^{-y} [\varphi_n(x,y)]^2\,dy < \frac{\Gamma}{n^2}, \qquad \int_G^\infty y^\alpha e^{-y} [\varphi_n(x,y)]^2\,dy < \frac{\Gamma}{n^2}.$$

**11. Conditions of development.** Suppose the given function $f(x)$ subject to the condition that both the integrals

$$\int_\gamma^\infty y^\alpha e^{-y} [f(y)]^2\,dy, \qquad \int_0^\beta y^\alpha e^{-y} [f(y)]^2\,dy$$

exist for certain values of $\gamma$ and $\beta$. From the preceding fundamental inequalities (27) it follows that

$$\left|\sqrt{(n+1)(n+\alpha+1)}\int_G^\infty y^\alpha e^{-y}\varphi_n(x,y)f(y)\,dy\right| < \frac{\sqrt{(n+1)(n+\alpha+1)}}{n}\sqrt{\Gamma}\sqrt{\int_G^\infty y^\alpha e^{-y}[f(y)]^2\,dy},$$

$$\left|\sqrt{(n+1)(n+\alpha+1)}\int_0^H y^\alpha e^{-y}\varphi_n(x,y)f(y)\,dy\right| < \frac{\sqrt{(n+1)(n+\alpha+1)}}{n}\sqrt{\Gamma}\sqrt{\int_0^H y^\alpha e^{-y}[f(y)]^2\,dy}.$$

Consequently, by taking $G$ sufficiently large and $H$ sufficiently small both the integrals

$$\sqrt{(n+1)(n+\alpha+1)}\int_G^\infty y^\alpha e^{-y}\varphi_n(x,y)f(y)\,dy$$

and

$$\sqrt{(n+1)(n+\alpha+1)}\int_0^H y^\alpha e^{-y}\varphi_n(x,y)f(y)\,dy$$

can be made numerically less than any given positive number, however small, independently of $n$.

The statement concerning the second integral remains true in case $\alpha > -\frac{1}{2}$ even if we do not admit the existence of the integral

$$\int_0^\beta y^\alpha e^{-y}[f(y)]^2\,dy$$

but suppose the existence of the integral

$$\int_0^\beta y^{\frac{\alpha}{2}-\frac{1}{4}}|f(y)|\,dy$$

for a certain $\beta$.

First we can write $\varphi_n(x,y)$ in the following from

$$\varphi_n(x,y) = \frac{[\Pi_{n+1}(x)-\Pi_n(x)]\,\Pi_n(y)}{y-x} - \frac{[\Pi_{n+1}(y)-\Pi_n(y)]\,\Pi_n(x)}{y-x}.$$

Further, taking into account the inequalities resulting from (18) and (19)

$$|\Pi_{n+1}(x)-\Pi_n(x)| < \frac{K}{n^{\frac{3}{4}}}, \qquad |\Pi_n(x)| < \frac{K}{n^{\frac{1}{4}}}$$

where $K$ is a certain constant, we obtain

$$\left|\sqrt{(n+1)(n+\alpha+1)}\int_0^H y^\alpha e^{-y}\varphi_n(x,y)f(y)\,dy\right|$$

$$<K\sqrt{(n+1)(n+\alpha+1)}\int_0^{\frac{1}{n}}\frac{y^\alpha e^{-y}}{x-y}\left\{\frac{|\Pi_n(y)|}{n^{\frac{3}{4}}}+\frac{|\Pi_{n+1}(y)-\Pi_n(y)|}{n^{\frac{1}{4}}}\right\}|f(y)|\,dy$$

$$+K\sqrt{(n+1)(n+\alpha+1)}\int_{\frac{1}{n}}^H\frac{y^\alpha e^{-y}}{x-y}\left\{\frac{|\Pi_n(y)|}{n^{\frac{3}{4}}}+\frac{|\Pi_{n+1}(y)-\Pi_n(y)|}{n^{\frac{1}{4}}}\right\}|f(y)|\,dy.$$

By (15) we have

$$|\Pi_n(y)|<Tn^{\frac{\alpha}{2}},\qquad |\Pi_{n+1}(y)-\Pi_n(y)|<Tn^{\frac{\alpha}{2}-\frac{1}{2}}$$

with a certain constant $T$, when $0\leqq y\leqq\frac{1}{n}$, so that

$$\sqrt{(n+1)(n+1+\alpha)}\int_0^{\frac{1}{n}}\frac{y^\alpha e^{-y}}{x-y}\left\{\frac{|\Pi_n(y)|}{n^{\frac{3}{4}}}+\frac{|\Pi_{n+1}(y)-\Pi_n(y)|}{n^{\frac{1}{4}}}\right\}|f(y)|\,dy$$

$$<Ln^{\frac{\alpha}{2}+\frac{1}{4}}\int_0^{\frac{1}{n}}|f(y)|y^\alpha\,dy<L\int_0^{\frac{1}{n}}y^{\frac{\alpha}{2}-\frac{1}{4}}|f(y)|\,dy$$

$L$ being a constant.

In the interval $\frac{1}{n}\leqq y\leqq H$ we have by (18) and (19)

$$|\Pi_n(y)|<\frac{My^{-\frac{\alpha}{2}-\frac{1}{4}}}{n^{\frac{1}{4}}},\qquad |\Pi_{n+1}(y)-\Pi_n(y)|<\frac{My^{-\frac{\alpha}{2}-\frac{1}{4}}}{n^{\frac{3}{4}}},$$

and consequently

$$\sqrt{(n+1)(n+1+\alpha)}\int_{\frac{1}{n}}^H\frac{y^\alpha e^{-y}}{x-y}\left\{\frac{|\Pi_n(y)|}{n^{\frac{3}{4}}}+\frac{|\Pi_{n+1}(y)-\Pi_n(y)|}{n^{\frac{1}{4}}}\right\}|f(y)|\,dy$$

$$<L'\int_{\frac{1}{n}}^H y^{\frac{\alpha}{2}-\frac{1}{4}}|f(y)|\,dy$$

$L'$ being another constant. We have now

$$\left|\sqrt{(n+1)(n+\alpha+1)}\int_0^H y^\alpha e^{-y}\varphi_n(x,y)f(y)\,dy\right|$$

$$<KL\int_0^{\frac{1}{n}}y^{\frac{\alpha}{2}-\frac{1}{4}}|f(y)|\,dy+KL'\int_{\frac{1}{n}}^H y^{\frac{\alpha}{2}-\frac{1}{4}}|f(y)|\,dy$$

$$<K(L+L')\int_0^H y^{\frac{\alpha}{2}-\frac{1}{4}}|f(y)|\,dy$$

and here the right hand member can be made as small as we please by taking $H$ sufficiently small and that independently of $n$.

From the preceding discussion it follows that we can confine our investigation to the integral with finite limits

$$K = \sqrt{(n+1)(n+\alpha+1)} \int_H^G y^\alpha e^{-y} \varphi_n(x,y) f(y)\, dy.$$

Now introducing the asymptotic expression (25) for $\varphi_n(x,y)$ we have to consider two integrals

$$K_1 = \frac{\sqrt{(n+1)(n+\alpha+1)}}{\pi n} \int_H^G e^{\frac{x-y}{2}} x^{-\frac{\alpha}{2}-\frac{1}{4}} y^{\frac{\alpha}{2}-\frac{1}{4}} \frac{T_n(x,y)}{y-x} f(y)\, dy,$$

$$K_2 = \frac{\sqrt{(n+1)(n+\alpha+1)}}{\pi n^{\frac{3}{2}}} \int_H^G e^{\frac{x-y}{2}} x^{-\frac{\alpha}{2}-\frac{1}{4}} y^{\frac{\alpha}{2}-\frac{1}{4}} \frac{U_n(x,y)}{y-x} f(y)\, dy.$$

Supposing $f(x)$ of a limited variation in a certain interval $x-\delta$, $x+\delta$ and both integrals

$$\int_{x+\delta}^G |f(y)|\, dy, \quad \int_H^{x-\delta} |f(y)|\, dy$$

existing, it can be easily shown by the second part of the lemma in 4 that

$$\lim K_2 = 0, \quad n \to \infty.$$

As to the integral $K_1$ it can be presented as a sum of two integrals

$$K' = \frac{\sqrt{(n+1)(n+\alpha+1)}}{\pi n} \int_H^G e^{\frac{x-y}{2}} x^{-\frac{\alpha}{2}-\frac{1}{4}} y^{\frac{\alpha}{2}-\frac{1}{4}} f(y) \times \frac{\sin\left(2\sqrt{ny} - \frac{2\alpha+1}{4}\pi\right) \cos\left(2\sqrt{nx} - \frac{2\alpha+1}{4}\pi\right)}{\sqrt{y}+\sqrt{x}}\, dy,$$

$$K'' = \frac{\sqrt{(n+1)(n+\alpha+1)}}{\pi n} \int_H^G e^{\frac{x-y}{2}} x^{-\frac{\alpha}{2}-\frac{1}{4}} y^{\frac{\alpha}{2}-\frac{1}{4}} x^{\frac{1}{2}} f(y) \times \frac{\sin 2\sqrt{n}(\sqrt{y}-\sqrt{x})}{y-x}\, dy$$

of which the first evidently has 0 as its limit and the second by substituting $x = \xi^2$, $y = \eta^2$ can be reduced to the Dirichlet's integral

$$K'' = \frac{\sqrt{(n+1)(n+\alpha+1)}}{\pi n} \int_{\sqrt{H}}^{\sqrt{G}} e^{\frac{\xi^2-\eta^2}{2}} \left(\frac{\eta}{\xi}\right)^{\alpha+\frac{1}{2}} \frac{2\xi}{\xi+\eta} f(\eta^2) \times \frac{\sin 2\sqrt{n}\,(\eta-\xi)}{\eta-\xi}\, d\eta$$

which has for its limit

$$\frac{f(x+0)+f(x-0)}{2}.$$

Summing up we arrive at the following conditions under which the development of an arbitrary function in series of Laguerre's polynomials is certainly possible. *If we suppose that the given function $f(x)$ satisfies the following conditions*:

1. *That the integral*

$$\int_\gamma^\infty y^\alpha e^{-y} [f(y)]^2\, dy$$

*exists for a certain $\gamma$.*

2. *That in case $-1 < \alpha \leqq -\frac{1}{2}$ the integral*

$$\int_0^\beta y^\alpha e^{-y} [f(y)]^2\, dy$$

*exists for a certain $\beta$, while in case $\alpha > -\frac{1}{2}$ only the existence of the integral*

$$\int_0^\beta y^{\frac{\alpha}{2}-\frac{1}{4}} |f(y)|\, dy$$

*is required.*

3. *That $f(x)$ is of limited variation in a certain interval $x-\delta$, $x+\delta$ and absolutely integrable in any finite interval, then the series*

$$A_0 \Pi_0(x) + A_1 \Pi_1(x) + A_2 \Pi_2(x) + \cdots$$

*where $x > 0$ and*

$$A_k = \int_0^\infty y^\alpha e^{-y} \Pi_k(y) f(y)\, dy,$$

*converges and has for its sum*

$$\frac{f(x+0)+f(x-0)}{2}.$$

As to the convergence of this series and its sum for $x = 0$ the author did not succeed in obtaining as simple and as general conditions as those stated above, and for this reason he préfers to postpone the treatment

of this case. It is very likely that the condition 1 may be improved, as this certainly is the case for Hermite's polynomials. But it seems hardly possible to substitute for the condition 2 anything less restrictive and of the same simplicity. It is easy to show that for $x$ different from 0 both the series

$$\sum_{n=1}^{\infty} \frac{\cos x\sqrt{n}}{n^{\sigma}} \text{ and } \sum_{n=1}^{\infty} \frac{\sin x\sqrt{n}}{n^{\sigma}}$$

converge for $\sigma > \frac{1}{2}$ and diverge for $\sigma \leqq \frac{1}{2}$.

Now, supposing $\alpha = 0$, let us consider the development of the function $f(x) = x^{-s}$ where $s < 1$. It is easy to show that in this case

$$A_n = \frac{\Gamma(1-s)}{\Gamma(s)} \left( \frac{1}{n^{1-s}} + \frac{\lambda_n}{n^{2-s}} \right),$$

$\lambda_n$ being a bounded function of $n$. Taking into account the asymptotic expression for $\Pi_n(x)$ we infer that both the series

$$\sum_{n=0}^{\infty} A_n \Pi_n(x)$$

and

$$\sum_{n=1}^{\infty} \frac{\cos\left(2\sqrt{nx} - \frac{\pi}{4}\right)}{n^{\frac{5}{4}-s}}$$

converge and diverge at the same time, that is converge when $s < \frac{3}{4}$ and diverge when $s \geqq \frac{3}{4}$. But the condition 2 is satisfied for $s < \frac{3}{4}$, and is no longer satisfied, when $s \geqq \frac{3}{4}$.

CARLETON COLLEGE,
November, 1926.

# PARALLELISM AND TRANSVERSALITY IN A SUB-SPACE OF A GENERAL (FINSLER) SPACE.*

By James Henry Taylor.

1. **Introduction.** The space here considered is an $n$-dimensional space for which an integral of the form

$$t = \int_{u_1}^{u} F(x, \dot{x})\, du$$

evaluated along a curve $C$ is taken as the definition of the arc length of the curve. The geometry of such a space was first considered by P. Finsler,† accordingly I have called such a space a *Finsler space.* The conditions which are imposed upon $F$ are essentially those which make the problem of minimizing such an integral, in the sense of the calculus of variations, a regular problem. The purpose of this paper is to show that parallelism of vectors in the sense developed by the author in an earlier paper‡ implies parallelism in any sub-space in which they lie, and to show that the angle between two vectors in a sub-space is the same whether computed with respect to the metric of the enveloping space or with respect to the metric of the sub-space. Orthogonality, with the meaning here used, includes transversality as a special case, and hence the transversality condition is shown to be preserved in a sub-space.

2. **Fundamental quantities of the space.** Let $x^1, x^2, \cdots, x^n$ be the coördinates of an $n$-dimensional space. As a basis of a geometry of such a space we take a function $F(x, dx)$ which is assumed to be an *invariant,* in the sense in which this term is used in tensor analysis,§ under point transformation of coördinates. The arc length $t$ of a curve $C$,

$$x^\alpha = x^\alpha(u), \quad u_1 \leqq u \leqq u_2, \qquad (\alpha = 1, 2, \cdots, n),$$

is taken to be the value of the integral

$$t = \int_{u_1}^{u} F(x, \dot{x})\, du$$

where we employ the vector notation $x$ and $\dot{x}$ to represent the quantities $(x^1, \cdots, x^n)$ and $(\dot{x}^1, \cdots, \dot{x}^n)$ respectively, and where the dots denote

* Presented to the American Mathematical Society, April 16, 1927; received April 21, 1927.

† *Über Kurven und Flächen in allgemeinen Räumen,* Dissertation, 1918.

‡ J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas,* Trans. Amer. Math. Soc., vol. 27 (1925), pp. 246–264.

§ Cf. L. P. Eisenhart, *Riemannian Geometry* (1926), p. 6.

derivatives with respect to $u$. For simplicity $C$ will be supposed to be a real, analytic curve, and the function $F(x, \dot{x})$ will be assumed to be a real, positive, analytic function of its $2n$ arguments in a domain $M$ to be considered. The set of values $\dot{x} = (0, 0, \cdots, 0)$ is excluded. In addition, $F$ will be required to possess the following properties in $M$.

(1) $$F(x, \varkappa\dot{x}) = \varkappa F(x, \dot{x}), \quad \varkappa > 0.$$

(2) $$\frac{\partial^2 F(x, \dot{x})}{\partial \dot{x}^\alpha \, \partial \dot{x}^\beta} \xi^\alpha \xi^\beta > 0, \quad \xi \neq \dot{x}.$$

(3) $F_1 > 0$, where $F_1$ denotes the $F_1$-function of the calculus of variations.

In (2), $\alpha$ and $\beta$ are summed from 1 to $n$, as will always be understood in the following whenever an index letter appears twice in the same term.

Let a function $f(x, \dot{x})$ be defined by

$$f(x, \dot{x}) = \frac{1}{2} F^2(x, \dot{x}).$$

Then as a result of (1), $f$ of course satisfies the homogeneity condition

$$f(x, \varkappa\dot{x}) = \varkappa^2 f(x, \dot{x}), \quad \varkappa > 0,$$

and the resulting identities obtained by differentiation,

(4) $$f_\alpha \dot{x}^\alpha = 2f; \quad f_{\alpha\beta}\dot{x}^\alpha = f_\beta; \quad f_{\alpha\beta\lambda}\dot{x}^\alpha = 0.$$

Here, as in the following, the subscripts applied to $f$ denote the partial derivatives with respect to the corresponding $\dot{x}$ variables. The quantities $f_\alpha, f_{\alpha\beta}$ and $f_{\alpha\beta\gamma}$ constitute covariant tensors of the ranks indicated by the subscripts.* The symmetric tensor $f_{\alpha\beta}$ plays the rôle of the fundamental tensor of the space. For, as a result of (4), the element of arc is given by

$$dt^2 = f_{\alpha\beta}(x, dx)\, dx^\alpha \, dx^\beta,$$

a homogeneous differential expression of the second order.

From the assumption (3), and that $F$ is positive, it results that the determinant†

$$|f_{\alpha\beta}| > 0.$$

* J. H. Taylor, loc. cit., p. 251.

† J. H. Taylor, *Reduction of Euler's equations to a canonical form*, Bull. Amer. Math. Soc., vol. 31 (1925), pp. 257–262.

A consequence which follows readily from (2) is that the form

$$f_{\alpha\beta}(x, r)\xi^\alpha \xi^\beta$$

is *positive definite.** The length $l$ of the contravariant vector $\xi$ is defined by

$$l^2 = f_{\alpha\beta}(x, r)\xi^\alpha \xi^\beta, \tag{5}$$

and the *angle* between the two vectors $\xi$ and $\eta$ is given by†

$$\cos(\xi, \eta) = \frac{f_{\alpha\beta}(x, r)\,\xi^\alpha \eta^\beta}{\sqrt{f_{\alpha\beta}(x, r)\,\xi^\alpha \xi^\beta}\ \sqrt{f_{\alpha\beta}(x, r)\,\eta^\alpha \eta^\beta}}. \tag{6}$$

We observe that the values of these invariants thus defined depend not only upon the position at which the forms are evaluated but also upon a vector $r$ at that point. However, they do not depend upon the magnitude of $r$ since $f_{\alpha\beta}$ is homogeneous of degree zero in the components of the latter. Hence (5) and (6) have the same values if $r$ be replaced by $dx$ where $dx$ has the same direction as $r$. The directions given by $\xi$ and $\eta$ are naturally said to be *orthogonal* in case $\cos(\xi, \eta) = 0$. As a consequence of the above form being positive definite the angle defined by (6) is real.‡

For future reference we record here the generalized Christoffel symbols which are defined by

$$[\alpha\beta, \mu] = \frac{1}{2}\left(\frac{\partial f_{\alpha\mu}}{\partial x^\beta} + \frac{\partial f_{\beta\mu}}{\partial x^\alpha} - \frac{\partial f_{\alpha\beta}}{\partial x^\mu}\right), \tag{7}$$

$$\Gamma^\lambda_{\alpha\beta} = f^{\lambda\mu}[\alpha\beta, \mu]$$

where $f^{\lambda\mu}$ is the element of the reciprocal matrix of $f_{\alpha\beta}$ corresponding to the term $f_{\mu\lambda}$.

3. **Fundamental quantities for a sub-space.** The equations

$$x^\alpha = x^\alpha(y^1, y^2, \cdots, y^m), \qquad 1 \leqq m < n) \tag{8}$$

where these functions are analytic in the $y$'s and such that the matrix $\| \partial x^\alpha / \partial y^i \|$ is of rank $m$, define an $m$-dimensional subspace $V_m$ which is immersed in the $n$-dimensional space $V_n$. From (8) we have by differentiation

* J. H. Taylor, Transactions, loc. cit., p. 252.

† J. L. Synge, *A generalization of the Riemannian line-element*, Trans. Amer. Math. Soc., vol. 27 (1925), p. 66.

‡ L. P. Eisenhart, loc. cit., p. 37.

$$\dot{x}^\alpha = \frac{\partial x^\alpha}{\partial y^i}\dot{y}^i; \quad \ddot{x}^\alpha = \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j}\dot{y}^i\dot{y}^j + \frac{\partial x^\alpha}{\partial y^i}\ddot{y}^i \tag{9}$$

$$\frac{\partial \dot{x}^\alpha}{\partial y^j} = \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j}\dot{y}^i; \quad \frac{\partial \dot{x}^\alpha}{\partial \dot{y}^i} = \frac{\partial x^\alpha}{\partial y^i}$$

where, as in the following, each Greek index is understood to have the range $1, 2, \cdots, n$ and each Latin index the range $1, 2, \cdots, m < n$.

By hypothesis, the length of arc of a curve in the sub-space $V_m$ is given by

$$t = \int_{u_1}^{u} F(x, \dot{x})\,du = \int_{u_1}^{u} H(y, \dot{y})\,du$$

where $H(y, \dot{y})$ is the resulting function in $y$ and $\dot{y}$ obtained from $F(x, \dot{x})$ by substituting (8) and the first of (9). An examination shows that all the properties postulated for $F$ likewise hold for $H$. We set $h(y, \dot{y}) = H^2/2$. Then from

$$h(y, \dot{y}) = f(x, \dot{x})$$

we have, making use of (9),

$$h_i = f_\alpha \frac{\partial x^\alpha}{\partial y^i}; \quad h_{ij} = f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}; \tag{10}$$

$$h_{ijk} = f_{\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}$$

where the subscripts applied to $h$ denote differentiation with respect to the $\dot{y}$ variables. Hence the metric of the subspace is given by

$$dt^2 = f_{\alpha\beta}\,dx^\alpha\,dx^\beta = f_{\alpha\beta}\frac{\partial y^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\,dy^i\,dy^j = h_{ij}\,dy^i\,dy^j.$$

By direct computation one finds†

$$\begin{aligned}[ij, k]^* = {} & f_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j}\frac{\partial x^\beta}{\partial y^k} + [\alpha\beta, \gamma]\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k} \\ & + \frac{1}{2}\left(\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\frac{\partial \dot{x}^\gamma}{\partial y^j}f_{\alpha\beta\gamma} + \frac{\partial x^\alpha}{\partial y^j}\frac{\partial x^\beta}{\partial y^k}\frac{\partial \dot{x}^\gamma}{\partial y^i}f_{\alpha\beta\gamma}\right. \\ & \left. - \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial \dot{x}^\gamma}{\partial y^k}f_{\alpha\beta\gamma}\right)\end{aligned}$$

where $[ij, k]^*$ is the Christoffel symbol of the first kind for the subspace. Introducing the factor $\dot{y}^i$ and contracting for $i$ gives

† J. H. Taylor, Transactions, loc. cit., p. 253–255.

$$[ij, k]^* \dot{y}^i = f_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} \frac{\partial x^\beta}{\partial y^k} \dot{y}^i + [\alpha\beta, \gamma] \dot{x}^\alpha \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k} + \frac{1}{2} f_{\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \frac{\partial \dot{x}^\gamma}{\partial y^i} \dot{y}^i$$

as each of the other two terms reduces to zero by the homogeneity condition (4). The last term of the right member may be further reduced by (9) as follows

$$\frac{1}{2} f_{\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \frac{\partial \dot{x}^\gamma}{\partial y^i} \dot{y}^i = \frac{1}{2} f_{\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \left( \ddot{x}^\gamma - \frac{\partial x^\gamma}{\partial y^i} \ddot{y}^i \right)$$
$$= \frac{1}{2} \ddot{x}^\gamma f_{\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} - \frac{1}{2} \ddot{y}^i h_{ijk}.$$

Hence we have

$$f_{\alpha\beta} \frac{\partial x^\beta}{\partial y^k} \left[ \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} \dot{y}^i + \Gamma^\alpha_{\sigma\varrho} \dot{x}^\sigma \frac{\partial x^\varrho}{\partial y^j} + \frac{1}{2} \ddot{x}^\gamma f_{\sigma\varrho\gamma} f^{\alpha\varrho} \frac{\partial x^\sigma}{\partial y^j} \right]$$
$$= [ij, k]^* \dot{y}^i + \frac{1}{2} \ddot{y}^i h_{ijk}. \tag{11}$$

**4. Invariance of parallelism in a subspace.** A contravariant vector $\xi$ defined along a curve is said to remain *parallel*, with respect to the curve, if it satisfies the system of differential equations†

$$\theta \xi^\alpha = 0 \tag{12}$$

where

$$\theta \xi^\alpha \equiv \frac{d\xi^\alpha}{du} + \left( \Gamma^\alpha_{\lambda\beta} \dot{x}^\beta + \frac{1}{2} \ddot{x}^\gamma f_{\lambda\beta\gamma} f^{\beta\alpha} \right) \xi^\lambda. \tag{13}$$

Let $\lambda^i$ be the components of any contravariant vector field in the subspace $V_m$. We denote the components in $V_n$ of the vector field by $\xi^\alpha$. Then‡

$$\xi^\alpha = \frac{\partial x^\alpha}{\partial y^j} \lambda^j. \tag{14}$$

Differentiating (14) with respect to $u$, the parameter along a curve $C$, we have

$$\frac{d\xi^\alpha}{du} = \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} \dot{y}^i \lambda^j + \frac{\partial x^\alpha}{\partial y^j} \frac{d\lambda^j}{du}.$$

Hence

$$\theta \xi^\alpha = \frac{d\lambda^j}{du} \frac{\partial x^\alpha}{\partial y^j} + \lambda^j \left[ \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} \dot{y}^i + \frac{\partial x^\varrho}{\partial y^j} \left( \Gamma^\alpha_{\varrho\beta} \dot{x}^\beta + \frac{1}{2} \ddot{x}^\gamma f_{\varrho\beta\gamma} f^{\beta\alpha} \right) \right].$$

† J. H. Taylor, Transactions, loc. cit., p. 259.

‡ Cf. L. P. Eisenhart, loc. cit., p. 46.

Then as a consequence of (11) we have

$$f_{\alpha\beta}(\theta\xi^\alpha)\frac{\partial x^\beta}{\partial y^k} = f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^j}\frac{dx^\beta}{dy^k}\frac{d\lambda^j}{du} + \lambda^j\left([ij,k]^*\dot{y}^i + \frac{1}{2}\ddot{y}^i h_{ijk}\right)$$
$$= h_{kr}\left[\frac{d\lambda^r}{du} + \left(\Gamma^{*r}_{ij}\dot{y}^i + \frac{1}{2}\ddot{y}^i h_{ijs}h^{rs}\right)\lambda^j\right]$$

Hence

$$f_{\alpha\beta}(\theta\xi^\alpha)\frac{\partial x^\beta}{\partial y^k} = h_{kr}(\theta^*\lambda^r) \tag{15}$$

where $\theta^*$ is the operator for the sub-space corresponding to (13). The condition that $\xi$ should remain parallel with respect to the enveloping space is $\theta\xi^\alpha = 0$. Thus we have the theorem:

*If a vector $\xi$ defined along a curve $C$ remains parallel with respect to a space, it remains parallel with respect to any sub-space in which it lies.*

We note two interesting corollaries of this theorem. Let $t$, the arc length of a curve $C$ be selected as parameter. Then the tangent vector $\xi^\alpha = \dot{x}^\alpha$ is a unit vector, and the condition† that $C$ shall be a geodesic of the space is $\theta\xi^\alpha = 0$. Hence:

*If a curve is a geodesic of a space, it is a geodesic of any sub-space in which it lies.*

By a vector at a point $P$ being *normal* to a sub-space we mean a vector say $\eta^\alpha$, which has the property that it is orthogonal to an *arbitrary* vector in the sub-space at that point. The condition for this is

$$f_{\alpha\beta}\,\delta x^\alpha\,\eta^\beta = 0$$

where $\delta x^\alpha$ is an arbitrary direction in the sub-space. By (9) this relation becomes

$$\left(f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^i}\eta^\beta\right)\delta y^i = 0,$$

which is to hold for arbitrary displacements $\delta y^i$, and hence

$$f_{\beta\alpha}\frac{\partial x^\alpha}{\partial y^i}\eta^\beta = 0 \tag{16}$$

is the desired condition that $\eta^\alpha$ shall be normal to the sub-space.

Let $t$ be the arc length of a curve $C$ which is not a geodesic in $V_n$. Denote by $\xi^\alpha$ the unit tangent vector; then $\theta\xi^\alpha$ is not a zero vector and it gives the direction of the principal normal of $C$ in $V_n$.‡ Suppose the

† J. H. Taylor, Bulletin, loc. cit.

‡ J. H. Taylor, Transactions, loc. cit., p. 261.

curve $C$ lies in a sub-space $V_m$, and denote the unit tangent vector $dy^i/du$ by $\lambda^i$. Then from (15) we have the theorem:

*If a curve $C$ is a geodesic in a sub-space $V_m$ but not a geodesic in an enveloping space $V_n$, the principal normals of $C$, regarded as a curve in $V_n$, are normal to the sub-space $V_m$ with respect to the line elements of $C$.**

5. **Invariancy of angle in a sub-space.** Let $dx$, $\delta_1 x$ and $\delta_2 x$ be three directions in a sub-space defined by (8). Then the angle between $\delta_1 x$ and $\delta_2 x$ with respect to the line element $(x, dx)$ is given by

$$\cos(\delta_1 x, \delta_2 x) = \frac{f_{\alpha\beta}(x, dx)\,\delta_1 x^\alpha \delta_2 x^\beta}{\sqrt{f_{\alpha\beta}\delta_1 x^\alpha \delta_1 x^\beta}\,\sqrt{f_{\alpha\beta}\delta_2 x^\alpha \delta_2 x^\beta}}.$$

By means of (9) and (10) this becomes

$$\cos(\delta_1 x, \delta_2 x) = \frac{h_{ij}(y, dy)\,\delta_1 y^i \delta_2 y^j}{\sqrt{h_{ij}\delta_1 y^i \delta_1 y^j}\,\sqrt{h_{ij}\delta_2 y^i \delta_2 y^j}} = \cos(\delta_1 y, \delta_2 y).$$

That is:

*The invariant $\cos\theta$, where $\theta$ is an angle between two directions of a sub-space with the respect to a line element in that sub-space, has the same value whether computed with respect to the metric of the enveloping space or with respect to the metric of the sub-space.*

We append as a corollary of this a theorem on transversality given by J. Douglas.† The notion of orthogonality here used includes transversality as a special case.‡ A direction $\eta$ is said to be *transversal* to a direction $\xi$ if

$$\eta^\alpha \frac{\partial F(x, \xi)}{\partial \dot{x}^\alpha} = 0.$$

By means of (4) we have

$$f_{\alpha\beta}(x, \xi)\xi^\alpha \eta^\beta = f_\beta(x, \xi)\eta^\beta = F(x, \xi)\frac{\partial F(x, \xi)}{\partial \dot{x}^\beta}\eta^\beta,$$

and hence the condition of $\eta$ being orthogonal to $\xi$ with respect to $\xi$, is equivalent to $\eta$ being transversal to $\xi$. We are now in position to state the theorem,

*If a direction $\eta$ is transversal to a direction $\xi$ with respect to a space, it is transversal with respect to any subspace in which these directions lie.*

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* For these corresponding theorems in Riemannian space, and others which could equally well be generalized for the space here under consideration, see L. P. Eisenhart, loc. cit., p. 75

† *The transversality relative to a surface of* $\int F(x, y, z, y', z')\,dx = \text{minimum}$, Bull. Amer. Math. Soc., vol. 32 (1926), pp. 669–674.

‡ L. Berwald, *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Math. Zeit., vol. 25 (1926), p. 56.

**6. Meusnier's Theorem. Geodesic curvature.** We suppose in this section that the sub-space $V_m$ is an $(n-1)$-dimensional space; such a space will be called a *hypersurface.*† Let $C$ be a curve, referred to arc length as parameter, which lies in $V_{n-1}$ and which is not a geodesic in $V_n$. Denote by $\lambda^j$ the components in the $y$'s of the unit tangent vector and by $\xi^\beta$ the components in the $x$'s of the same vector. Then equation (15) holds which we write in the form

$$f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^k}(\theta\,\xi^\beta) = h_{kj}(\theta^*\lambda^j) = f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^k}\frac{\partial x^\beta}{\partial y^j}(\theta^*\lambda^j)$$

or

$$f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^k}\left[\theta\,\xi^\beta - \frac{\partial x^\beta}{\partial y^j}(\theta^*\lambda^j)\right] = 0. \tag{17}$$

Let $\eta^\alpha$ be the unit vector *normal* to the hypersurface. Then by (16) its components satisfy the equations

$$f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^k}\eta^\beta = 0. \tag{18}$$

Since by hypothesis the matrix $\|f_{\alpha\beta}\|$ is non-singular and the matrix $\|\partial x^\alpha/\partial y^k\|$ is of rank $n-1$, it follows that the matrix of coefficients of the equations (18) is of rank $(n-1)$. Hence from (17) and (18) it follows that

$$\theta\,\xi^\beta = \frac{1}{R}\eta^\beta + \frac{\partial x^\beta}{\partial y^j}(\theta^*\lambda^j), \tag{19}$$

where $1/R$ is a scalar. As $\eta^\beta$ was chosen as a unit vector we see that $1/R$ is the component on the normal to the hypersurface of the principal normal (given by $\theta\,\xi^\beta$) of $C$ regarded as a curve in $V_n$. We shall solve (19) for $1/R$ in order to show that this quantity depends upon the line element of $C$ but does not involve the second derivatives. Remembering that $\xi^\beta = dx^\beta/dt$ and $\lambda^j = dy^j/dt$, we have on performing the operations indicated in (19)

$$\ddot{x}^\beta + \Gamma^\beta_{\lambda\alpha}\dot{x}^\lambda\dot{x}^\alpha = \frac{1}{R}\eta^\beta + \frac{\partial x^\beta}{\partial y^j}[\ddot{y}^j + \Gamma^{*j}_{ik}\dot{y}^i\dot{y}^k]$$

as the other terms of (13) reduce to zero as a result of the homogeneity condition (4). By means of (9) this equation reduces to

$$\frac{\partial^2 x^\beta}{\partial y^i\,\partial y^j}\dot{y}^i\dot{y}^j + \Gamma^\beta_{\lambda\alpha}\dot{x}^\lambda\dot{x}^\alpha = \frac{1}{R}\eta^\beta + \frac{\partial x^\beta}{\partial y^j}\Gamma^{*j}_{ik}\dot{y}^i\dot{y}^k.$$

† L. P. Eisenhart, loc. cit., p. 41.

By introducing the factor $f_{\beta\gamma}\,\eta^{\gamma}$ and summing for $\beta$ and $\gamma$ one obtains, since $\eta^{\gamma}$ satisfies (18)

$$f_{\beta\gamma}\,\eta^{\gamma}\frac{\partial^2 x^{\beta}}{\partial y^{i}\,\partial y^{j}}\dot{y}^{i}\,\dot{y}^{j}+[\lambda\alpha,\gamma]\,\dot{x}^{\lambda}\,\dot{x}^{\alpha}\,\eta^{\gamma}=\frac{1}{R}.$$

This shows that the value of $1/R$ at a point $P$ for a given $V_{n-1}$ is the same for all curves in $V_{n-1}$ passing through $P$ and having the same direction. We call $1/R$ the *normal curvature* of $V_{n-1}$ with respect to a given direction, i. e. with respect to the line element of the curve $C$. If $C$ is a geodesic in $V_{n-1}$ then $\theta^{*}\lambda^{j}$ vanishes and the normal curvature has for its value the value of the first curvature of $C$ regarded as a curve in $V_n$. Then we have the theorem

*The normal curvature of a hypersurface for a given direction is the first curvature in the enveloping space of the geodesic of the hypersurface in this direction.*

The first curvature of $C$ in $V_n$ and $V_{n-1}$ are given respectively by

$$\frac{1}{\varrho}=\sqrt{f_{\alpha\beta}\,(\theta\xi^{\alpha})\,(\theta\xi^{\beta})}\,,\qquad \frac{1}{\varrho_g}=\sqrt{h_{ij}\,(\theta^{*}\lambda^{i})\,(\theta^{*}\lambda^{j})}\,.$$

Now define unit vectors $\mu^{\beta}$ and $\zeta^{\beta}$ by the equations

$$\mu^{\beta}=\varrho(\theta\xi^{\beta}),\qquad \zeta^{\beta}=\varrho_g\frac{\partial x^{\beta}}{\partial y^{j}}(\theta^{*}\lambda^{j}).$$

Then (19) may be written in the form

$$\frac{\mu^{\beta}}{\varrho}=\frac{\eta^{\beta}}{R}+\frac{\zeta^{\beta}}{\varrho_g}.$$

We note that the vector $\mu^{\beta}$ is expressed linearly in terms of the orthogonal vectors $\eta^{\beta}$ and $\zeta^{\beta}$. Denote by $\sigma$ the angle between $\mu^{\beta}$ and $\eta^{\beta}$. Then

$$\frac{1}{R}=\frac{\cos\sigma}{\varrho},\qquad \frac{1}{\varrho_g}=\frac{\sin\sigma}{\varrho}.$$

The first of these equalities is a generalization of the theorem of Meusnier, and from the second we see that $1/\varrho_g$ is a generalization of the geodesic curvature.†

† For these corresponding theorems in the Riemannian geometry see L. P. Eisenhart, loc. cit., pp. 151–152.

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# ON THE FORM OF DIFFERENTIAL EQUATIONS OF A SYSTEM OF PATHS.*

By Alonzo Church.

In a recent paper† Veblen and Thomas have proposed, as a generalization of the affine geometry of paths studied by them, the study of paths determined by a system of differential equations of the form

$$\delta^{ij}_{ab}\frac{d^2x^a}{ds^2}\frac{dx^b}{ds}+\Pi^{ij}_{pqr}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^r}{ds}=0 \tag{1}$$

in which the $\Pi$'s are functions of the $x$'s, symmetric in the indices $p, q, r$ and alternating in the indices $i$ and $j$, and subject to an appropriate law of transformation.

It is the purpose of this paper to show that, if equations (1) are not the differential equations of a system of paths which has an affine connection, then the curves which satisfy (1) fail to constitute a system of paths at all, in that they do not have the property that, locally, two points are joined by one and only curve.‡

Since equations (1) are invariant under all analytic transformations of the parameter $s$, we are at liberty to choose the parameter as we please. Let us, therefore, fix a particular integer $k$ such that $1 \leqq k \leqq n$, and choose $s = \alpha x^k + \beta$ where $\alpha$ and $\beta$ are constants. Then $d^2x^k/ds^2 = 0$. Of the equations (1) choose those in which $i = k$ and solve them for $d^2x^j/ds^2$, so obtaining

$$\frac{d^2x^j}{ds^2}=-\Pi^{kj}_{pqr}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^r}{ds}\left(\frac{dx^k}{ds}\right)^{-1} \tag{2}$$

where the index $k$ has the fixed value just chosen. Since the right hand side is homogeneous of the second degree in the first derivatives, these equations determine a system of paths§ (that is, a system of curves which has the property that, locally, two points are joined by one and only one curve). Consequently if equations (1) are to determine a system of paths it must be that the remaining equations of (1), in which $i \neq k$, are none of them independent of equations (2).

* Received May 23, 1927.

† O. Veblen and J. M. Thomas, *Projective invariants of affine geometry of paths*, Annals of Math. (2), vol. 27 (1926), p. 295.

‡ O. Veblen, *Remarks on the foundations of geometry*, Bull. Amer. Math. Soc., vol. 31 (1925), p. 128. O. Veblen and J. M. Thomas, loc. cit.

§ J. Douglas, *The most general geometry of paths*, read before the American Mathematical Society May 7, 1927.

If we substitute in (1) the values of the second derivatives given by (2), we obtain a set of equations which, after multiplication by $dx^k/ds$, can be written in the form

$$(3) \qquad (\delta^j_b \Pi^{ki}_{pqr} + \delta^i_b \Pi^{jk}_{pqr} + \delta^k_b \Pi^{ij}_{pqr}) \frac{dx^b}{ds} \frac{dx^p}{ds} \frac{dx^q}{ds} \frac{dx^r}{ds} = 0.$$

Since these equations, unless the left hand side vanishes identically, are clearly independent of (2), a necessary condition that (1) determine a system of paths is that the left hand side of (3) vanish identically; that is

$$\begin{aligned} &\delta^j_b \Pi^{ki}_{pqr} + \delta^i_b \Pi^{jk}_{pqr} + \delta^k_b \Pi^{ij}_{pqr} + \delta^j_r \Pi^{ki}_{bpq} + \delta^i_r \Pi^{jk}_{bpq} + \delta^k_r \Pi^{ij}_{bpq} + \delta^j_q \Pi^{ki}_{brp} \\ &\quad + \delta^i_q \Pi^{jk}_{brp} + \delta^k_q \Pi^{ij}_{brp} + \delta^j_p \Pi^{ki}_{bqr} + \delta^i_p \Pi^{jk}_{bqr} + \delta^k_p \Pi^{ij}_{bqr} = 0. \end{aligned}$$

Contracting this for $k$ and $b$ and setting

$$(4) \qquad \Pi^{ki}_{kpq} = -\frac{n+1}{3} \Pi^i_{pq},$$

an equation which implies that the quantities $\Pi^i_{pq}$ transform like a projective connection, we obtain, after dividing through by $n+1$, the condition

$$(5) \qquad \Pi^{ij}_{pqr} - \frac{1}{3}(\delta^{ij}_{kp} \Pi^k_{qr} + \delta^{ij}_{kq} \Pi^k_{rp} + \delta^{ij}_{kr} \Pi^k_{pq}) = 0.$$

Now this is the condition obtained by Veblen and Thomas that the paths determined by (1) have an affine connection. In fact, if we substitute for $\Pi^{ij}_{pqr}$ in (1) the expression $\frac{1}{3}(\delta^{ij}_{kp} \Pi^k_{qr} + \delta^{ij}_{kq} \Pi^k_{rp} + \delta^{ij}_{kr} \Pi^k_{pq})$, we obtain the equations

$$\delta^{ij}_{ab} \frac{d^2 x^a}{ds^2} \frac{dx^b}{ds} + \delta^{ij}_{kp} \Pi^k_{qr} \frac{dx^p}{ds} \frac{dx^q}{ds} \frac{dx^r}{ds} = 0$$

which are known to represent a system of paths which has an affine connection.

Hence we infer that a system of differential equations of the form (1) cannot be used to determine a system of paths which does not have an affine connection.

It is pointed out by Veblen and Thomas that the left hand side of (5) is a tensor. From this it follows that $\Pi^{ij}_{pqr}$ can be expressed uniquely in the form

$$\Pi^{ij}_{pqr} = T^{ij}_{pqr} + \frac{1}{3}(\delta^{ij}_{kp} \Pi^k_{qr} + \delta^{ij}_{kq} \Pi^k_{rp} + \delta^{ij}_{kr} \Pi^k_{pq})$$

where $T^{ij}_{pqr}$ is a tensor and $\Pi^k_{pq}$ is the projective connection determined by (4). Hence the theory of the invariants of $\Pi^{ij}_{pqr}$ is equivalent to the theory of the simultaneous invariants of a projective connection and a tensor.

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# DIFFERENTIAL INVARIANTS OF RELATIVE QUADRATIC DIFFERENTIAL FORMS.*

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**Historical introduction.** The differential equations which define the differential invariants of affinely connected manifolds were established and studied by us in a previous paper.‡ On the basis of the differential equations we deduced the number of functionally independent absolute scalar differential invariants and parameters of any order. The normal tensors $A$ and tensor extensions $F$ of the set of arbitrary functions in case of the parameters were taken as independent variables. This selection of independent variables enabled us to write down the differential equations of the invariants and parameters in a very brief explicit manner and resulted in considerable simplification and clarity of treatment over all foregoing related investigations. It was shown by making use of the number of

* Received May 17, 1927.

† National Research Fellow in Mathematics.

‡ Annals of Math., vol. 28 (1927), pp. 196–236.

functionally independent scalar invariants that a certain set of scalar invariants $I$ involving invariants of order $\omega+1$ $(\leqq n+1)$ at most was complete in the sense that necessary and sufficient algebraic (i. e. non-differential) conditions for the equivalence of two affinely connected manifolds could be expressed in terms of the invariants of the set. In the present paper our work on affine differential invariants and parameters is extended to the invariants of the non-singular relative quadratic differential form (see § 1). Throughout the paper it has been possible to reduce the treatment very much on account of the analogy with our previous work. As the theory of the non-singular relative form is reducible to that of the ordinary quadratic differential form (see § 1) we shall first give a short historical account of the work on the invariants of quadratic differential forms treated by means of differential equations. This will make it possible to compare our own work more easily with that of others.

At the end of this introduction is appended a bibliography of the more important investigations on this subject arranged according to the chronological order of their appearance. References to papers in the bibliography on page 641 are denoted by Arabic numerals.

In 1884 the first systematic treatment of the differential invariants of infinite continuous groups was published by Sophus Lie[1].* This work of Lie has proved itself fundamental in practically all of the later developments of the general theory of differential invariants. On this account we shall note a few of the underlying ideas in Lie's theory.

An infinite continuous group of transformations is defined as follows:†

A family of transformations

$$y^i = f^i(\overline{y}^1, \cdots, \overline{y}^m) \qquad (i = 1, \cdots, m)$$

is an infinite continuous group of transformations if $f^1, \cdots, f^m$ is the most general solution of a system of partial differential equations

$$(1)\quad W_k\left(y^1, \cdots, y^m;\ \overline{y}^1, \cdots, \overline{y}^m;\ \frac{\partial y^1}{\partial \overline{y}^1}, \cdots, \frac{\partial y^m}{\partial \overline{y}^m};\ \frac{\partial^2 y^1}{\partial \overline{y}^{1^2}}, \cdots\right) = 0 \ (k = 1, 2, \ldots)$$

which possesses the following two properties.

First, the most general solution of the system (1) does not depend on a finite number of arbitrary constants.

Second, if

$$y^i = f^i(\overline{y}^1, \cdots, \overline{y}^m) \qquad (i = 1, \cdots, m)$$

---

* Lie had already presented an outline of his theory to the Scientific Academy at Christiania during the previous year. See, Gesell. d. Wiss. zu Christiania, 12 (1883).

† This definition is taken directly from S. Lie, Leipz. Ber., vol. 43 (1891), p. 317.

and

$$y^i = \varphi^i(\overline{y}^1, \cdots, \overline{y}^m) \qquad (i = 1, \cdots, m)$$

are any two solutions of the differential equations (1), then

$$y^i = \varphi^i(f^1(\overline{y}), \cdots, f^m(\overline{y})) \qquad (i = 1, \cdots, m)$$

is also a system of solutions of these differential equations.

In this definition of an infinite continuous group the variables $y^1, \cdots, y^m$ may denote a set of variables $x^1, \cdots, x^n, z^1, \cdots, z^q$ of which the $x^i$ are the independent and the $z^i$ the dependent variables.

A differential invariant in the sense of Lie[1] of the group of transformations of the above variables $x^1, \cdots, x^n, z^1, \cdots, z^q$ is a function

$$\Omega\left(x^1, \cdots, x^n;\ z^1, \cdots, z^q;\ \frac{\partial z^1}{\partial x^1}, \cdots, \frac{\partial z^q}{\partial x^n};\ \frac{\partial^2 z^1}{\partial x^{1^2}}, \cdots\right)$$

which retains its form under all transformations of the group, i. e.

$$\Omega\left(x^1, \cdots, z^q, \frac{\partial z^1}{\partial x^1}, \cdots;\ \frac{\partial^2 z^1}{\partial x^{1^2}}, \cdots\right) = \Omega\left(\overline{x}^1, \cdots, \overline{z}^q;\ \frac{\partial \overline{z}^1}{\partial \overline{x}^1}; \cdots;\ \frac{\partial^2 \overline{z}^1}{\partial \overline{x}^{1^2}}, \cdots\right).$$

On the basis of these definitions Lie proves the following remarkable theorem.

*Every infinite continuous group determines an infinite set of differential invariants which are given as solutions of complete systems of partial differential equations.*

Somewhat more precisely we may say that the differential equations of an invariant $\Omega$ of order $p$, i. e., an invariant $\Omega$ depending on derivatives of the $z^i$ up to and including those of order $p$, are represented by

$$Xf\Omega = 0 \tag{2}$$

provided that $Xf$ denotes the symbol of the extended group containing derivatives of the $z^i$ up to those of order $p$ inclusive. Owing to the arbitrary functions in the equation (2) this breaks up into a system of homogeneous partial differential equations which are complete as stated in the above theorem.

As an application of his general theory Lie[1] considers the binary quadratic differential form

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2 \tag{3}$$

and a scalar function $\varphi(x, y)$. On account of the requirement of absolute invariance of $ds^2$ and the function $\varphi$ under arbitrary transformations

$$x = f(\bar{x}, \bar{y}); \qquad y = g(\bar{x}, \bar{y}) \tag{4}$$

we have the relations

$$\bar{\varphi}(\bar{x}, \bar{y}) = \varphi(x, y) \tag{5a}$$

and

$$\begin{aligned}
\bar{E} &= E\left(\frac{\partial x}{\partial \bar{x}}\right)^2 + 2F\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{x}} + G\left(\frac{\partial y}{\partial \bar{x}}\right)^2, \\
\bar{F} &= E\frac{\partial x}{\partial \bar{x}}\frac{\partial x}{\partial \bar{y}} + F\left(\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}} + \frac{\partial y}{\partial \bar{x}}\frac{\partial x}{\partial \bar{y}}\right) + G\frac{\partial y}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}}, \qquad (5b) \\
\bar{G} &= E\left(\frac{\partial x}{\partial \bar{y}}\right)^2 + 2F\frac{\partial x}{\partial \bar{y}}\frac{\partial y}{\partial \bar{y}} + G\left(\frac{\partial y}{\partial \bar{y}}\right)^2.
\end{aligned}$$

The equations (5) which are of the type (1) are the differential equations defining the group of transformations in the independent variables $x$, $y$ and the dependent variables $E$, $F$, $G$, and $\varphi$. Since the equations (5) express no conditions on the independent variables $x$ and $y$ these transform by the arbitrary transformation (4). Lie, then, indicates how the differential equations of the parameters corresponding to (2) are determined and it is evident that a similar treatment can be given for the general quadratic differential form in $n$ variables $x^i$ and a set of scalar functions $\varphi^i$. While Lie thus showed that the problem of determining such differential parameters is an instance of his general theory, he did not consider the problem further.

The special treatment of this problem was taken up later in the thesis of Lie's student K. Zorawski[2] who confined himself, however, to the binary quadratic form. Using Lie's method, Zorawski constructed the complete system of partial differential equations satisfied by scalar invariants (Gaussian invariants) of order $p\,(\geqq 1)$ in the derivatives of the coefficients of the form (3) and by differential parameters (Beltramian invariants) of order $p\,(\geqq 1)$ in the derivatives of a set of scalar functions $\varphi^1, \cdots, \varphi^w$ involving derivatives of order $p-1$ in the coefficients of the form (3). He refers to the latter as differential parameters of order $p$. This unnecessary restriction in the relation between the order of the derivatives of the functions $\varphi^i$ and the derivatives of the coefficients of the form (3) has not been made in our paper but seems to have been imposed by all other writers without apparent justification. The number of differential equations in the complete system given by Zorawski for the determination of invariants and parameters of any order $p$ *increases* with $p$. A corresponding

increase in the number of differential equations occurs in the work of all other writers on this subject. We mention this point explicitly inasmuch as the number of differential equations defining the invariants or parameters of any order remains the same in our theory.

The determination of the number of functionally independent scalar invariants and parameters of the binary form (3) which is the principal contribution due to Zorawski, is equivalent to the determination of the number of independent equations in the complete system of differential equations by which these invariants and parameters are defined. In order to find the number of these equations Zorawski proved the following theorem:

*If the differential equations for the determination of the invariants (Gaussian or Beltramian) of order p are independent, those for the determination of the invariants (Gaussian or Beltramian) of order $p+1$ are also independent.*

| Order | Number of invariants ($n = 2$) | |
|---|---|---|
| | Gaussian | Beltramian |
| 0 | 0 | .. |
| 1 | 0 | $2w-1$ |
| 2 | 1 | $3w$ |
| 3 | 1 | $4w$ |
| 4 | 3 | $5w+1$ |
| 5 | 4 | $6w$ |
| 6 | 5 | $7w$ |
| $p$ | $p-1$ | $(p+1)w$ |

Table I. (Zorawski.)

In view of this theorem it is possible by a consideration of the independence of the differential equations for several special cases to determine the number of independent invariants and parameters of any order. The results* thus obtained by Zorawski are given in Table I. The number of Gaussian invariants of order $p (\geqq 2)$ given in this table was obtained by Zorawski by taking the difference between the numbers of functionally independent solutions of the differential equations defining the invariants of orders $p$ and $p-1$ respectively, while the number of Beltramian invariants of order $p (\geqq 2)$ was taken as the difference

$$(6)\ [\mathfrak{D}(p-1, p, w, n) - \mathfrak{N}(n, p)] - [\mathfrak{D}(p-2, p-1, w, n) - \mathfrak{N}(n, p-1)]\ (n = 2)$$

* After having determined the number of Beltramian invariants of orders one and four in table I, Zorawski replaced these by $w$ and $6w$ respectively in order to conform to a particular method of construction of his invariants.

in terms of our notation (see § 20). For $p = 0$, or 1 the number of invariants in the table can be regarded merely as the number of functionally independent solutions of the differential equations of the invariants of orders zero or one respectively. To say with Zorawsky that the numbers in the table I represent the number of functionally independent invariants of any order $p$ is somewhat misleading for actually it is possible to construct $\mathfrak{N}(2, p)$ functionally independent Gaussian invariants and $\mathfrak{D}(p-1, p, w, 2)$ functionally independent Beltramian invariants each of which is of order $p$ for the binary form (3). Most of the later writers, following Zorawski, have used his terminology.

Zorawski also treated the Minding invariants which, however, can be reduced to Beltrami's differential parameters.* On this account we do not consider the Minding type of invariant.

In the thesis of T. Levi-Civita[3] written at the suggestion of Ricci we find the first treatment of the general quadratic differential form. By counting the number $M_p$ of differential equations for the determination of the scalar invariants of order $p$ and the corresponding number $N_p$ of independent variables which were taken to be the coefficients of the quadratic form and their derivatives up to those of order $p$ inclusive, he found the number $N_p - M_p$ which he pointed out gives the minimum number of functionally independent invariants of order $p$. For the form in two variables the number

$$N_p - M_p = (p+1)(p-2)/2 \qquad (p \geq 3)$$

and is therefore equal to the number of independent invariants of order $p$ obtainable by adding up the numbers in the Gaussian column in Table I. Levi-Civita, however, did not determine the number of independent differential equations as is necessary for the exact determination of the number of independent invariants.

The determination of the exact number of absolute scalar invariants for the general form was made in 1902 by C. N. Haskins[4]. His work can be regarded as the extension of that on the binary form by Zorawski. Using the coefficients of the form and their derivatives as the independent variables he showed that the differential equations satiesfied by the invariants of order $p$ can be obtained by adding certain terms to the system of differential equations for the invariants of order $p-1$, and also by adding certain equations to this system. These latter equations he refers to as the "final equations of order $p$" and states the following theorem regarding them:

* See (5) p. 333; also (6) p. 168.

*If all the equations of order $p-1$ are independent and if the final equations of order $p$ are independent, then all the equations of order $p$ are independent.*

Haskins next proves that *the final equations of any order $p (\geqq 1)$ are independent.* By showing the independence of the differential equations for the invariants of a particular order $p = \varrho$ the independence of the equations for the invariants of orders $p > \varrho$ will therefore follow. As we see, these theorems which pertain to the general quadratic form in $n$ variables $x^i$ take the place of that given by Zorawski for the binary form. The essential step taken by Haskins beyond the work of Zorawski is his demonstration of the following theorem:

| Order | Number of Gaussian invariants ($n \geqq 3$) |
|---|---|
| 0 | 0 |
| 1 | 0 |
| 2 | $(n-2)(n-1)n(n+3)/12$ |
| 3 | $(n-1)n^2(n+1)(n+2)/24$ |
| $p$ | $n(p-1)\ (n+p-1)!/2\ (n-2)!\ (p+1)!$ |

Table II. (Haskins.)

| Order | Number of functions | Number of Beltramian invariants ($n \geqq 3$) |
|---|---|---|
| 1 | $w \leqq n$ | $w(w+1)/2$ |
| 1 | $w \geqq n$ | $wn - [n(n-1)/2]$ |
| 2 | $w \leqq n$ | $[wn(n+1)/2] - [(n-w)(n-w-1)/2]$ |
| 2 | $w \geqq n$ | $wn(n+1)/2$ |
| 3 | $w$ | $[wn(n+1)(n+2)/6] + [n(n-1)/2]$ |
| 4 | $w$ | $wn(n+1)(n+2)(n+3)/24$ |
| $p$ | $w$ | $w(n+p-1)!/(n-1)!\ p!$ |

Table III. (Haskins.)

*If the differential equations satisfied by the invariants of order two are dependent for every form in $n$ variables they are dependent for every form in $n-1$ variables.*

In virtue of the above theorems the demonstration of the independence in general of the differential equations is reduced to the consideration of several particular cases as treated by us, for example, in § 12.

The number of functionally independent absolute scalar or Gaussian invariants found by Haskins is given in Table II. The significance of the numbers in Table II is the same as that in Table I.

The paper of Haskins containing the above results was followed by a second one by him in 1904 in which he gave the corresponding results for the differential parameters or Beltramian invariants[6]. He first proved theorems which take the place of cases (a) and (b) in § 19 of our paper after which the number of differential parameters of any order $p$ can be obtained immediately. The same restriction in the order of the differential parameter which was imposed by Zorawski also occurs in the work of Haskins. His results are given in Table III.

Later Th. De Donder[8] gave a very cursory treatment of this problem independently of Haskins.

Another writer to follow up the work of Zorowski was A. R. Forsyth. By solving the differential equations he actually constructed sets of all functionally independent differential parameters of the first three orders (subject to the usual restriction of order) involving one and two scalar functions, both for the two fundamental forms of a two-dimensional surface[5] and for the quadratic form of ordinary Euclidean space[7].*

The question of the determination of the differential parameters of the general quadratic form in $n$ variables was taken up by J. E. Wright† but the explicit determination of the parameters in our sense which would constitute the only real solution of this problem, was not given by him.

A determination of the number of functionally independent differential invariants and parameters for the binary form (3) was undertaken by J. Knoblauch‡ by the process of algebraic elimination of the derivatives of the transformation (4) from the equations of transformation of the independent variables i. e., the coefficients of the form (3), the scalars $\varphi^i$, and their derivatives. This paper was the beginning of a rather extended discussion between Knoblauch and Zorawski in regard to the intrinsic merits of their two methods.§ While the problem can obviously be considered as an algebraic one, it does not seem to us to have been treated satisfactorily as such by Knoblauch. For example, he has not considered the possibility that the elimination processes may lead to equations between the original and transformed variables in which these quantities are inextricably bound together, nor the independence of his equations which is necessary for the rigorous determination of the number of independent invariants or parameters. Such criticism can not be made against the

* For the plane two-dimensional case, see A. R. Forsyth, Rend. Circ. Matem. Palermo, vol. 21 (1906), pp. 115–124.

† Amer. Journ. of Math., vol. 27 (1905), p. 323–342.

‡ Crelle, vol. 131 (1906), pp. 247–264.

§ See, Leipz. Ber., vol. 59 (1907), pp. 372–377; pp. 160–186; ibid., vol. 60 (1908), pp. 20–52; and ibid., vol. 61 (1909), pp. 3–30.

treatment given by Zorawski and it is our point of view that this problem can be handled very satisfactorily by the general method of Lie's theory of differential invariants, which lies at the basis of the work of Zorawski as well as our own.

We shall now give a brief resume of the more important features of our own work. If the coefficients $G_{\alpha\beta}$ of the relative form (1.1) are taken to be real functions of a set of real variables $x^1, \cdots, x^n$ this form will go over into one of which the coefficients can be taken real or pure imaginary depending on the sign of the Jacobian of the transformation (1.3) and the weight $\varkappa$ of the form (1.1) itself. As we shall wish to leave the transformation (1.3) and the (real) constant $\varkappa$ as unrestricted as possible we shall consider all the coefficients $G_{\alpha\beta}$ with respect to a set of variables $x^i$ to be real simultaneously or pure imaginary simultaneously. Of course in certain special cases the extension to pure imaginary coefficients is not necessary, e. g., if $\varkappa$ has the rational form $p/q$ where $p$ is even.

It is shown in § 1 that every non-singular relative form $G$, i. e., one for which $n\varkappa+2$ does not vanish, can be associated with an absolute quadratic form $H$ such that the equivalence of two relative forms implies the equivalence of their associated forms and conversely. Hence the invariant theory of the relative form $G$ is reduced to the invariant theory of its associated form $H$ and is developed by us on this basis.

An important step in the simplification of the work was taken in selecting the tensor coefficients $H_{\alpha\beta}$ of the form $H$ and its tensor extensions $H_{\alpha\beta,\,\gamma\delta\,\cdots\,\sigma}$ as the variables to which the (scalar or tensor) invariants are referred. The number of these variables which are functionally independent is easily calculated from the complete sets of identities (9.1), (9.2), and (9.3). For the case of the differential parameters the scalars $F^{(i)}$ and their extensions $F^{(i)}_{\alpha\beta\,\cdots\,\eta}$ are taken as the additional variables. By the replacement Theorems (3.I) and (3.II) any scalar or tensor differential invariant and parameter can be expressed as a function of these variables. On account of the tensor character of the finite transformation of the variables their infinitesimal transformation is linear and homogeneous in the first derivatives of the arbitrary functions $\xi^i$ (see § 10) with the result that the differential equations satiesfied by the invariants and parameters are in all cases $n^2$ in number, and can be written down in a simple explicit manner. In this paper however, we discuss only the differential equations of the scalar invariants and parameters.* The completeness of the latter equations follows from Lie's general theory but we have given the direct proof of this by which we have found the structural constants

* Logically this is all that is necessary as the scalar invariants alone form a complete set.

of the $n^2$-parameter group of transformations of the variables (see § 10). In general the treatment of the independence of the differential equations is quite analogous to that of previous writers but it has been shortened and improved considerably by our methods.

We have found that the functionally independent solutions of the differential equations satiesfied by the absolute scalar invariants of order $p\,(\geqq 0)$ are $\mathfrak{N}(n, p)$ in number (see § 13). The corresponding result for the differential parameters of order $p\,(\geqq 0)$ in the derivatives of the coefficients of the form $G$ and of order $q\,(\geqq 0)$ in the derivatives of the set of scalars $F^{(1)}, \cdots, F^{(w)}$, i. e., a parameter of order $(p, q)$ in our notation, is given by the number $\mathfrak{D}(p, q, w, n)$ defined in § 20.

As we have already remarked the statement by certain writers of their results on the numbers of independent scalar invariants and parameters has been quite confusing and has necessitated a more or less detailed reading of their work for its correct interpretation. We have attempted to eliminate such a difficulty in our own work by a precise definition of two *fundamental* sets of scalar invariants and parameters (see § 13 and § 20). What most writers have called the number of independent scalar invariants of order $p$ is the difference between the numbers of invariants in two (first or second) fundamental sets of invariants of orders $p$ and $p-1$ respectively, i. e., the difference $\mathfrak{N}(n, p) - \mathfrak{N}(n, p-1)$ which is the number of invariants of order $p$ in a *second* fundamental set. Correspondingly $\mathfrak{D}(p, q, w, n)$ is the number of parameters in a first fundamental set of order $(p, q)$. It is the difference (6) without the restriction $n = 2$ which represents the least number of parameters of order $(p-1, p)$ in a complete set of solutions of the differential equations satisfied by the parameters of order $(p-1, p)$ that is the number of independent parameters of order $p$ in the terminology of most writers. This difference (6) is the number of parameters of order $(p-1, p)$ in a second fundamental set of parameters of order $(p-1, p)$.

In § 4 it is shown that the tensor $H_{\alpha\beta}$ and the totality of its extensions constitute a complete set of invariants of the form $G$. The finiteness of this complete set is proved in § 8 by making use of Theorem (5.II) on the relationship between the extensions $H_{\alpha\beta,\gamma\delta\cdots\sigma}$ and the curvature tensor $B_{\alpha\beta\gamma\delta}$ with its successive covariant derivatives, and a theorem on differential equations in § 7. In fact we establish in § 8 that the tensor $H_{\alpha\beta}$ and its extensions $H_{\alpha\beta,\gamma_1\cdots\gamma_r}$ $(r = 2, \cdots, N+2)$ where $N \leqq n^2+n$, are complete. It is also shown in § 8 that the integer $N$ is an arithmetical invariant of the form $G$. In § 15 it is proved that *the* $\mathfrak{N}(n, \omega+1)$ *absolute scalar invariants of a second fundamental set of order* $\omega+1$ *where* $\omega \leqq n+1$ *constitute a complete set of invariants of the form* $G$. By

establishing the equality $\omega = N+1$ we then show the invariant nature of the number $\omega$ and considerably reduce the upper bound of $N$.

The problem of the equivalence of two absolute quadratic differential forms was treated in 1869 by E. B. Christoffel* who gave a *sufficient* condition for equivalence in terms of the equality of the scalar invariants of the two forms. Later writers have also considered this problem but do not seem to have gone beyond the work of Christoffel and it is our opinion that we are the first to give a definite proof of the completeness of the absolute scalar invariants of a quadratic form in the sense that both necessary and sufficient conditions for the transformation of two such forms can be expressed in terms of these invariants.

A complete set of scalar invariants of the binary form $G$ has been constructed in § 18. For additional remarks on special forms $G$ the reader is referred to § 6 and § 16.

**Bibliography.** The following is a list of the more important papers treating differential invariants and parameters of quadratic differential forms by means of differential equations.†

[1] *Lie, S.*: Ueber Differentialinvarianten; Math. Annalen, vol. 24 (1884), pp. 537–578.

[2] *Zorawski, K.*: Ueber Biegungsinvarianten; Acta Math., vol. 16 (1892), pp. 1–64.

[3] *Levi-Civita, T.*: Sugli invarianti assoluti; Atti del R. Istituto Veneto, vol. 52 (1894), pp. 1447–1523.

[4] *Haskins, C. N.*: On the invariants of quadratic differential forms; Trans. Amer. Math. Soc., vol. 3 (1902), pp. 71–91.

[5] *Forsyth, A. R.*: The differential invariants of a surface and their geometric significance; Phil. Trans. A., vol. 201 (1903), pp. 329–402.

[6] *Haskins, C. N.*: On the invariants of quadratic differential forms, II; Trans. Amer. Math. Soc., vol. 5 (1904), pp. 167–192.

[7] *Forsyth, A. R.*: The differential invariants of space; Phil. Trans. A., vol. 202 (1904), pp. 277–333.

[8] *De Donder, Th.*: Application nouvelle des invariants intégraux (Deuxième note); Acad. R. de Belgique, Classe des Sciences, Mémoires, Deux. Série, vol. 1 (1904), pp. 1–18.

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* Crelle, vol. 70 (1869), pp. 46–70; and ibid., pp. 241–245.

† For other references see J. E. Wright, Invariants of Quadratic Differential Forms, Cambridge Tract No. 9 (1908); R. Weitzenböck, Encyclop. d. Math. Wissensch., vol. $III_3$; R. Weitzenböck, Invariantentheorie, Groningen (1923); O. Veblen, Invariants of Quadratic Differential Forms, Cambridge Tract; and the footnotes in the introduction of the present paper.

**1. Relative quadratic differential forms.** The quadratic differential form*

$$G_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1.1}$$

is called a relative form of weight $\varkappa$ if it transforms in the manner given by

$$\bar{G}_{\alpha\beta}\, d\bar{x}^\alpha\, d\bar{x}^\beta = (x\,\bar{x})^\varkappa\, G_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1.2}$$

under an arbitrary (real) analytic transformation

$$\mathfrak{G}: \quad x^i = f^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n) \qquad (i = 1, 2, \cdots, n) \tag{1.3}$$

of the (real) variables $x^i$ with non-vanishing Jacobian

$$\begin{vmatrix} \dfrac{\partial x^1}{\partial \bar{x}^1} & \cdots & \dfrac{\partial x^1}{\partial \bar{x}^n} \\ \vdots & & \vdots \\ \dfrac{\partial x^n}{\partial \bar{x}^1} & \cdots & \dfrac{\partial x^n}{\partial \bar{x}^n} \end{vmatrix}$$

which we denote by $(x\,\bar{x})$ for brevity. It is furthermore assumed that

$$G_{\alpha\beta} = G_{\beta\alpha} \tag{1.4}$$

and that the determinant $|G_{\alpha\beta}|$ does not vanish identically in the set of variables $x^i$.

From (1.2) the quantities $G_{\alpha\beta}$ transform according to the law

$$\bar{G}_{\alpha\beta} = (x\,\bar{x})^\varkappa\, G_{\sigma\tau}\, \frac{\partial x^\sigma}{\partial \bar{x}^\alpha}\, \frac{\partial x^\tau}{\partial \bar{x}^\beta} \tag{1.5}$$

and are thus the components of a relative tensor of weight $\varkappa$. On account of the assumption regarding the determinant $|G_{\alpha\beta}|$ we can define in the ordinary manner a set of quantities $G^{\alpha\beta}$ which are the components of a relative tensor of weight $-\varkappa$. By taking the determinant of both sides of (1.5) we have

$$|\bar{G}_{\alpha\beta}| = (x\,\bar{x})^{n\varkappa+2}\, |G_{\alpha\beta}|. \tag{1.6}$$

* Unless a statement to the contrary is made, subscripts and superscripts can take on integer values $(1, 2, \cdots, n)$ and it is to be understood that an index which occurs twice in a term is to be summed over these values of the index.

Hence under the condition

$$n\varkappa + 2 \neq 0 \tag{1.7}$$

we can eliminate the Jacobian $(x\,\bar{x})$ between (1.5) and (1.6) so as to obtain

$$\bar{H}_{\alpha\beta} = H_{\sigma\tau}\frac{\partial x^{\sigma}}{\partial \bar{x}^{\alpha}}\frac{\partial x^{\tau}}{\partial \bar{x}^{\beta}} \tag{1.8}$$

where

$$H_{\sigma\tau} = \frac{G_{\sigma\tau}}{|G_{\alpha\beta}|^{r}}; \qquad r = \varkappa/(n\varkappa+2). \tag{1.9}$$

It follows conversely that the equations (1.8) imply the equations (1.5). In other words *the invariant theory of the relative tensor $G_{\alpha\beta}$ of weight $\varkappa$ satisfying the condition* (1.7) *is completely equivalent to that of an associated absolute tensor $H_{\alpha\beta}$ defined by* (1.9). This circumstance brings the general theory of the relative form (1.1) into relationship with the theory of the ordinary quadratic differential form and is developed by us from this standpoint.

If the condition (1.7) is not satisfied a relationship of the above sort does not hold. For this *singular* case the invariant theory of the form (1.1) is equivalent in its nature to that of the conformal Riemann geometry and consequently offers special difficulties in the way of its development.*

Throughout our paper we shall restrict ourselves to the *nonsingular* form (1.1) and for brevity shall refer to this simply as the form $G$. Similarly the letter $H$ will have reference to the associated tensor $H_{\alpha\beta}$ or to the extensions derivable from it, (see § 3).

For future reference it is to be noted that the symmetry condition

$$H_{\sigma\tau} = H_{\tau\sigma} \tag{1.10}$$

is satisfied. If we denote by $\Gamma^{i}_{\alpha\beta}$ and $\left\{\begin{matrix} i \\ \alpha\,\beta \end{matrix}\right\}$ the Christoffel symbols of the second kind based on the functions $H_{\alpha\beta}$ and $G_{\alpha\beta}$ respectively, then

$$\Gamma^{i}_{\alpha\beta} = \left\{\begin{matrix} i \\ \alpha\,\beta \end{matrix}\right\} - r\left(\delta^{i}_{\alpha}\left\{\begin{matrix} v \\ v\,\beta \end{matrix}\right\} + \delta^{i}_{\beta}\left\{\begin{matrix} v \\ v\,\alpha \end{matrix}\right\} - G_{\alpha\beta}\,G^{i\sigma}\left\{\begin{matrix} v \\ v\,\sigma \end{matrix}\right\}\right). \tag{1.11}$$

The quantities $\Gamma^{i}_{\alpha\beta}$ transform in the ordinary manner

$$\bar{\Gamma}^{\sigma}_{\alpha\beta}\frac{\partial x^{i}}{\partial \bar{x}^{\sigma}} = \frac{\partial^{2} x^{i}}{\partial \bar{x}^{\alpha}\,\partial \bar{x}^{\beta}} + \Gamma^{i}_{\sigma\tau}\frac{\partial x^{\sigma}}{\partial \bar{x}^{\alpha}}\frac{\partial x^{\tau}}{\partial \bar{x}^{\beta}} \tag{1.12}$$

* See T. Y. Thomas, Proc. Nat. Acad. Sci., vol. 11 (1925), pp. 722–725; and ibid., vol. 12 (1926), pp. 352–359.

but the $\left\{ \begin{matrix} i \\ \alpha\beta \end{matrix} \right\}$ have a somewhat more complicated law of transformation. However this need not be considered, as the transformation (1.12) suffices for the construction of the scalar or tensor invariants of the form $G$.

It is possible for a set of real coefficients of the form $G$ with respect to variables $x^i$ to go over into a set of pure imaginary coefficients with respect to variables $\bar{x}^i$ on account of the generality of the group of transformations $\mathfrak{G}$ and the admissible values of the constant $\varkappa$. Accordingly we shall consider the coefficients $G_{\alpha\beta}$ all to be real or all to be pure imaginary simultaneously with respect to variables $x^i$ as such a set of functions $G_{\alpha\beta}$ is closed under the group of transformations $\mathfrak{G}$. For special values of the constant $\varkappa$, including in particular the case of the ordinary quadratic form ($\varkappa = 0$), the extension to pure imaginary coefficients is unnecessary as we have already mentioned in the introduction. The character of the tensor $H_{\alpha\beta}$ together with the contravariant forms $G^{\alpha\beta}$ and $H^{\alpha\beta}$ is similar to that of the tensor $G_{\alpha\beta}$. On the other hand the functions $\Gamma^i_{\alpha\beta}$ are real in all cases with the result that the transformation to a system of "normal coördinates" is a transformation $\mathfrak{G}$.

2. **Normal coördinates.** It is now possible to define a set of variables $y^i$ identical with the normal coördinates of Riemann* which were introduced by him from the geometrical point of view as the system of coördinates such that the equations of the geodesic lines through the origin take the simple parametric form: $y^i = \xi^i s$. Since our present discussion is analytical rather than geometrical we shall introduce these variables $y^i$ from a slightly different viewpoint than that of Riemann. However it will be convenient to retain the designation "normal coördinates" for the variables $y^i$ as well as the related terminology which has grown up in this connection.

An important step in the development of our theory of invariants might be expected if we could replace the underlying group of arbitrary transformations $\mathfrak{G}$ by a comparatively simple associated group of linear homogeneous transformations:

$$y^i = a^i_\alpha \bar{y}^\alpha \tag{2.1}$$

where the sets of variables $y^i$ and $\bar{y}^i$ are to be thought of as related to the variables $x^i$ and $\bar{x}^i$ respectively in a definitely determined manner. Under the assumption of the existence of the transformation (2.1) the equations (1.12) become

$$\bar{C}^\sigma_{\alpha\beta} \frac{\partial y^i}{\partial \bar{y}^\sigma} = C^i_{\sigma\tau} \frac{\partial y^\sigma}{\partial \bar{y}^\alpha} \frac{\partial y^\tau}{\partial \bar{y}^\beta} \tag{2.2}$$

* See Weyl's commentary to B. Riemann: Über die Hypothesen, welche der Geometrie zu Grunde liegen, Berlin (1923).

with respect to the transformation between the $y^i$ and $\bar{y}^i$ variables. From (2.1) and (2.2) we obtain

$$\bar{C}^{\sigma}_{\alpha\beta}\,\bar{y}^{\alpha}\,\bar{y}^{\beta}\,a^{i}_{\sigma} = C^{i}_{\sigma\tau}\,y^{\sigma}\,y^{\tau},$$

a set of equations which will be satisfied if

$$C^{i}_{\sigma\tau}\,y^{\sigma}\,y^{\tau} = 0. \tag{2.3}$$

With the equations (2.3) the contact with a method of definition of a system of normal coördinates $y^i$ is established.* We need not enter upon a discussion of these coördinates here as their properties are well known. For our purpose it is sufficient to note that they transform by (2.1) where

$$a^{i}_{\alpha} = \left(\frac{\partial x^i}{\partial \bar{x}^{\alpha}}\right)_{x^i = q^i} \tag{2.4}$$

and that

$$\begin{aligned} x^i &= q^i \text{ for } y^i = 0; \qquad \bar{x}^i = \bar{q}^i \text{ for } \bar{y}^i = 0; \\ \frac{\partial x^i}{\partial y^{\alpha}} &= \delta^{i}_{\alpha} \text{ for } y^i = 0; \qquad \frac{\partial \bar{x}^i}{\partial \bar{y}^{\alpha}} = \delta^{i}_{\alpha} \text{ for } \bar{y}^i = 0. \end{aligned} \tag{2.5}$$

The $r$th partial derivatives of the components of a scalar or tensor $T$ define, when evaluated at the origin of a system of normal coördinates $y^i$, a set of functions of the variables $x^i$ which constitute the components of a tensor called the $r$th extension of $T$.† If $r = 1$, the extension is identical with the first covariant derivative. It is evident from (2.2) that this process of extension can also be applied to the functions $\Gamma^{i}_{\alpha\beta}$ which gives rise to the set of normal tensors $A$. When the process of extension is applied to the tensor $H_{\alpha\beta}$ we obtain the sequence of (absolute) tensors $H_{\alpha\beta,\gamma\delta\cdots\sigma}$, the components of which are all real or all pure imaginary depending on the real or pure imaginary character of the components $H_{\alpha\beta}$ themselves. In particular $H_{\alpha\beta,\gamma}$ vanishes identically. For future reference we shall give the explicit formula for the second extension $H_{\alpha\beta,\gamma\delta}$ which depends on the components $H_{\alpha\beta}$ together with their first and second derivatives.

$$\begin{aligned} H_{\alpha\beta,\gamma\delta} = {} & \frac{\partial^2 H_{\alpha\beta}}{\partial x^{\gamma}\,\partial x^{\delta}} - \frac{\partial H_{\alpha\beta}}{\partial x^{\sigma}}\,\Gamma^{\sigma}_{\gamma\delta} - \frac{\partial H_{\sigma\beta}}{\partial x^{\gamma}}\,\Gamma^{\sigma}_{\alpha\delta} - \frac{\partial H_{\alpha\sigma}}{\partial x^{\gamma}}\,\Gamma^{\sigma}_{\beta\delta} \\ & - \frac{\partial H_{\sigma\beta}}{\partial x^{\delta}}\,\Gamma^{\sigma}_{\alpha\gamma} - \frac{\partial H_{\alpha\sigma}}{\partial x^{\delta}}\,\Gamma^{\sigma}_{\beta\gamma} - H_{\sigma\beta}\,\Gamma^{\sigma}_{\alpha\gamma\delta} - H_{\alpha\sigma}\,\Gamma^{\sigma}_{\beta\gamma\delta} \\ & + H_{\sigma\tau}\,\Gamma^{\sigma}_{\alpha\gamma}\,\Gamma^{\tau}_{\beta\delta} + H_{\sigma\tau}\,\Gamma^{\sigma}_{\alpha\delta}\,\Gamma^{\tau}_{\beta\gamma} \end{aligned} \tag{2.6}$$

* O. Veblen and T. Y. Thomas, Trans. Amer. Math. Soc., vol. 25 (1923), p. 563.

† O. Veblen and T. Y. Thomas, loc. cit., p. 571.

where

$$\Gamma^i_{\alpha\beta\gamma} = \frac{1}{3}\left(\frac{\partial \Gamma^i_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial \Gamma^i_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial \Gamma^i_{\gamma\alpha}}{\partial x^\beta} - 2\Gamma^i_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma} - 2\Gamma^i_{\alpha\beta}\Gamma^\sigma_{\gamma\alpha} - 2\Gamma^i_{\sigma\gamma}\Gamma^\sigma_{\alpha\beta}\right). \tag{2.7}$$

*The point of view which we have adopted in introducing normal coördinates $y^i$ on the basis of the transformation* (2.1) *is precisely that which is of most significance from the invariant theoretic standpoint. Only on account of the particular type of relation* (2.1) *is it possible to develop a complete invariant theory of the form G on the basis of the elegant methods of the tensor analysis.*

3. **Differential invariants.** *Definition.* The set of functions

$$E^{ab\cdots f}_{gh\cdots k}\left(G_{\alpha\beta};\ \frac{\partial G_{\alpha\beta}}{\partial x^\gamma};\ \cdots;\ \frac{\partial^p G_{\alpha\beta}}{\partial x^\gamma \cdots \partial x^\varepsilon}\right) \tag{3.1}$$

will be called a differential invariant of order $p$ of the form $G$ if each function of the set retains its form as a function of $G_{\alpha\beta}$ and its derivatives under the transformation of the functions $E$ which is associated with a transformation $\mathfrak{G}$ of the variables $x^i$.

This notion of a differential invariant of the form $G$ can evidently be extended to the set of functions

$$\Phi^{ab\cdots f}_{gh\cdots k}\left(G_{\alpha\beta};\ \cdots;\ \frac{\partial^p G_{\alpha\beta}}{\partial x^\gamma \cdots \partial x^\varepsilon};\ F^{(i)};\ \frac{\partial F^{(i)}}{\partial x^\lambda};\ \cdots;\ \frac{\partial^q F^{(i)}}{\partial x^\lambda \cdots \partial x^\nu}\right) \tag{3.2}$$

where $F^{(i)}$ denotes any one of a set of arbitrary scalar functions, i. e.,

$$\bar{F}^{(i)}(\bar{x}^1, \cdots, \bar{x}^n) = F^{(i)}(x^1, \cdots, x^n) \qquad (i = 1, 2, \cdots, w) \tag{3.3}$$

under a transformation $\mathfrak{G}$. The set of functions $\Phi$ of the type (3.2) will be called a differential parameter of the form $G$.

*In this paper we shall restrict the term invariant and parameter to absolute or relative scalars and tensors.*

From (1.9) we have

$$G_{\alpha\beta} = |H_{ij}|^{x/2} H_{\alpha\beta}$$

and hence any set of functions (3.1) and (3.2) can be put into the form

$$M^{ab\cdots f}_{gh\cdots k}\left(H_{\alpha\beta};\ \cdots;\ \frac{\partial^p H_{\alpha\beta}}{\partial x^\gamma \cdots \partial x^\varepsilon}\right) \tag{3.4}$$

and

$$\Omega^{ab\cdots f}_{gh\cdots k}\left(H_{\alpha\beta};\ \cdots;\ \frac{\partial^p H_{\alpha\beta}}{\partial x^\gamma \cdots \partial x^\varepsilon};\ F^{(i)};\ \cdots;\ \frac{\partial^q F^{(i)}}{\partial x^\lambda \cdots \partial x^\nu}\right) \tag{3.5}$$

respectively. Conversely the sets of functions (3.4) and (3.5) can be expressed in the form (3.1) and (3.2) respectively by making use of the relation (1.9). Obviously (3.4) is a differential invariant if (3.1) is a differential invariant, and conversely. A similar remark applies to expressions (3.2) and (3.5).

If we transform a differential invariant (3.4) to a system of normal coördinates $y^i$ and evaluate at the origin of this system, the components of the invariant (3.4) take the form

$$M^{ab\cdots f}_{gh\cdots k}(H_{\alpha\beta};\ 0;\ H_{\alpha\beta,\gamma\delta};\ \cdots;\ H_{\alpha\beta,\gamma\delta\cdots\varepsilon}) \tag{3.6}$$

which enables us to state the following replacement theorem.

THEOREM (3.I). *Any differential invariant* (3.4) *of the form* $G$ *can be put into the form* (3.6) *by replacing the first derivatives of* $H_{\alpha\beta}$ *by zero, and higher derivatives by the corresponding extensions* $H_{\alpha\beta,\gamma\delta\cdots\varrho}$.

In a similar manner the set of functions (3.5) can be given the form

$$\Omega^{ab\cdots f}_{gh\cdots k}(H_{\alpha\beta};\ 0;\ H_{\alpha\beta,\gamma\delta};\ \cdots;\ H_{\alpha\beta,\gamma\delta\cdots\varepsilon};\ F^{(i)};\ F^{(i)}_{\lambda};\ \cdots;\ F^{(i)}_{\lambda\cdots\nu}) \tag{3.7}$$

where the quantities $F^{(i)}_{\lambda\cdots\eta}$ denote the extensions of the scalars $F^{(i)}$. This gives the replacement theorem for differential parameters.

THEOREM (3.II). *Any differential parameter* (3.5) *of the form* $G$ *can be put into the form* (3.7) *by replacing the derivatives of* $H_{\alpha\beta}$ *and* $F^{(i)}$ *by the corresponding extensions.*

The symbol $(p, q)$ may be used conveniently to denote the order of the parameter (3.2) or (3.5). It is clear that we need not consider parameters of order $(1, q)$.

4. **A complete set of tensor invariants.** *Definition.* A set of differential invariants (or parameters) of the form $G$ will be said to be complete if algebraic (i. e. non-differential) necessary and sufficient conditions for the equivalence of $G$ with any other form $\bar{G}$ can be expressed in terms of the members of this set.

We shall now demonstrate the following theorem.

THEOREM (4.I). *The tensor* $H_{\alpha\beta}$ *and its extensions* $H_{\alpha\beta,\gamma\delta\cdots\sigma}$ *constitute a complete set of differential invariants of the form* $G$.

We have shown in § 1 that the equivalence of the associated forms $H_{\alpha\beta}\,dx^\alpha\,dx^\beta$ and $\bar{H}_{\alpha\beta}\,d\bar{x}^\alpha\,d\bar{x}^\beta$ implies that of the forms $G$ and $\bar{G}$ them-

selves. In order to show the equivalence of $H$ and $\overline{H}$ let us consider the following infinite sequence:

$$
\begin{aligned}
H_{\alpha\beta} &= H_{\mu\nu}\, u^{\mu}_{\alpha} u^{\nu}_{\beta}, \\
H_{\alpha\beta,\gamma\delta} &= H_{\mu\nu,\sigma\tau}\, u^{\mu}_{\alpha} u^{\nu}_{\beta} u^{\sigma}_{\gamma} u^{\tau}_{\delta}, \\
H_{\alpha\beta,\gamma\delta\varepsilon} &= H_{\mu\nu,\sigma\tau v}\, u^{\mu}_{\alpha} u^{\nu}_{\beta} u^{\sigma}_{\gamma} u^{\tau}_{\delta} u^{v}_{\varepsilon}, \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots
\end{aligned}
\tag{4.1}
$$

Obviously a necessary condition for the equivalence of the two forms $H$ and $\overline{H}$ is that the infinite sequence of equations (4.1) possess a numerical solution

$$
x^i = q^i; \quad \overline{x}^i = \overline{q}^i, \quad u^i_j = q^i_j. \tag{4.2}
$$

We shall show that this condition is also sufficient. For this purpose erect normal coördinates $y^i$ and $\overline{y}^i$ with origins at $x^i = q^i$ and $\overline{x}^i = \overline{q}^i$ respectively. Relate these normal coördinates based on the two forms $H$ and $\overline{H}$ by

$$
y^i = q^i_j \overline{y}^j. \tag{4.3}
$$

Let the components $H_{\alpha\beta}$ be denoted by $h_{\alpha\beta}$ and the components $\overline{H}_{\alpha\beta}$ by $\overline{h}_{\alpha\beta}$ when referred to the normal coördinates $y^i$ and $\overline{y}^i$ respectively. Expanding $h_{\mu\nu}$ and $\overline{h}_{\alpha\beta}$ about the origins of the coördinates $y^i$ and $\overline{y}^i$ respectively, we have

$$
\begin{aligned}
h_{\mu\nu} &= H_{\mu\nu}(q) + \frac{1}{2} H_{\mu\nu,\sigma\tau}(q)\, y^{\sigma} y^{\tau} + \cdots + \cdots, \\
\overline{h}_{\alpha\beta} &= \overline{H}_{\alpha\beta}(\overline{q}) + \frac{1}{2} \overline{H}_{\alpha\beta,\gamma\delta}(\overline{q})\, \overline{y}^{\gamma} \overline{y}^{\delta} + \cdots + \cdots.
\end{aligned}
\tag{4.4}
$$

On multiplying each term of the expansion for $h_{\mu\nu}$ by $q^{\mu}_{\alpha} q^{\nu}_{\beta}$ and summing on $\mu$ and $\nu$ we obtain

$$
\overline{h}_{\alpha\beta} = h_{\mu\nu}\, q^{\mu}_{\alpha} q^{\nu}_{\beta}
$$

which in virtue of (4.3) becomes

$$
\overline{h}_{\alpha\beta} = h_{\mu\nu} \frac{\partial y^{\mu}}{\partial \overline{y}^{\alpha}} \frac{\partial y^{\nu}}{\partial \overline{y}^{\beta}}.
$$

Hence there exists a transformation $\mathfrak{G}$ between the variables $x^i$ and $\overline{x}^i$ defined by (4.3) and the equations of transformation to the normal coördinates $y^i$ and $\overline{y}^i$ which will take the form $H$ into $\overline{H}$, or in other words which will satisfy the differential equations (1.8). The theorem is therefore proved.

**5. The curvature tensor and its covariant derivatives.** The integrability conditions of the differential equations (1.12) are

$$B^{\lambda}_{ijk}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\lambda}} = B^{\alpha}_{\mu\nu\varrho}\frac{\partial x^{\mu}}{\partial \bar{x}^{i}}\frac{\partial x^{\nu}}{\partial \bar{x}_{j}}\frac{\partial x^{\varrho}}{\partial \bar{x}^{k}} \tag{5.1}$$

where

$$B^{i}_{\alpha\beta\gamma} = \frac{\partial \Gamma^{i}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{i}_{\alpha\gamma}}{\partial x^{\beta}} + \Gamma^{i}_{j\gamma}\Gamma^{j}_{\alpha\beta} - \Gamma^{i}_{j\beta}\Gamma^{j}_{\alpha\gamma} \tag{5.2}$$

and there is a similar expression for $\bar{B}^{i}_{\alpha\beta\gamma}$. The tensor $B^{i}_{\alpha\beta\gamma}$ is called the curvature tensor based on the quadratic form $H$. It admits the covariant form given by

$$B_{\alpha\beta\gamma\delta} = H_{\alpha\sigma}B^{\sigma}_{\beta\gamma\delta}. \tag{5.3}$$

By a slight calculation $B_{\alpha\beta\gamma\delta}$ may be given the form

$$\begin{aligned} B_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 H_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\delta}} + \frac{\partial^2 H_{\beta\delta}}{\partial x^{\alpha}\partial x^{\gamma}} - \frac{\partial^2 H_{\alpha\delta}}{\partial x^{\beta}\partial x^{\gamma}} - \frac{\partial^2 H_{\beta\gamma}}{\partial x^{\alpha}\partial x^{\delta}}\right) \\ + H_{\lambda\tau}(\Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\tau}_{\beta\delta} - \Gamma^{\lambda}_{\alpha\delta}\Gamma^{\tau}_{\beta\gamma}). \end{aligned} \tag{5.4}$$

Transforming (5.4) to a system of normal coördinates and evaluating at the origin we obtain

$$2B_{\alpha\beta\gamma\delta} = H_{\alpha\gamma,\beta\delta} + H_{\beta\delta,\alpha\gamma} - H_{\alpha\delta,\beta\gamma} - H_{\beta\gamma,\alpha\delta}. \tag{5.5}$$

On making use of the identities (9.4) and (9.5) of § 9 the equations (5.5) become

$$B_{\alpha\beta\gamma\delta} = H_{\alpha\beta,\gamma\delta} + 2H_{\alpha\gamma,\beta\delta}. \tag{5.6}$$

By continuing this process, i. e., by transforming the $m$th ($m \geqq 1$) covariant derivative $B_{\alpha\beta\gamma\delta\ldots\tau}$ of the curvature tensor $B_{\alpha\beta\gamma\delta}$ as given by (5.4) to a system of normal coördinates and evaluating at the origin of this system, the tensor $B_{\alpha\beta\gamma\delta\ldots\tau}$ can be expressed in terms of $H_{\alpha\beta}$ and the extensions $H_{\alpha\beta,\gamma\delta\ldots\varrho}$ up to and including the $(m+2)$th extension. In other words

$$B_{\alpha\beta\gamma\delta\ldots\tau} = \varphi(H_{ab};\ H_{ab,cd};\ \cdots;\ H_{ab,cd\ldots j}) \tag{5.7}$$

where $\varphi$ denotes a set of functions depending on indices $\alpha\beta\cdots\tau$ and the subscripts $cd\cdots j$ appearing in the extension of highest order $H_{ab,cd\ldots j}$ in the expressions $\varphi$ are $m+2$ in number. In the expressions $\varphi$ there can obviously occur no term depending on $H_{\alpha\beta}$ alone nor one which involves the covariant

derivative $H_{\alpha\beta,\gamma}$ since this vanishes identically. On account of this and the method of construction of the functions $\varphi$ the following theorem describing the nature of the relation (5.7) is evident.

THEOREM (5.I). *The differential invariant $B_{\alpha\beta\gamma\delta\ldots\lambda}$ of order $p(\geqq 2)$ can be expressed as a sum of homogeneous polynomials in the components of the extensions*

$$H_{ab,c_1c_2\cdots c_k} \qquad (k = 2, \cdots, p)$$

*the coefficients of which depend on the components $H_{\alpha\beta}$.*

The simplest way of deriving the expressions for the extensions $H_{\alpha\beta,\lambda\ldots\nu}$ in terms of the curvature tensor $B_{\alpha\beta\gamma\delta}$ and its covariant derivatives is to begin by writing down the equations (1.11) in the form

$$\frac{\partial H_{\alpha\beta}}{\partial x^\gamma} = H_{\sigma\beta}\,\Gamma^\sigma_{\alpha\gamma} + H_{\alpha\sigma}\,\Gamma^\sigma_{\beta\gamma}, \tag{5.8}$$

which express the fact that the covariant derivative $H_{\alpha\beta,\gamma}$ vanishes identically. If we differentiate (5.8) and evaluate at the origin of a system of normal coördinates we obtain

$$H_{\alpha\beta,\gamma\delta} = H_{\sigma\beta}\,A^\sigma_{\alpha\gamma\delta} + H_{\alpha\sigma}\,A^\sigma_{\beta\gamma\delta}, \tag{5.9}$$

where $A^i_{\alpha\beta\gamma}$ is a normal tensor. Using (5.3) and the relations*

$$3A^i_{\alpha\beta\gamma} = B^i_{\alpha\beta\gamma} + B^i_{\beta\alpha\gamma}$$

equations (5.9) become

$$3H_{\alpha\beta,\gamma\delta} = B_{\beta\alpha\gamma\delta} + B_{\beta\gamma\alpha\delta} + B_{\alpha\beta\gamma\delta} + B_{\alpha\gamma\beta\delta}. \tag{5.10}$$

By the identity (5.4) the first and third terms of (5.10) cancel, giving

$$H_{\alpha\beta,\gamma\delta} = \frac{1}{3}\,(B_{\alpha\gamma\beta\delta} + B_{\beta\gamma\alpha\delta}). \tag{5.11}$$

By repeated differentiation of (5.8) and evaluation at the origin of a system of normal coördinates we can express the $p(\geqq 2)$th extension $H_{\alpha\beta,\gamma\delta\ldots\eta}$ as a sum of homogeneous polynomials in extensions $H_{\alpha\beta,\gamma\delta\ldots\zeta}$ of order less than $p$ and normal tensors $A^i_{\alpha\beta\gamma\delta\ldots\xi}$ up to and including those of order $p$. Consequently it is possible to express the extension $H_{\alpha\beta,\gamma\delta\ldots\eta}$ of order $p(\geqq 2)$ as a sum of homogeneous polynomials in the normal tensors $A^i_{\alpha\beta\gamma\delta\ldots\xi}$ up to and including those of order $p$ with coefficients which depend on the quantities $H_{\alpha\beta}$. Now it can be shown that the normal tensor $A^i_{\alpha\beta\gamma\delta\ldots\zeta}$ of order $p$ can itself be expressed as a sum of homogeneous

* O. Veblen, Proc. Nat. Acad. Sci., vol. 8 (1922), p. 195.

polynomials in the invariants $B_{\alpha\beta\gamma\delta\ldots\varepsilon}$ up to and including those of order $p$. This follows quite simply by observing that the process of covariant differentiation leads to equations of the form

$$B^i_{\alpha\beta\gamma\delta\ldots\zeta} = \frac{\partial^{p-1}\Gamma^i_{\alpha\beta}}{\partial x^\gamma\,\partial x^\delta\cdots\partial x^\zeta} - \frac{\partial^{p-1}\Gamma^i_{\alpha\gamma}}{\partial x^\beta\,\partial x^\delta\cdots\partial x^\zeta} + \star \tag{5.12}$$

where the star indicates derivatives of the $\Gamma^i_{\alpha\beta}$ of order less than $p-1$ and covariant derivatives $B^i_{\alpha\beta\gamma\delta\ldots\varepsilon}$ involving a smaller number of subcripts than those in the set $\alpha\beta\gamma\delta\cdots\zeta$. Evaluating (5.12) at the origin of a system of normal coördinates, the derivatives of the $\Gamma^i_{\alpha\beta}$ in the right hand member of (5.12) go over into the corresponding normal tensors and these equations may then be solved for the normal tensor $A^i_{\alpha\beta\gamma\delta\ldots\zeta}$ of order $p$ when use is made of the complete set of identities satisfied by this tensor. In this manner we arrive at an expression for the normal tensor $A^i_{\alpha\beta\gamma\delta\ldots\zeta}$ of order $p$ which consists of the sum of homogeneous polynomials of normal tensors $A$ of order less than $p$ and tensors $B^i_{\alpha\beta\gamma\delta\ldots\varepsilon}$ up to and including those of order $p$. Using this last expression for $A^i_{\alpha\beta\gamma\delta\ldots\zeta}$ as a recursion formula we can express $A^i_{\alpha\beta\gamma\delta\ldots\zeta}$ as a sum of homogeneous polynomials in $B^i_{\alpha\beta\gamma\delta\ldots\varepsilon}$ up to and including those of order $p$.* Finally on making use of relations

$$\begin{aligned} B^i_{\alpha\beta\gamma} &= H^{i\sigma}\,B_{\sigma\alpha\beta\gamma}, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ B^i_{\alpha\beta\gamma\delta\ldots\zeta} &= H^{i\sigma}\,B_{\sigma\alpha\beta\gamma\delta\ldots\zeta} \end{aligned} \tag{5.13}$$

we arrive at an expression for the $p\,(\geqq 2)$th extension $H_{\alpha\beta,\,\gamma\delta\ldots\eta}$ which can be represented by

$$H_{\alpha\beta,\gamma\delta\ldots\eta} = \psi\,(H_{ab};\ B_{abcd};\ \cdots;\ B_{abcd\ldots k}) \tag{5.14}$$

where $\psi$ denotes a set of functions depending on indices $\alpha\beta\cdots\eta$ and the subscripts $ab\cdots k$ appearing in the highest covariant derivative $B_{abcd\ldots k}$ in the expressions $\psi$ are $p+2$ in number. Corresponding to theorem (5.I) we now arrive at the following theorem.

THEOREM (5.II). *The extension $H_{\alpha\beta,\,\gamma\delta\ldots\eta}$ of order $p\,(\geqq 2)$ can be expressed as a sum of homogeneous polynomials in the components of the invariant.*

$$B_{abc_1c_2\cdots c_k} \qquad (k = 2, 3, \cdots, p)$$

*the coefficients of which depend on the components $H_{\alpha\beta}$.*

* Cf. O. Veblen, loc. cit., p. 196.

If the second extension $H_{\alpha\beta,\,\gamma\delta}$ vanishes identically, it follows from (5.6) that the curvature tensor $B_{\alpha\beta\gamma\delta}$ also vanishes identically with the consequence that all covariant derivatives $B_{\alpha\beta\gamma\delta\ldots\eta}$ do likewise. Hence Theorem (5.II) admits the following corollary.

*Corollary. If the second extension $H_{\alpha\beta,\,\gamma\delta}$ (or the curvature tensor $B_{\alpha\beta\gamma\delta}$) vanishes identically then all the extensions $H_{\alpha\beta,\,\gamma\delta\ldots\eta}$ likewise vanish identically.*

This corollary is one of a number of important results which can evidently be obtained from Theorems (5.I) and (5.II) of this paragraph.

**6. Reduction of the form $G$ to a form with constant coefficients.** Let us assume that the form $G$ has real coefficients $G_{\alpha\beta}$ with respect to the $x^i$ variables. If the coefficients $G_{\alpha\beta}$ are constant, then

$$B_{\alpha\beta\gamma\delta} = 0, \text{ or } H_{\alpha\beta,\,\gamma\delta} = 0 \tag{6.1}$$

identically. Now taking the condition (6.1) to be satisfied and transforming to a system of normal coordinates $y^i$, the associated tensor components $H_{\alpha\beta}$ go over into a set of components $h_{\alpha\beta}$ (= const.) since all the terms, except the constant term, in the expansion for $h_{\alpha\beta}$ about the point $y^i = 0$, are zero by the corollary of § 5. The components $h_{\alpha\beta}$ may be real or pure imaginary. Also the components $g_{\alpha\beta}$ representing the coefficients of the form $G$ in the $y^i$ coördinate system, are constant by (1.9) and moreover are real since the Jacobian of the transformation to the $y^i$ coördinates is positive by (2.5). By the theorem on the transformation of algebraic quadratic forms we can transform the components $h_{\alpha\beta}$ by a real orthogonal transformation (2.1) of the normal coördinates into a set of components $\bar{h}_{\alpha\beta} = \pm\delta^\alpha_\beta$ or $\pm i\delta^\alpha_\beta$ according as the components $h_{\alpha\beta}$ are real or pure imaginary. Under this transformation (2.1) the components $g_{\alpha\beta}$ become $\bar{g}_{\alpha\beta} = \pm\delta^\alpha_\beta$ as is seen immediately from the relation between the quantities $\bar{g}_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$ (see (1.9)) and the fact that the Jacobian of this transformation is unity. As a consequence we can state the following theorem.

THEOREM (6.I). *A necessary and sufficient condition that the form $G$ having real coefficients $G_{\alpha\beta}$ be reducible by a transformation $\mathfrak{G}$ with positive Jacobian to a form with coefficients $\pm\delta^\alpha_\beta$ is that the curvature tensor $B_{\alpha\beta\gamma\delta}$ or second extension $H_{\alpha\beta,\gamma\delta}$ vanish identically.*

If the form $G$ can not be transformed into one with real coefficients $G_{\alpha\beta}$ then $G_{\alpha\beta}$ must have the form $i\psi_{\alpha\beta}$ where the functions $\psi_{\alpha\beta}$ are real. The modification of the above theorem for this case is obvious.

We shall next consider the problem of expressing the equivalence of two forms $G$ and $\bar{G}$ in terms of a finite number of algebraic conditions.

For this purpose we must make use of a theorem on partial differential equations given in the following paragraph.

7. **A theorem on partial differential equations.** Consider the system of differential equations

$$\frac{\partial Z^\alpha}{\partial x^i} = \psi_i^\alpha(Z, x) \qquad (\alpha = 1, 2, \cdots, R;\ i = 1, 2, \cdots, n) \tag{7.1}$$

in which the $Z$ are to be considered as functions of the independent variables $x^i$ and the $\psi_i^\alpha$ are analytic functions of their arguments. We seek solutions

$$Z^\alpha = Z^\alpha(x^1, \cdots, x^n) \tag{7.2}$$

of (7.1) which satisfy a system of equations

$$F_\lambda^{(0)}(Z, x) = 0. \tag{7.3}$$

It can be shown that the condition for the existence of such a solution is given by the following theorem.*

THEOREM (7.I). *A necessary and sufficient condition that the system of differential equations* (7.1) *admit a solution* (7.2) *satisfying equations* (7.3) *is that there exist an integer* $N(\geqq 1)$ *such that the first* $N$ *sets of equations of the sequence*

$$F_\lambda^{(1)}(Z, x) = 0; \quad F_\mu^{(2)}(Z, x) = 0; \quad F_\nu^{(3)}(Z, x) = 0; \cdots \tag{7.4}$$

*are algebraically consistent considered as equations for the determination of the* $Z^\alpha$ *as functions of independent variables* $x^i$ *and that all their solutions satisfy the* $(N+1)$st *set of equations in* (7.4) *where* $F_\lambda^{(1)} = 0$ *is the set of equations consisting of* (7.3), *the equations of integrability of* (7.1) *and the equations obtained by differentiating* (7.3) *with respect to* $x^i$ *and eliminating the derivatives of* $Z^\alpha$ *which occur by means of* (7.1); *and* $F_\tau^{(k+1)} = 0$ *for* $k \geqq 1$ *is the set of equations obtained by differentiating the set of equations* $F_\tau^{(k)} = 0$ *with respect to* $x^i$ *and eliminating the derivatives of* $Z^\alpha$ *by means of* (7.1).

By the conditions of integrability of equations (7.1) we mean of course the set of conditions obtained by differentiating (7.1) with respect to $x^j$ and then eliminating the second derivatives of $Z^\alpha$ by means of

$$\frac{\partial^2 Z^\alpha}{\partial x^i \partial x^j} = \frac{\partial^2 Z^\alpha}{\partial x^j \partial x^i}$$

and first derivatives of $Z^\alpha$ by means of (7.1).

* See reference to J. E. Wright and O. Veblen in the footnote to the bibliography. See also T. Levi-Civita, Lezioni di calcolo differenziale assoluto, Roma (1925), p. 40, § 8; and O. Veblen and J. M. Thomas, Ann. of Math. vol. 27 (1926), p. 288.

The necessity condition of the theorem is evident: in case the differential equations (7.1) possess a solution (7.2) satisfying (7.3) it is clear that the number $N$ must exist inasmuch as it is not possible to have an infinite number of independent equations (7.4) in a finite number of variables.

The sufficiency condition of the theorem can be proved quite readily. The first $N$ sets of equations (7.4) can be written in the solved form

$$Z^\alpha = \zeta^\alpha(Z^{M+1}, \cdots, Z^R, x) \quad (\alpha = 1, \cdots, M \leqq R) \tag{7.5}$$

provided that we assume, which we may do without any loss of generality, that the dependent $Z$'s are those of the set $Z^1, \cdots, Z^M$. Differentiation of (7.5) with respect to $x^i$ and elimination of the derivatives of the $Z$'s by (7.1) give the equations

$$\psi_i^\alpha = \frac{\partial \zeta^\alpha}{\partial Z^\sigma}\psi_i^\sigma + \frac{\partial \zeta^\alpha}{\partial x^i} \qquad (\sigma = M+1, \cdots, R). \tag{7.6}$$

Let us denote by the symbol $[\varphi]$ the expression obtained by eliminating the quantities $Z^1, \cdots, Z^M$ in the function $\varphi(Z, x)$ by means of (7.5). Thus the function $[\varphi]$ depends on the set of arguments $Z^{M+1}, \cdots, Z^R, x^1, \cdots, x^n$. From (7.6) we then obtain

$$[\psi_i^\alpha] = \frac{\partial \zeta^\alpha}{\partial Z^\sigma}[\psi_i^\sigma] + \frac{\partial \zeta^\alpha}{\partial x^i} \qquad (\sigma = M+1, \cdots, R) \tag{7.7}$$

and this equation is satisfied identically in the arguments $Z^{M+1}, \cdots, Z^R$, $x^1, \cdots, x^n$ as is evident from the method of formation of the sequence (7.4) and the fact that all solutions of the first $N$ sets of these equations, which are equivalent to (7.5), satisfy the $(N+1)$st set of equations of the sequence (7.4). On account of this it follows that the equations

$$\frac{\partial \zeta^\alpha}{\partial x^i} + \frac{\partial \zeta^\alpha}{\partial Z^\sigma}\frac{\partial Z^\sigma}{\partial x^i} = [\psi_i^\alpha] \qquad (\alpha = 1, \cdots, M \leqq R), \tag{7.8}$$

$$\frac{\partial Z^\sigma}{\partial x^i} = [\psi_i^\sigma] \qquad (\sigma = M+1, \cdots, R) \tag{7.9}$$

which are obtained by substituting (7.5) in (7.1) are such that any solution $Z^\sigma = Z^\sigma(x)$ of (7.9) is a solution of (7.8). The integrability conditions of (7.9) are

$$\frac{\partial [\psi_i^\sigma]}{\partial x^j} + \frac{\partial [\psi_i^\sigma]}{\partial Z^\varrho}[\psi_j^\varrho] = \frac{\partial [\psi_j^\sigma]}{\partial x^i} + \frac{\partial [\psi_j^\sigma]}{\partial Z^\varrho}[\psi_i^\varrho] \qquad (\sigma, \varrho = M+1, \cdots, R)$$

or

$$\left[\frac{\partial \psi_i^\sigma}{\partial x^j} + \frac{\partial \psi_i^\sigma}{\partial Z^\alpha}\frac{\partial \zeta^\alpha}{\partial x^j}\right] + \left[\frac{\partial \psi_i^\sigma}{\partial Z^\varrho} + \frac{\partial \psi_i^\sigma}{\partial Z^\alpha}\frac{\partial \zeta^\alpha}{\partial Z^\varrho}\right][\psi_j^\varrho]$$
$$= \left[\frac{\partial \psi_j^\sigma}{\partial x^i} + \frac{\partial \psi_j^\sigma}{\partial Z^\alpha}\frac{\partial \zeta^\alpha}{\partial x^i}\right] + \left[\frac{\partial \psi_j^\sigma}{\partial Z^\varrho} + \frac{\partial \psi_j^\sigma}{\partial Z^\alpha}\frac{\partial \zeta^\alpha}{\partial Z^\varrho}\right][\psi_i^\varrho].$$

By (7.6) these last equations become

$$\left[\frac{\partial \psi_i^\sigma}{\partial x^j} + \frac{\partial \psi_i^\sigma}{\partial Z^\alpha}\,\psi_j^\alpha + \frac{\partial \psi_i^\sigma}{\partial Z^\varrho}\,\psi_j^\varrho\right] = \left[\frac{\partial \psi_j^\sigma}{\partial x^i} + \frac{\partial \psi_j^\sigma}{\partial Z^\alpha}\,\psi_i^\alpha + \frac{\partial \psi_j^\sigma}{\partial Z^\varrho}\,\psi_i^\varrho\right]$$

and hence are satisfied identically in the variables $Z^{M+1}, \cdots, Z^R, x^1, \cdots, x^n$ since they are the result of eliminating $Z^1, \cdots, Z^M$ by (7.5) in certain of the equations of integrability of (7.1). This completes the proof of the theorem.

In the process of obtaining the value of the integer $N$ in Theorem (7.I) so as to make $N$ have a least value we observe that each set of equations $F_\eta^{(\sigma)} = 0$ in the sequence (7.4) must contribute at least one additional condition on the quantities $Z^\alpha$ over the conditions imposed on them by all preceding sets. Hence

$$1 \leqq N \leqq R \tag{7.10}$$

where $R$ is the total number of variables $Z^\alpha$. By the integer $N$ in the following we shall mean the least value of this integer, which will therefore satisfy (7.10).

8. **Finiteness of the complete set of tensor invariants.** We saw in § 4 that the tensor $H_{\alpha\beta}$ and its extensions constitute a complete set of differential invariants of the form $G$. The question arises whether it is possible to give an algebraic characterization of the form $G$ by the tensor $H_{\alpha\beta}$ and a *finite* number of its extensions. This question can be answered in the affirmative by the following theorem.

THEOREM (8.I). *A necessary and sufficient condition for the equivalence of two forms $G$ and $\bar{G}$ of weight $\varkappa$ is that there exist a number $N(\geqq 1)$ such that the equations*

$$\begin{aligned}
\bar{H}_{\alpha\beta} &= H_{\mu\nu}\, u_\alpha^\mu\, u_\beta^\nu, \\
\bar{H}_{\alpha\beta,\gamma\delta_1} &= H_{\mu\nu,\sigma\tau_1}\, u_\alpha^\mu\, u_\beta^\nu\, u_\gamma^\sigma\, u_{\delta_1}^{\tau_1}, \\
\bar{H}_{\alpha\beta,\gamma\delta_1\delta_2} &= H_{\mu\nu,\sigma\tau_1\tau_2}\, u_\alpha^\mu\, u_\beta^\nu\, u_\gamma^\sigma\, u_{\delta_1}^{\tau_1}\, u_{\delta_2}^{\tau_2}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\bar{H}_{\alpha\beta,\gamma\delta_1\cdots\delta_N} &= H_{\mu\nu,\sigma\tau_1\cdots\tau_N}\, u_\alpha^\mu\, u_\beta^\nu\, u_\gamma^\sigma\, u_{\delta_1}^{\tau_1} \cdots u_{\delta_N}^{\tau_N}
\end{aligned} \tag{8.1}$$

*are compatible considered as equations for the determination of $x^i$ and $u_\alpha^i$ as functions of the independent variables $\bar{x}^i$, and that all their solutions*

$$x^i = f^i(\bar{x}^1, \cdots, \bar{x}^n); \qquad u_\alpha^i = \varphi_\alpha^i(\bar{x}^1, \cdots, \bar{x}^n) \tag{8.2}$$

*satisfy the equations*

$$\bar{H}_{\alpha\beta,\gamma\delta_1\cdots\delta_{N+1}} = H_{\mu\nu,\sigma\tau_1\cdots\tau_{N+1}} u^{\mu}_{\alpha} u^{\nu}_{\beta} u^{\sigma}_{\gamma} u^{\tau_1}_{\delta_1} \cdots u^{\tau_{N+1}}_{\sigma_{N+1}}. \tag{8.3}$$

The necessity condition of this theorem follows by the observation made in the demonstration of the necessity of the condition of Theorem (7.I).

To prove the sufficiency condition of the above theorem let us write the equations (1.12) in the form

$$\frac{\partial u^i_\alpha}{\partial \bar{x}^\beta} = \bar{\Gamma}^{\sigma}_{\alpha\beta} u^i_\sigma - \Gamma^i_{\sigma\tau} u^\sigma_\alpha u^\tau_\beta; \qquad \frac{\partial x^i}{\partial \bar{x}^\alpha} = u^i_\alpha(\bar{x}^1, \cdots, \bar{x}^n). \tag{8.4}$$

Let us also write the equations (1.8) as

$$\bar{H}_{\alpha\beta} = H_{\sigma\tau} u^\sigma_\alpha u^\tau_\beta. \tag{8.5}$$

The differential equations (8.4) are of the form (7.1) as is evident when we denote the $n(n+1)$ variables $x^i$ and $u^i_\alpha$ in (8.4) by $n(n+1)$ variables $Z^\alpha$ aud replace the variables $\bar{x}^i$ in (8.4) by the $n$ variables $x^i$. Under the same change in notation equations (8.5) go over into equations involving the variables $Z^\alpha$ and $x^i$ which are of the form (7.3). Corresponding to the sequence of equations (7.4) we now have

$$\begin{aligned}
\bar{H}_{\alpha\beta} &= H_{\lambda\mu} u^\lambda_\alpha u^\mu_\beta, \\
\bar{B}^{\sigma}_{\alpha\beta\gamma} u^i_\sigma &= B^i_{\lambda\mu\nu} u^\lambda_\alpha u^\mu_\beta u^\nu_\gamma, \\
\bar{B}^{\sigma}_{\alpha\beta\gamma\delta} u^i_\sigma &= B^i_{\lambda\mu\nu\varrho} u^\lambda_\alpha u^\mu_\beta u^\nu_\gamma u^\varrho_\delta, \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{8.6}$$

Making use of the equations (5.3) we can evidently write the sequence (8.6) in the equivalent form

$$\begin{aligned}
\bar{H}_{\alpha\beta} &= H_{\lambda\mu} u^\lambda_\alpha u^\mu_\beta, \\
\bar{B}_{\alpha\beta\gamma\delta} &= B_{\lambda\mu\nu\varrho} u^\lambda_\alpha u^\mu_\beta u^\nu_\gamma u^\varrho_\delta, \\
\bar{B}_{\alpha\beta\gamma\delta\varepsilon} &= B_{\lambda\mu\nu\varrho\sigma} u^\lambda_\alpha u^\mu_\beta u^\nu_\gamma u^\varrho_\delta u^\sigma_\varepsilon, \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{8.7}$$

Now consider the relations (5.14) in the barred variables $(\bar{H})$ and $(\bar{B})$. In the right hand member of these equations eliminate the $\bar{H}_{\alpha\beta}$ and the

quantities $(\bar{B})$ by means of (8.7) and then eliminate the quantities $(B)$ which have been introduced by means of (5.7). There results a sequence of equations expressing the extensions of $\bar{H}_{\alpha\beta}$ in terms of the quantities $(H)$ and $u_\alpha^i$. These equations when combined with the equations (8.5) give a sequence of equations by means of which the quantities $(\bar{H})$ are expressed in terms of the quantities $(H)$ and $u_\alpha^i$ and hence must be identical with the sequence (4.1) since (4.1) or an equivalent form derivable from (4.1) by means of the identities of the following paragraph constitute the only relations that can exist between the quantities $(H)$, $(\bar{H})$ and $u_\alpha^i$. Conversely we can pass from the sequence (4.1) to the sequence (8.6). This proves the sufficiency condition of Theorem (8.I) since if this condition is satisfied for the sequence (4.1) it is likewise satisfied for the sequence (8.7) which by Theorem (7.I) suffices for the transformation of the two differential forms $G$ and $\bar{G}$.

By the inequality (7.10) we have that the integer $N$ appearing in the statement of Theorem (8.I) will satisfy the inequality

$$1 \leqq N \leqq n^2+n. \tag{8.8}$$

It can be shown quite readily that the *integer $N$ is an arithmetical invariant of the form $G$.* For simplicity we shall denote the equations of the sequence (4.1) by

$$(H) = (\bar{H}, u). \qquad (u_\alpha^i \text{ now corresponds to } \partial\bar{x}^i/\partial x^\alpha.) \tag{8.9}$$

In order to prove the invariant character of the integer $N$ we must show that the value of this integer determined by (8.9) is the same as its value when determined by

$$(H) = (\bar{\bar{H}}, v) \tag{8.10}$$

where the quantities $(\bar{\bar{H}})$ are based on the form $\bar{\bar{G}}$ which is obtainable from the form $\bar{G}$ by an arbitrary transformation

$$x^i = f^i(\bar{\bar{x}}). \tag{8.11}$$

Hence between the quantities $(\bar{H})$ and $(\bar{\bar{H}})$ the equations

$$(\bar{H}) = (\bar{\bar{H}}, w); \quad w_\alpha^i = \frac{\partial\bar{\bar{x}}^i}{\partial\bar{x}^\alpha} \tag{8.12}$$

hold.

Let

$$x^i = g^i(\bar{x}, \cdots, \bar{x}^n); \quad u^i_\alpha = h^i_\alpha(\bar{x}^1, \cdots, \bar{x}^n) \tag{8.13}$$

denote any solution of the first $N = \bar{N}$ sets of equations (8.9).

Form the relations

$$x^i = r^i(\bar{\bar{x}}); \quad v^i_j = s^i_j(\bar{\bar{x}}) \tag{8.14}$$

where $r^i(\bar{\bar{x}})$ is obtained from $g^i(\bar{x})$ by the substitution (8.11) and $s^i_j$ is defined by

$$s^i_j = w^i_\sigma u^\sigma_j. \tag{8.15}$$

By the equations (8.12) and (8.15) the equations (8.9) go over into equations (8.10). Hence (8.14) satisfies the first $\bar{N}+1$ sets of equations (8.10). The set $S$ of all solutions of the first $\bar{N}$ sets of equations (8.9) defines a set $S'$ of solutions (8.14) of the first $\bar{N}$ sets of equations (8.10) each of which satisfies the $(\bar{N}+1)$st set of these equations. The solutions (8.14) of the set $S'$ constitute all solutions of the first $\bar{N}$ sets of equations (8.10) for if there existed any solution of these equations not in the set $S'$ we could use this solution and the relation (8.11) to define a relation (8.13) by the process by which (8.14) was obtained, and this relation would constitute a solution of the first $\bar{N}$ sets of equations (8.9) which can not be in the set $S$ contrary to hypothesis. Let the integer $N = \bar{\bar{N}}$ be determined by (8.10). We have just shown that $\bar{\bar{N}}$, which is the least value of the integer $N$ determined by the equations (8.10), can not be greater than $\bar{N}$. Also $\bar{\bar{N}}$ can not be less than $\bar{N}$, for then it would follow by an argument similar to the one which we have just made that the integer $N$ determined by the equations (8.9) would be less than $\bar{N}$. Hence $\bar{N} = \bar{\bar{N}}$. This complets the proof that the integer $N$ is an invariant of the form $G$.

As a consequence of the invariant nature of the integer $N$ we can say that the *tensor $H_{\alpha\beta}$ and its extensions $H_{\alpha\beta,\gamma_1\cdots\gamma_r}(r = 2, \cdots, N+2)$ give a complete algebraic characterization of the form $G$.*

We shall see in § 15 that the upper bound of the integer $N$ can be reduced considerably.

**9. Complete sets of identities of $H$-invariants.** By (1.9) the tensor $H_{\sigma\tau}$ is determined when the relative tensor $G_{\sigma\tau}$ is given subject to the condition that the determinant $|G_{\alpha\beta}|$ does not vanish identically; conversely, the relative tensor $G_{\sigma\tau}$ is determined when the tensor $H_{\sigma\tau}$ is given provided that the determinant $|H_{\alpha\beta}|$ does not vanish identically. Hence the components $H_{\sigma\tau}$ can be taken as arbitrary functions subject only to the symmetry condition (1.10) for any particular set of independent variables $x^i$. Con-

sequently the complete sets of identities satisfied by the tensor $H_{\sigma\tau}$ and its extensions $H_{\alpha\beta,\gamma\delta\cdots\sigma}$ must be identical with the complete sets of identities satisfied by the fundamental metric tensor and its extensions in a Riemann geometry.* We may therefore say that the identities

$$H_{\alpha\beta} = H_{\beta\alpha} \tag{9.1}$$

are the complete set of identities for the tensor $H_{\alpha\beta}$ itself; also the identities

$$H_{\alpha\beta,\gamma_1\cdots\gamma_p} = H_{\beta\alpha,\gamma_1\cdots\gamma_p}; \qquad H_{\alpha\beta,\gamma_1\cdots\gamma_p} = H_{\alpha\beta,\delta_1\cdots\delta_p} \tag{9.2}$$

and

$$S(H_{\alpha\beta,\gamma_1\cdots\gamma_p}) = 0 \tag{9.3}$$

constitute the complete set of identities of the $p(\geqq 2)$th extension $H_{\alpha\beta,\gamma_1\cdots\gamma_p}$ where $\delta_1 \cdots \delta_p$ is any permutation of $\gamma_1 \cdots \gamma_p$ and $S(\ )$ denotes the sum of all terms obtainable from the one in parenthesis by a cyclic permutation of the subscripts $\beta\gamma_1 \cdots \gamma_p$.

In particular if $p = 2$ the identities (9.2) and (9.3) become

$$H_{\alpha\beta,\gamma\delta} = H_{\beta\alpha,\gamma\delta}; \qquad H_{\alpha\beta,\gamma\delta} = H_{\alpha\beta,\delta\gamma} \tag{9.4}$$

and

$$H_{\alpha\beta,\gamma\delta} + H_{\alpha\gamma,\delta\beta} + H_{\alpha\delta,\beta\gamma} = 0. \tag{9.5}$$

By an interchange of indices in (9.5) we have

$$H_{\gamma\delta,\alpha\beta} + H_{\gamma\alpha,\beta\delta} + H_{\gamma\beta,\delta\alpha} = 0. \tag{9.6}$$

Subtracting (9.6) from (9.5) we obtain

$$(H_{\alpha\beta,\gamma\delta} - H_{\gamma\delta,\alpha\beta}) + (H_{\alpha\delta,\beta\gamma} - H_{\beta\gamma,\alpha\delta}) = 0 \tag{9.7}$$

which by an interchange of indices may be written as

$$(H_{\beta\alpha,\delta\gamma}) - H_{\delta\gamma,\beta\alpha}) + (H_{\beta\gamma,\alpha\delta} - H_{\alpha\delta,\beta\gamma}) = 0. \tag{9.8}$$

Finally subtracting (9.8) from (9.7) we find the interesting identity

$$H_{\alpha\beta,\gamma\delta} = H_{\gamma\delta,\alpha\beta}. \tag{9.9}$$

Let us now calculate the number of independent components of the $(H)$ tensors. We shall denote by $N(n, 0)$ the number of independent components

* T. Y. Thomas, Math. Zeit., vol. 25 (1926), pp. 714–722.

of the tensor $H_{\alpha\beta}$ and by $N(n, p)$ the number of independent components of the $p$th extension $H_{\alpha\beta,\gamma_1\cdots\gamma_p}$. Since $H_{\alpha\beta,\gamma}$ vanishes identically, the number $N(n, 1)$ is zero. In order to calculate the number $N(n, p)$ we must first define the symbol $K(n, p)$ as the number of combinations with repetitions of $n$ things taken $p$ at a time. We have

$$(9.10)\quad K(n,p) = \frac{n(n+1)(n+2)\cdots(n+p-1)}{p!}; \qquad K(n,0) = 1.$$

By (9.1) the number of independent components $H_{\alpha\beta}$ is

$$(9.11)\quad N(n,0) = K(n,2) = n(n+1)/2.$$

By the identities (9.2) alone the number of independent components of $H_{\alpha\beta,\gamma_1\cdots\gamma_p}$ is given by $K(n, 2)\ K(n, p)$. Also the independent identities of (9.3) are $nK(n, p+1)$ in number. Hence

$$(9.12)\quad N(n,p) = K(n,2)\ K(n,p) - nK(n,p+1) \qquad (p \geqq 2)$$

which by (9.10) becomes

$$(9.13)\quad N(n,p) = \frac{n}{2}\cdot\frac{(n+p-1)!}{(n-2)!}\cdot\frac{(p-1)}{(p+1)!} \qquad (p \geq 2)$$

For the purpose of convenient reference we shall write down the following special cases of the above formulas:

$$(9.14)\quad \begin{aligned} N(n,2) &= n^2(n^2-1)/12,\\ N(n,3) &= n^2(n^2-1)(n+2)/24,\\ N(n,4) &= n^2(n^2-1)(n+2)(n+3)/80. \end{aligned}$$

It can easily be seen that *the number $N(n, p)$ gives the number of independent components of the tensor* $B_{\alpha\beta\gamma_1\cdots\gamma_p}$ which is the covariant curvature tensor if $p = 2$ and the $(p-2)$nd covariant derivative of the curvature tensor $B_{\alpha\beta\gamma_1\gamma_2}$ for $p > 2$. Consider the two sets of invariants

$$\begin{aligned} &\text{I:} \quad H_{\alpha\beta};\ H_{\alpha\beta,\gamma_1\gamma_2};\ \cdots;\ H_{\alpha\beta,\gamma_1\cdots\gamma_p},\\ &\text{II:} \quad H_{\alpha\beta};\ B_{\alpha\beta\gamma_1\gamma_2};\ \cdots;\ B_{\alpha\beta\gamma_1\cdots\gamma_p} \end{aligned}$$

which are connected by the formulas (5.7) and (5.14). Let us suppose that all dependent quantities have been eliminated from equations (5.7) and (5.14) so that these equations express relations between the independent components of the two sets I and II. Suppose that the number of independent components of the set I were greater than the number of independent components of the set II. Then a certain number of the equations

(5.14), less than the total number of these equations, can be solved for the independent components of the set II and when these latter relations are used to eliminate the quantities of the set II from the remaining equations in (5.14) we obtain relations between certain of the independent quantities of the set I which is contrary to the hypothesis that these quantities are independent. In a similar manner the independent components of the set II can not be greater than those of the set I. Hence the number of the independent components of the sets I and II are equal. It follows immediately by a recurrence consideration that the number of independent components of corresponding tensors in the sets I and II are also equal, i. e., the number of independent components of $B_{\alpha\beta\gamma_1\cdots\gamma_p}$ is the same as the number of independent components of $H_{\alpha\beta,\gamma_1\cdots\gamma_p}$.

10. **Differential equations of absolute scalar invariants.** We saw in § 3 that a differential invariant of order $p$ of the form $G$ can be expressed as a function of the tensor $H_{\alpha\beta}$ and its extensions $H_{\alpha\beta,\gamma_1\cdots\gamma_r}$ $(r=2,\cdots,p)$*. In particular if $I(H)$ represents an absolute scalar invariant of order $p$ expressed in terms of $H_{\alpha\beta}$ and its extensions then

$$I(H)=I(\bar{H}) \tag{10.1}$$

under any transformation $\mathfrak{G}$.

Let us now derive the system of partial differential equations satisfied by an absolute scalar invariant $I(H)$ of order $p$. A necessary and sufficient condition that $I(H)$ be an absolute scalar invariant of the form $G$ is that (10.1) hold under the infinitesimal transformations

$$x^i=\bar{x}^i+\varepsilon\,\xi^i(\bar{x}^1,\cdots,\bar{x}^n) \tag{10.2}$$

where the functions $\xi^i$ are arbitrary†. But the necessary and sufficient condition that (10.1) hold under the infinitesimal transformations (10.2) is that

$$\left[\frac{dI(\bar{H})}{d\varepsilon}\right]_{\varepsilon=0}=0. \tag{10.3}$$

We have

$$\left[\frac{dI(\bar{H})}{d\varepsilon}\right]_{\varepsilon=0}=\frac{\partial I(H)}{\partial H_{\alpha\beta}}\left(\frac{d\bar{H}_{\alpha\beta}}{d\varepsilon}\right)_{\varepsilon=0}+\frac{\partial I(H)}{\partial H_{\alpha\beta,\gamma_1\gamma_2}}\left(\frac{d\bar{H}_{\alpha\beta,\gamma_1\gamma_2}}{d\varepsilon}\right)_{\varepsilon=0}+\cdots$$
$$+\frac{\partial I(H)}{\partial H_{\alpha\beta,\gamma_1\cdots\gamma_p}}\left(\frac{d\bar{H}_{\alpha\beta,\gamma_1\cdots\gamma_p}}{d\varepsilon}\right)_{\varepsilon=0}.$$

* We remind the reader that we understand by the general differential invariant a relative tensor of arbitrary weight. See § 3.

† T. Y. Thomas and A. D. Michal, Annals of Math., vol. 28 (1927), p. 204.

But

$$\left(\frac{d\bar{H}_{\alpha\beta}}{d\varepsilon}\right)_{\varepsilon=0} = (\delta^\tau_\alpha H_{\sigma\beta} + \delta^\tau_\beta H_{\alpha\sigma})\frac{\partial \xi^\sigma}{\partial x^\tau}$$

and in general*

$$\left(\frac{dH_{\alpha\beta,\gamma_1\cdots\gamma_r}}{d\varepsilon}\right)_{\varepsilon=0} = (\delta^\tau_\alpha H_{\sigma\beta,\gamma_1\cdots\gamma_r} + \delta^\tau_\beta H_{\alpha\sigma,\gamma_1\cdots\gamma_r} + \delta^\tau_{\gamma_1} H_{\alpha\beta,\sigma\gamma_2\cdots\gamma_r} + \cdots + \delta^\tau_{\gamma_r} H_{\alpha\beta,\gamma_1\cdots\gamma_{r-1}\sigma})\frac{\partial \xi^\tau}{\partial x^\tau}.$$

Hence on account of the arbitrariness of the functions $\xi^i$ the equations (10.3) become

$$\binom{\tau}{\sigma\alpha\beta}\frac{\partial I(H)}{\partial H_{\alpha\beta}} + \binom{\tau}{\sigma\alpha\beta\gamma_1\gamma_2}\frac{\partial I(H)}{\partial H_{\alpha\beta,\gamma_1\gamma_2}} + \cdots + \binom{\tau}{\sigma\alpha\beta\gamma_1\cdots\gamma_p}\frac{\partial I(H)}{\partial H_{\alpha\beta,\gamma_1\cdots\gamma_p}} = 0 \tag{10.4}$$

where

$$\binom{\tau}{\sigma\alpha\beta} = \delta^\tau_\alpha H_{\sigma\beta} + \delta^\tau_\beta H_{\alpha\sigma} \tag{10.5}$$

and

$$\binom{\tau}{\sigma\alpha\beta\ldots\xi\eta} = \delta^\tau_\alpha H_{\sigma\beta,\ldots\xi\eta} + \delta^\tau_\beta H_{\alpha\sigma,\ldots\xi\eta} + \cdots + \delta^\tau_\eta H_{\alpha\beta,\ldots\xi\sigma}. \tag{10.6}$$

As thus defined the expressions $\binom{\tau}{\sigma\alpha\beta\gamma}$ are zero since $H_{\alpha\beta,\gamma}$ vanishes identically.

In the following we shall consider that the scalar invariant $I(H)$ in the differential equations (10.4) is expressed in terms of arguments $(H)$ which are independent when regard is taken of the complete sets of identities (9.1), (9.2) and (9.3).

Let us denote the system of differential equations (10.4) by

$$X^\tau_\sigma(p)I = 0 \tag{10.7}$$

where

$$X^\tau_\sigma(p)I = \sum_{r=2}^{p+2}\binom{\tau}{\sigma\gamma_1\gamma_2\gamma_3\cdots\gamma_r}\frac{\partial I}{\partial H_{\gamma_1\gamma_2,\gamma_3\cdots\gamma_r}}. \tag{10.8}$$

The completeness of the differential equations (10.4) can be proved quite simply by the method of induction. First take the case when $p = 0$ in equations (10.7). We have

$$X^\tau_\sigma(0)I = \binom{\tau}{\sigma\alpha\beta}\frac{\partial I}{\partial H_{\alpha\beta}} = 0. \tag{10.9}$$

* T. Y. Thomas and A. D. Michal, loc. cit., p 203.

Forming the alternants of the equations (10.9) we obtain after a slight calculation

$$(X^\tau_\sigma(0),\ X^\lambda_\mu(0))\,I = \delta^\tau_\mu X^\lambda_\sigma(0)\,I - \delta^\lambda_\sigma X^\tau_\mu(0)\,I \tag{10.10}$$

which proves the completeness of the differential equations (10.9). Now assume that the relations

$$(X^\tau_\sigma(p),\ X^\lambda_\mu(p))\,I = \delta^\tau_\mu X^\lambda_\sigma(p)\,I - \delta^\lambda_\sigma X^\tau_\mu(p)\,I \tag{10.11}$$

hold for $p = r$. We wish to show that (10.11) holds for $p = r+1$. We have

$$X^\tau_\sigma(r+1)\,I = X^\tau_\sigma(r)\,I + Z^\tau_\sigma I \tag{10.12}$$

where

$$Z^\tau_\sigma I = \binom{\tau}{\sigma\gamma_1 \cdots \gamma_{r+1}} \frac{\partial I}{\partial H_{\gamma_1 \cdots \gamma_{r+1}}}. \tag{10.13}$$

Obviously

$$(X^\tau_\sigma(r+1),\ X^\lambda_\mu(r+1))\,I = (X^\tau_\sigma(r),\ X^\lambda_\mu(r))\,I + (Z^\tau_\sigma,\ Z^\lambda_\mu)\,I. \tag{10.14}$$

We have

$$\begin{aligned}(Z^\tau_\sigma, Z^\lambda_\mu)\,I = \Bigg[ &\binom{\tau}{\sigma\delta_1 \cdots \delta_{r+1}} \frac{\partial \binom{\lambda}{\mu\gamma_1 \cdots \gamma_{r+1}}}{\partial H_{\delta_1 \cdots \delta_{r+1}}} \\ &- \binom{\lambda}{\mu\delta_1 \cdots \delta_{r+1}} \frac{\partial \binom{\tau}{\sigma\gamma_1 \cdots \gamma_{r+1}}}{\partial H_{\delta_1 \cdots \delta_{r+1}}} \Bigg] \frac{\partial I}{\partial H_{\gamma_1 \cdots \gamma_{r+1}}} \\ = \Bigg[ &\delta^\lambda_{\gamma_1} \binom{\tau}{\sigma\mu\gamma_2 \cdots \gamma_{r+1}} + \delta^\lambda_{\gamma_2} \binom{\tau}{\sigma\gamma_1\mu\gamma_3 \cdots \gamma_{r+1}} + \cdots + \delta^\lambda_{\gamma_{r+1}} \binom{\tau}{\sigma\gamma_1 \cdots \gamma_r\mu} \\ &- \delta^\tau_{\gamma_1} \binom{\lambda}{\mu\sigma\gamma_2 \cdots \gamma_{r+1}} - \delta^\tau_{\gamma_2} \binom{\lambda}{\mu\gamma_1\sigma\gamma_3 \cdots \gamma_{r+1}} - \cdots - \delta^\tau_{\gamma_{r+1}} \binom{\lambda}{\mu\gamma_1 \cdots \gamma_r\sigma} \Bigg] \\ &\times \frac{\partial I}{\partial H_{\gamma_1 \cdots \gamma_{r+1}}}.\end{aligned}$$

On making use of the equations (10.6) the above equations become

$$(Z^\tau_\sigma, Z^\lambda_\mu)\,I = \delta^\tau_\mu Z^\lambda_\sigma\,I - \delta^\lambda_\sigma Z^\tau_\mu\,I. \tag{10.15}$$

By (10.14), (10.15) and the assumption that (10.11) holds for $p = r$ it follows that (10.11) holds for $p = r+1$. Hence (10.11) is true for all values of $p$ since by (10.10) it holds for $p = 0$, i. e., the system of differential equations (10.7) is always complete.

The results obtained so far in this paragraph may now be summed up in the following theorem.

THEOREM (10.I). *A necessary and sufficient condition that a function*

$$I(H_{\alpha\beta};\ H_{\alpha\beta,\gamma_1\gamma_2};\ \cdots;\ H_{\alpha\beta,\gamma_1\cdots\gamma_r})$$

*be an absolute scalar differential invariant of the form G is that it satisfy the complete system of partial differential equations* (10.4).

The relations (10.11) can be written in the form

$$(X^{\tau}_{\sigma}(p),\ X^{\lambda}_{\mu}(p))\, I = C^{\tau\lambda\varrho}_{\sigma\mu\nu}\, X^{\nu}_{\varrho}(p)\, I$$

where

$$C^{\tau\lambda\varrho}_{\sigma\mu\nu} = \delta^{\tau}_{\mu}\,\delta^{\lambda}_{\nu}\,\delta^{\varrho}_{\sigma} - \delta^{\lambda}_{\sigma}\,\delta^{\tau}_{\nu}\,\delta^{\varrho}_{\mu}$$

are the structural constants of the underlying $n^2$-parameter group. While it is evident that the group in question is furnished by the equations (8.1) for $N = p-1$ in the $n^2$ parameters $u^i_\alpha$ this can be shown in a formal way as in our paper on affine differential invariants (loc. cit., p. 219) where the underlying group is the group of transformations of the normal tensors $A$.

It follows immediately from the differential equations (10.9) that there exist no absolute scalar invariants $I(H)$ which depend on the components $H_{\alpha\beta}$ alone. Consider the expanded form of the equations (10.9), i. e.,

$$(\delta^{\tau}_{\alpha}\, H_{\sigma\beta} + \delta^{\tau}_{\beta}\, H_{\alpha\sigma})\,\frac{\partial I}{\partial H_{\alpha\beta}} = 0. \tag{10.16}$$

Multiplying the equations (10.16) by $H^{\sigma i}$ we obtain

$$\frac{\partial I}{\partial H_{\alpha\beta}} + \frac{\partial I}{\partial H_{\beta\alpha}} = 0. \tag{10.17}$$

If we consider $I$ to be expressed in terms of the $n(n+1)/2$ independent components $H_{\alpha\beta}\,(\alpha \leqq \beta)$ then the equations (10.17) reduce to

$$\frac{\partial I}{\partial H_{\alpha\beta}} = 0 \qquad (\alpha \leqq \beta) \tag{10.18}$$

and hence *there exist no absolute scalar invariants which depend on the coefficients $G_{\alpha\beta}$ of the form G alone, i.e., there exists no invariant of order zero.*

We have already seen on the basis of the replacement theorem (3.I) that there exists no differential invariant of order one. However absolute scalar differential invariants of order two or greater exist in general as will be shown in the following paragraphs.

11. **General theorems on the independence of the differential equations.** In order to find the number of functionally independent absolute scalar invariants $I(H)$ considered as functions of the tensor $H_{\alpha\beta}$ and its extensions it is necessary to calculate the number of independent equations (10.7). For this purpose we shall demonstrate two general theorems regarding these equations.

THEOREM (11.I). *If the differential equations* (10.7) *for the determination of the absolute scalar differential invariants of order $p$ are independent for any form $G$, those for the determination of the invariants of order $p+1$ are also independent.*

This theorem is an immediate consequence of the differential equations (10.7) themselves.

THEOREM (11.II). *If the differential equations for the determination of the absolute scalar differential invariants of order two are dependent for every form $G$ in $n$ variables, they are dependent for every form in $n-1$ variables.*

By the hypothesis of Theorem (11.II) the equations

$$\binom{\tau}{\sigma\,\alpha\,\beta}\frac{\partial I}{\partial H_{\alpha\beta}}+\binom{\tau}{\sigma\,\alpha\,\beta\,\gamma\,\delta}\frac{\partial I}{\partial H_{\alpha\beta,\gamma\delta}}=0 \tag{11.1}$$

are dependent. Consequently the linear homogeneous equations

$$\binom{\tau}{\sigma\,\alpha\,\beta}\eta_\tau^\sigma=0;\qquad \binom{\tau}{\sigma\,\alpha\,\beta\,\gamma\,\delta}\eta_\tau^\sigma=0 \tag{11.2}$$

admit solutions $\eta$ which are not all identically zero. In equations (11.2) the indices $\alpha, \beta$ in $\binom{\tau}{\sigma\,\alpha\,\beta}$ and the indices $\alpha, \beta, \gamma, \delta$ in $\binom{\tau}{\sigma\,\alpha\,\beta\,\gamma\,\delta}$ correspond to the indices of the independent components $H_{\alpha\beta}$ and $H_{\alpha\beta,\gamma\delta}$ respectively. But from the identities (9.1), (9.2) and (9.3) the expressions $\binom{\tau}{\sigma\,\alpha\,\beta}$ and $\binom{\tau}{\sigma\,\alpha\,\beta\,\gamma\,\delta}$ satisfy identities

$$\binom{\tau}{\sigma\,\alpha\,\beta}=\binom{\tau}{\sigma\,\beta\,\alpha};\qquad \binom{\tau}{\sigma\,\alpha\,\beta\,\gamma\,\delta}=\binom{\tau}{\sigma\,\beta\,\alpha\,\gamma\,\delta}=\binom{\tau}{\sigma\,\alpha\,\beta\,\delta\,\gamma};$$

$$\binom{\tau}{\sigma\,\alpha\,\beta\,\gamma\,\delta}+\binom{\tau}{\sigma\,\alpha\,\gamma\,\beta\,\delta}+\binom{\tau}{\sigma\,\alpha\,\delta\,\beta\,\gamma}=0.$$

As the above identities in $\binom{\tau}{\sigma\,\alpha\,\beta}$ and $\binom{\tau}{\sigma\,\alpha\,\beta\,\gamma\,\delta}$ correspond exactly to the identities satisfied by the tensors $H_{\alpha\beta}$ and $H_{\alpha\beta,\gamma\delta}$ respectively it follows that the equations (11.2) hold for all values of the indices $\alpha, \beta, \gamma, \delta$.

The expanded form of (11.2) is

$$H_{\sigma\beta}\,\eta^{\sigma}_{\alpha}+H_{\alpha\sigma}\,\eta^{\sigma}_{\beta}=0 \tag{11.3}$$

and

$$H_{\sigma\beta,\gamma\delta}\,\eta^{\sigma}_{\alpha}+H_{\alpha\sigma,\gamma\delta}\,\eta^{\sigma}_{\beta}+H_{\alpha\beta,\sigma\delta}\,\eta^{\sigma}_{\gamma}+H_{\alpha\beta,\gamma\sigma}\,\eta^{\sigma}_{\delta}=0. \tag{11.4}$$

Let us now choose

$$H_{\alpha\beta}=\psi_{\alpha\beta}(x^1,\cdots,x^{n-1})\quad(\alpha,\beta=1,2,\cdots,n-1) \tag{11.5}$$

and

$$H_{n\alpha}=\begin{cases}0 \text{ if } \alpha\neq n,\\ 1 \text{ if } \alpha=n,\end{cases} \tag{11.6}$$

where $\psi_{\alpha\beta}$ denotes an arbitrary function of its arguments $x^1,\cdots,x^{n-1}$. With this selection of the tensor $H_{\alpha\beta}$ it results that the components

$$H_{\alpha\beta,\gamma\delta}\qquad(\alpha,\beta,\gamma,\delta=1,2,\cdots,n-1)$$

are derivable as the second extension of a tensor $H_{\alpha\beta}$ $(\alpha,\beta=1,\cdots,n-1)$ whose components are given by (11.5). It also follows that

$$\begin{aligned}&H_{n\alpha,\beta\gamma}=H_{\alpha\beta,n\gamma}=0\quad(\alpha,\beta,\gamma=1,2,\cdots,n-1),\\ &H_{nn,\beta\gamma}=H_{n\beta,n\gamma}=H_{\beta\gamma,nn}=0\qquad(\beta,\gamma=1,2,\cdots,n-1),\end{aligned} \tag{11.7}$$

when use is made of the formula (2.6). From (11.3) and (11.6) we obtain

$$\sum_{\sigma=1}^{n-1}(H_{\sigma\beta}\,\eta^{\sigma}_{\alpha}+H_{\alpha\sigma}\,\eta^{\sigma}_{\beta})=0\qquad(\alpha,\beta=1,2,\cdots,n-1) \tag{11.8}$$

and

$$\sum_{\sigma=1}^{n-1}H_{\sigma\beta}\,\eta^{\sigma}_{n}+\eta^{n}_{\beta}=0\qquad(\beta=1,2,\cdots,n-1) \tag{11.9}$$

where the latter equations are obtained by putting $\alpha=n$ in (11.3). Similarly from (11.4) and (11.7) we find that

$$\sum_{\sigma=1}^{n-1}(H_{\sigma\beta,\gamma\delta}\,\eta^{\sigma}_{\alpha}+H_{\alpha\sigma,\gamma\delta}\,\eta^{\sigma}_{\beta}+H_{\alpha\beta,\sigma\delta}\,\eta^{\sigma}_{\gamma}+H_{\alpha\beta,\gamma\sigma}\,\eta^{\sigma}_{\delta})=0 \tag{11.10}$$

$$(\alpha,\beta,\gamma,\delta=1,\cdots,n-1)$$

and

$$\sum_{\sigma=1}^{n-1}H_{\sigma\beta,\gamma\delta}\,\eta^{\sigma}_{n}=0\qquad(\beta,\gamma,\delta=1,2,\cdots,n-1). \tag{11.11}$$

Finally by putting $\alpha = \beta = n$ in (11.3) we have

$$\eta_n^n = 0 \quad \text{(not summed for } n\text{)}. \tag{11.12}$$

From what has been said we see that the equations (11.8) and (11.10) correspond to the equations (11.3) and (11.4) for the general form $G$ in $n-1$ variables. If the differential equations (11.1) are independent for a form $G$ in $n-1$ variables then the only solution of the equations (11.8) and (11.10) is

$$\eta_\alpha^i = 0 \qquad (i, \alpha = 1, 2, \cdots, n-1). \tag{11.13}$$

It is evident that in general there exists a determinant $D$ of order $n-1$ formed from the coefficients of $\eta_n^\sigma$ in (11.11) which does not vanish identically. In general then the quantities $\eta_n^\sigma$ are zero from which it follows by (11.9) that the quantities $\eta_\sigma^n$ are also zero. Under (11.13) and the assumption of the existence of a determinant $D$ it follows that all quantities $\eta_j^i = 0$ which is contrary to the hypothesis that the equations (11.1) are dependent. Finally the apparent restriction in the assumption of the existence of the determinant $D$ can be removed by the same argument used by us in our paper on affine differential invariants.* This completes the proof of Theorem (11.II).

12. **Number of independent differential equations.** To employ the theorems of the preceding paragraph to calculate the number of functionally independent absolute scalar invariants of the form $G$ we must first determine a least value of $n$ and a least value of $p$ for which the equations (10.4) are independent. When this has been done the theorems of the preceding paragraph assert that for greater values of $n$ and $p$ the equations are likewise independent.

The differential equations (11.1) for the determination of the absolute scalar invariants of order two of the form $G$ in two variables are dependent. In fact let

$$H_{11} = \alpha; \quad H_{12} = \beta; \quad H_{22} = \gamma; \quad H_{12,12} = \delta \tag{12.1}$$

be the four† independent components in terms of which all the other $H_{\alpha\beta}$ and $H_{\alpha\beta,\gamma\delta}$ can be expressed. The remaining components $H_{\alpha\beta}$ and $H_{\alpha\beta,\gamma\delta}$ which are not identically zero are given by

$$H_{21} = \beta; \quad H_{21,12} = H_{21,21} = H_{12,21} = \delta; \quad H_{11,22} = H_{22,11} = -2\delta.$$

* Loc. cit., p. 212.

† Cf. formulas (9.5) and (9.8); and also the complete sets of identies (9,1), (9.2) and (9.3).

Let us consider the invariant $I(H)$ in (11.1) as well as the quantities

$$\begin{Bmatrix} \tau \\ \sigma\, \alpha\, \beta \end{Bmatrix} \text{ and } \begin{Bmatrix} \tau \\ \sigma\, \alpha\, \beta\, \gamma\, \delta \end{Bmatrix}$$

to be expressed in terms of the independent components $H_{\alpha\beta}$ and $H_{\alpha\beta,\gamma\delta}$ in (12.1) alone. Then all the derivatives of $I(H)$ with respect to the dependent components $H_{\alpha\beta}$ and $H_{\alpha\beta,\gamma\delta}$ vanish and the equations (11.1) for the form $G$ in two variables become

$$\begin{aligned}
&2\alpha\frac{\partial I}{\partial \alpha}+\beta\frac{\partial I}{\partial \beta}+(\quad)+2\delta\frac{\partial I}{\partial \delta}=0,\\
&2\beta\frac{\partial I}{\partial \alpha}+\gamma\frac{\partial I}{\partial \beta}+(\quad)+(\quad)=0,\\
&(\quad)+\alpha\frac{\partial I}{\partial \beta}+2\beta\frac{\partial I}{\partial \gamma}+(\quad)=0,\\
&(\quad)+\beta\frac{\partial I}{\partial \beta}+2\gamma\frac{\partial I}{\partial \gamma}+2\delta\frac{\partial I}{\partial \delta}=0.
\end{aligned} \tag{12.2}$$

The determinant of the equations (12.2) vanishes identically which shows the *dependence* of these equations. However the determinant

$$\begin{vmatrix} 2\alpha & \beta & 0 \\ 2\beta & \gamma & 0 \\ 0 & \alpha & 2\beta \end{vmatrix} = 4\beta(\alpha\gamma-\beta^2)$$

does not vanish in general. Hence *the first three equations in* (12.2) *are independent for the general form G in two variables.*

If $n=3$ the independent components $H_{\alpha\beta}$ and $H_{\alpha\beta,\gamma\delta}$ may be selected as

$$\begin{aligned}
&H_{11}=\alpha;\quad H_{12}=\beta;\quad H_{13}=\gamma;\quad H_{22}=\delta;\quad H_{23}=\varepsilon;\quad H_{33}=\zeta;\\
&H_{12,12}=a;\qquad H_{12,13}=b;\qquad H_{13,13}=c;\\
&H_{13,22}=d;\qquad H_{13,23}=e;\qquad H_{23,23}=f.
\end{aligned} \tag{12.3}$$

The remaining components which do not vanish identically and which are not identical among themselves on account of (9.1), (9.4) and (9.9) are then given by

$$\begin{aligned}
&H_{21}=\beta;\qquad H_{31}=\gamma;\qquad H_{32}=\varepsilon;\\
&H_{11,22}=-2a;\qquad H_{11,23}=-2b;\qquad H_{11,33}=-2c;\\
&H_{12,23}=-2d;\qquad H_{12,33}=-2e;\qquad H_{22,33}=-2f.
\end{aligned} \tag{12.4}$$

By a method entirely similar to the one employed in getting equations (12.2) we can arrive at the final form of the equations (11.1) for $n = 3$. It will be sufficient however to write down only the matrix of these equations which is given in Table IV. The quantities $I_\alpha, \cdots, I_f$ in this table denote the partial derivatives of $I$ with respect to the components $\alpha, \cdots, f$ respectively. The rank of the matrix in Table IV is nine in general as may be seen most easily by assigning special numerical values to the components $\alpha, \beta, \cdots, f$ and then selecting a determinant of order nine from the matrix which does not vanish. Thus if we put $\beta = c = d = 1$ and take the remaining components in the set $\alpha, \cdots, f$ as zero, the determinant

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{vmatrix}$$

| $\binom{\tau}{\sigma}$ | $I_\alpha$ | $I_\beta$ | $I_\gamma$ | $I_\delta$ | $I_\varepsilon$ | $I_\zeta$ | $I_a$ | $I_b$ | $I_c$ | $I_d$ | $I_e$ | $I_f$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\binom{1}{1}$ | $2a$ | $\beta$ | $\gamma$ | 0 | 0 | 0 | $2a$ | $2b$ | $2c$ | $d$ | $e$ | 0 |
| $\binom{1}{2}$ | $2\beta$ | $\delta$ | $\varepsilon$ | 0 | 0 | 0 | 0 | $-d$ | $2e$ | 0 | $f$ | $f$ |
| $\binom{1}{3}$ | $\gamma$ | $\varepsilon$ | $\varepsilon$ | 0 | 0 | 0 | $-4d$ | $-e$ | 0 | $-2f$ | 0 | 0 |
| $\binom{2}{1}$ | 0 | $\beta$ | 0 | $\beta$ | $\gamma$ | 0 | 0 | 0 | 0 | $2b$ | $c$ | $2e$ |
| $\binom{2}{2}$ | 0 | $\beta$ | 0 | $2\delta$ | $\varepsilon$ | 0 | $2a$ | $b$ | 0 | $2d$ | $e$ | $2f$ |
| $\binom{2}{3}$ | 0 | $\gamma$ | 0 | $2\varepsilon$ | $\zeta$ | 0 | $2b$ | $c$ | 0 | $2e$ | 0 | 0 |
| $\binom{3}{1}$ | 0 | 0 | $\gamma$ | 0 | $\beta$ | $2\gamma$ | 0 | 0 | 0 | $-2a$ | $-b$ | $-4d$ |
| $\binom{3}{2}$ | 0 | 0 | $\beta$ | 0 | $\delta$ | $2\varepsilon$ | 0 | $a$ | $2b$ | 0 | $-d$ | 0 |
| $\binom{3}{3}$ | 0 | 0 | $\gamma$ | 0 | $\varepsilon$ | $2\zeta$ | 0 | $b$ | $2c$ | $d$ | $2e$ | $2f$ |

Table IV.

formed from columns 1, 2, 3, 4, 5, 7, 8, 9, 10 of the Table IV has the value $-16$. This establishes the following theorem.

THEOREM (12.I). *The differential equations* (11.1) *are independent for the general form G in three variables.*

Finally we must consider the equations (10.4) for $n = 2$ and $p = 3$, i. e., the equations

$$(12.5)\quad \binom{\tau}{\sigma\alpha\beta}\frac{\partial I}{\partial H_{\alpha\beta}} + \binom{\tau}{\sigma\alpha\beta\gamma\delta}\frac{\partial I}{\partial H_{\alpha\beta,\gamma\delta}} + \binom{\tau}{\sigma\alpha\beta\gamma\delta\varepsilon}\frac{\partial I}{\partial H_{\alpha\beta,\gamma\delta\varepsilon}} = 0$$

where the indices take on the values 1 and 2. For this case the independent components will be taken as

$$(12.6)\quad \begin{aligned} &H_{11} = \alpha; \quad H_{12} = \beta; \quad H_{22} = \gamma; \\ &H_{12,12} = \delta; \quad H_{12,122} = \varepsilon; \quad H_{12,112} = \zeta. \end{aligned}$$

Then the dependent components which are not zero and which are not equal among themselves on account of (9.1), (9.9) and (9.2) for $p = 2$ or $p = 3$ are

$$\begin{aligned} &H_{21} = \beta; \quad H_{11,22} = -2\delta; \quad H_{22,112} = -\varepsilon; \\ &H_{11,122} = -\zeta; \quad H_{22,111} = 3\varepsilon; \quad H_{11,222} = 3\zeta. \end{aligned}$$

Corresponding to the Table IV we now have the Table V.

| $\binom{\tau}{\sigma}$ | $I_\alpha$ | $I_\beta$ | $I_\gamma$ | $I_\delta$ | $I_\varepsilon$ | $I_\zeta$ |
|---|---|---|---|---|---|---|
| $\binom{1}{1}$ | $2\alpha$ | $\beta$ | 0 | $2\delta$ | $2\varepsilon$ | $2\zeta$ |
| $\binom{1}{2}$ | $2\beta$ | $\gamma$ | 0 | 0 | 0 | $\varepsilon$ |
| $\binom{2}{1}$ | 0 | $\alpha$ | $2\beta$ | 0 | $\zeta$ | 0 |
| $\binom{2}{2}$ | 0 | $\beta$ | $2\gamma$ | $2\delta$ | $2\varepsilon$ | $2\zeta$ |

Table V.

The rank of the matrix in Table V is four in general since for example the determinant formed from columns 1, 2, 3, 5 does not vanish identically. Thus we arrive at the following theorem.

THEOREM (12.II). *The differential equations* (12.5) *are independent for the general form G in two variables.*

On the basis of Theorems (11.I), (11.II), (12.I) and (12.II) we have the following general result.

THEOREM (12.III). *For the general form $G$ in $n$ variables there are $n^2$ independent differential equations* (10.4) *if $n \geqq 2$, $p \geqq 3$ or if $n \geqq 3$, $p \geqq 2$. If $n = 2$, $p = 2$ there are three independent equations* (10.4).

**13. Fundamental sets of absolute scalar invariants.** We are now in a position to give the number of functionally independent scalar invariants of all orders. In this connection we shall define two sets of differential invariants which will be called the *first and second fundamental sets of differential invariants* respectively.

*Definition.* A first fundamental set of scalar differential invariants of order $p$ is a set of functionally independent scalar differential invariants of the form $G$ *each of which is of order* $p$ such that any other scalar differential invariant of $G$ of order $p$ can be expressed as a function of the invariants of the set.

The actual existence of a first fundamental set of scalar invariants can easily be inferred. Let us denote by $\mathfrak{N}(n, p)$ the number of functionally independent solutions of the complete system of differential equations (10.4). It follows that the number $\mathfrak{N}(n, p)$ is likewise the number of invariants in a first fundamental set of invariants of order $p$. The detailed proof of this statement is similar to that given by us for the case of affine differential invariants.*

By Theorem (12.III) and the definition of the quantities $N(n, p)$ in § 9, the number $\mathfrak{N}(n, p)$ is given by

$$\mathfrak{N}(n, p) = \sum_{\alpha=0}^{p} N(n, \alpha) - n^2 \tag{13.1}$$

with the exception of the following combinations:

$$n = n,\ p = 0;\quad n = n,\ p = 1;\quad n = 2,\ p = 2. \tag{13.2}$$

For the exceptional cases (13.2) we have

$$\mathfrak{N}(n, 0) = 0;\quad \mathfrak{N}(n, 1) = 0;\quad \mathfrak{N}(2, 2) = 1. \tag{13.3}$$

In considering the algebraic characterization of the form $G$ by means of its absolute scalar differential invariants it seems somewhat desirable to us to make use of another fundamental set of scalar invariants which possesses the property that the fundamental set of invariants of order $p$ is contained in the fundamental set of invariants of order $p+1$.

* Loc. cit., p. 217.

*Definition.* A second fundamental set of scalar invariants of order two is a fundamental set of solutions* of the differential equations (11.1).

*Definition.* A second fundamental set of scalar invarians of order $p>2$ is a fundamental set of solutions of the differential equations (10.4) such that a subset of these solutions considered as a set of scalar invariants constitute a second fundamental set of scalar invariants of order $p-1$.

Since there are no absolute scalar invariants of order zero or one it follows from the above definition that a second fundamental set of scalar invariants of order two is likewise a first fundamental set of scalar invariants of order two and vice versa. However a first and second fundamental set of scalar invariants of order $p>2$ can never be identical.

It is of course evident that $\mathfrak{N}(n, p)$ as defined by (13.1) and (13.3) also gives the number of invariants in a second fundamental set of invariants.

**14. Completeness and finiteness of absolute scalar invariants.** It was seen in § 8 that the tensor $H_{\alpha\beta}$ and its extensions $H_{\alpha\beta,\gamma_1\cdots\gamma_r}$ $(r = 2, \cdots, N+2)$ give a complete characterization of the form $G$ or in other words that these tensors constitute a complete set of invariants of the form $G$. We shall now show that a complete set of invariants of the form $G$ can also be given by a finite number of absolute scalar invariants.

Let

$$I_i(H) \qquad (i = 1, 2, \cdots, \mathfrak{N}(n, p);\ n \geqq 2,\ p \geqq 2) \tag{14.1}$$

be a second fundamental set of absolute scalar differential invariants of order $p$ for the form $G$ in $n$ variables. Then form the relations

$$I_i(\bar{H}) = \underline{I}_i(H) \qquad (i = 1, 2, \cdots, \mathfrak{N}(n, p);\ n \geqq 2,\ p \geqq 2) \tag{14.2}$$

where the $I_i(\bar{H})$ are the invariants of the form $\bar{G}$ which correspond respectively to the invariants $I_i(H)$ of the form $G$. It is possible to solve the equations (14.2) for $\mathfrak{N}(n, p)$ of the quantities $(\bar{H})$. We shall denote the solved form of the equations (14.2) by

$$\{\bar{H}\} = \Phi(H,\ R(\bar{H})) \tag{14.3}$$

where $\{\bar{H}\}$ stands for a set of $\mathfrak{N}(n, p)$ independent components of the tensors $(\bar{H})$ in (14.2) and $R(\bar{H})$ is a set of components $(\bar{H})$ in (14.2) not

* By a fundamental set of solutions of a system of differential equations we understand a set of functionally independent solutions such that any solution of these equations is a function of the members of this set.

contained in $\{\overline{H}\}$. Now let $r(\overline{H})$ denote the set of independent components of the tensors $(\overline{H})$ which are not in the set $\{\overline{H}\}$ and form the algebraic equations which are derived from the tensor equations of transformation by which the components $(H)$ are transformed into the components of the set $r(\overline{H})$ by putting $u_\alpha^i$ in place of the derivatives $\partial x^i/\partial \overline{x}^\alpha$. We shall represent these equations by

$$r(\overline{H}) = \varphi(H, u). \tag{14.4}$$

All the $n^2$ quantities $u_\alpha^i$ must occur in the equations (14.4). This is seen by the following consideration. If we eliminate the members of the set $R(\overline{H})$ from (14.3) by means of (14.4) we shall have a set of equations which together with (14.4) constitutes a set of relations between all of the components $(\overline{H})$ of the tensors $H_{\alpha\beta}$ and $H_{\alpha\beta,\gamma_1\cdots\gamma_r}$ $(r = 2, \cdots, p)$, the components $(H)$ in (14.3) and (14.4), and the quantities $u_\alpha^i$ in (14.4). If all the $n^2$ quantities $u_\alpha^i$ did not occur in (14.4) we should therefore have a relation between the above quantities $(\overline{H})$, $(H)$ and $u_\alpha^i$ which is not identical with the algebraic form of the tensor relation in virtue of the complete sets of identities that the components $(H)$ satisfy, whereas in fact the tensor relationship is the only one which can exist in general between them.

With the exception of the case $n = 2$, $p = 2$ there are $n^2$ components in the set $r(\overline{H})$ and we can therefore solve the equations (14.4) uniquely (in a sufficiently small neighborhood) for the $n^2$ quantities $u_\alpha^i$. Thus

$$u_\alpha^i = \psi_\alpha^i(H, r(\overline{H})). \tag{14.5}$$

In case $n = 2$, $p = 2$ there is one relation (14.2) and three equations (14.4). Hence one of the four quantities $u_\alpha^i$ can be taken as arbitrary after which the remaining three quantities $u_\alpha^i$ are uniquely determined by (14.4). Thus we may write

$$u_\alpha^i = \psi_\alpha^i(H, r(\overline{H}), w) \qquad (n = 2, p = 2) \tag{14.6}$$

where $w$ in the right member of these equations denotes any one of the quantities $u_\alpha^i$.

After these preliminary remarks we shall proceed directly to the proof of the following theorem.

THEOREM (14.I). *A necessary and sufficient condition that two quadratic differential forms $G$ and $\overline{G}$ be transformable into one another, i. e., that there exist a transformation*

$$\mathfrak{G}: \quad x^i = f^i(\bar{x}^1, \cdots, \bar{x}^n) \tag{14.7}$$

*which satisfies the equation*

$$\bar{G} = (x\bar{x})^x G, \tag{14.8}$$

*is that an integer* $\omega \geqq 2$ *exist such that the equations* (14.2) for $p = \omega$ *are compatible in the variables* $x^i$ *as functions of the independent variables* $\bar{x}^i$ *and that all their solutions*

$$x^i = \varphi^i(\bar{x}^1, \cdots, \bar{x}^n) \tag{14.9}$$

*satisfy the equations* (14.2) *for* $p = \omega + 1$.

By the integer $\omega$ we shall mean the *least* value that this integer can assume.

As in the Theorem (8.I) the necessity part of the above theorem is immediate. To prove the sufficiency part of this theorem let us consider first of all that (14.9) denotes any particular solution of the equations (14.2) for $p = \omega$ which satisfies the equations (14.2) for $p = \omega + 1$. Then the equations (14.9) and (14.5) determine the quantities

$$u^i_\alpha = \theta^i_\alpha(\bar{x}^1, \cdots, \bar{x}^n) \tag{14.10}$$

uniquely as functions of the variables $\bar{x}^i$. In case $n = 2$, $p = 2$ the equations (14.9) and (14.6) give a determination of the quantities $u^i_\alpha$ which may likewise be represented by (14.10), but this determination is not unique owing to the presence of the parameter $w$ which can be selected as an arbitrary function of the variables $\bar{x}^i$. Since the equations (14.3) and (14.4) for $p = \omega$ are equivalent to the equations

$$\begin{aligned}
\bar{H}_{\alpha\beta} &= H_{\lambda\mu}\, u^\lambda_\alpha\, u^\mu_\beta, \\
\bar{H}_{\alpha\beta,\gamma_1\gamma_2} &= H_{\lambda\mu,\nu_1\nu_2}\, u^\lambda_\alpha\, u^\mu_\beta\, u^{\nu_1}_{\gamma_1}\, u^{\nu_2}_{\gamma_2}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\bar{H}_{\alpha\beta,\gamma_1\cdots\gamma_\omega} &= H_{\lambda\mu,\nu_1\cdots\nu_\omega}\, u^\lambda_\alpha\, u^\mu_\beta\, u^{\nu_1}_{\gamma_1} \cdots u^{\nu_\omega}_{\gamma_\omega}
\end{aligned} \tag{14.11}$$

it follows that (14.9) and (14.10) give a solution of the above equations (14.11).

It is fairly evident that (14.9) and (14.10) as defined above can be *any* solution of the equations (14.11), for, let

$$x^i = \lambda^i(\bar{x}^1, \cdots, \bar{x}^n) \tag{14.12}$$

and

$$w_j^i = \mu_j^i(\overline{x}^1, \cdots, \overline{x}^n) \tag{14.13}$$

denote any solution of the equations (14.11). Inasmuch as the equations (14.2) for $p = \omega$ must be a consequence of (14.11), they must be satisfied by (14.12) which can therefore be selected as the solution (14.9). In general (i. e., $n \neq 2$ if $p = 2$) the functions $\theta_\alpha^i$ in (14.10) are uniquely determined by (14.4) and (14.9) and hence must be identical with the functions $\mu_\alpha^i$ in (14.13). For the case $n = p = 2$ the functions $\psi_\alpha^i$ in (14.6) can evidently be made identical with the functions $\mu_\alpha^i$ by a proper choice of the parameter $w$.

We wish to show now that (14.9) and (14.10) will likewise satisfy the set of equations

$$\overline{H}_{\alpha\beta,\gamma_1\cdots\gamma_{\omega+1}} = H_{\lambda\mu,\nu_1\cdots\nu_{\omega+1}} u_\alpha^\lambda u_\beta^\mu u_{\gamma_1}^{\nu_1} \cdots u_{\gamma_{\omega+1}}^{\nu_{\omega+1}} \tag{14.14}$$

for when this has been done the sufficiency part of Theorem (14.I) will then follow by Theorem (8.I). If the components $(H)$ of the tensors

$$H_{\alpha\beta} \text{ and } H_{\alpha\beta,\gamma_1\cdots\gamma_r} \qquad (r = 2, \cdots, p) \tag{14.15}$$

for $p = \omega + 1$ are given and likewise the components $(\overline{H})$ of the set $r(\overline{H})$, corresponding to $p = \omega$ then the determination (14.10) of the functions $u_\alpha^i$ by (14.4) is the same as though the components (14.15) are given only for $p = \omega$. That the above set $r(\overline{H})$ corresponding to $p = \omega$ can be selected as a set $r(\overline{H})$ corresponding to $p = \omega+1$ in general (i. e., $n \neq 2$, if $\omega = 2$) is seen from the fact that if the components $(H)$ in (14.15) for $p = \omega+1$ and the components of the set $r(\overline{H})$ for $p = \omega$ are given the remaining components $(\overline{H})$ corresponding to the given components $(H)$ are uniquely determined. Hence by the argument that has already been used we have in general (i. e. $n \neq 2$, if $\omega = 2$) that the equations (14.2) for $p = \omega+1$ and the equations (14.4) are equivalent to the equations (14.11) and (14.14) from which it follows that (14.9) and (14.10) satisfy (14.14). If $n = 2$ and $\omega = 2$ we consider the components $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}$ given by (12.6) as independent parameters in the equations (14.2) for $n = 2$, $p = 3$. We can then solve (14.2) uniquely for the remaining components $\overline{\varepsilon}, \overline{\zeta}$ in (12.6) obtaining

$$\overline{\varepsilon} = P(H; \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}); \quad \overline{\zeta} = Q(H; \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}). \tag{14.16}$$

However we have seen that the equations (14.2) for $n = 2$, $p = 2$ and (14.4) are equivalent to the tensor equations of transformation of the

components $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$. Using these latter equations, to eliminate the quantities $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ in (14.16) we obtain the tensor equations of transformation of the components $\bar{\varepsilon}$, $\bar{\zeta}$ which are equivalent to (14.14) for $n = 2$, $\omega = 2$. Hence (14.9) and (14.10) likewise satisfy (14.14) for the exceptional case $n = 2$, $\omega = 2$. As we have already observed this completes the proof of Theorem (14.I).

It is clear that the $\omega - 1$ sets of equations (14.2) which are obtained by allowing $p$ in (14.2) to take on the $\omega - 1$ values $2, \cdots, \omega$ successively, are such that each set of equations after the first contains at least one more independent equation in the variables $\bar{x}^i$ than the sets of equations which immediately precede it. Hence the inequality

$$2 \leqq \omega \leqq n+1 \tag{14.17}$$

is satisfied by the integer $\omega$.

In the above characterization of the form $G$ we can of course replace the second fundamental set of invariants (14.1) for $p = \omega$ by a set of functionally independent invariants $J_i(x)$ depending on the variables $x^i$ such that every invariant (14.1) for $p = \omega$ considered as a function of the variables $x^i$ is functionally dependent on the invariants $J_i(x)$. If the invariants $J_i(x)$ are $\theta$ in number it is evident that the inequality

$$\omega - 1 \leqq \theta \leqq n \tag{14.18}$$

holds.

15. **Certain arithmetical invariants of the form $G$.** One of the most obvious arithmetical invariants of the form $G$ is the number $n$ of different values that can be taken on by the indices $\alpha$ and $\beta$ in this form. Less obvious perhaps is the invariant nature of the integer $N$ which was proved in § 8. On the basis of the invariance of the integer $N$ the invariance of the integer $\omega$ defined in the preceding paragraph follows immediately. In the proof of Theorem (14.I) we saw that if all solutions (14.9) of (14.2) for $p = \omega$ satisfy (14.2) for $p = \omega + 1$ then all solutions (14.12) and (14.13) of the equations (14.11) satisfy (14.14) or in other words that all solutions (14.14) and (14.13) of the first $\omega$ sets of equations (4.1) satisfy the $(\omega+1)$st set of these equations. This shows that the integer $N$ determined by the sequence (4.1) on the assumption that the two forms $G$ and $\bar{G}$ are equivalent is less than or equal to $\omega - 1$. If however, $N$ is less than $\omega - 1$ where $\omega > 2$ then there must exist an integer $L$ less than $\omega$ such that all solutions (14.9) of the equations (14.2) for $p = L$ satisfy (14.2) for $p = L+1$ which is contrary to our hypothesis. In case $\omega = 2$, $N$ must of course have the value 1. Hence in all cases we have

$$N = \omega - 1. \tag{15.1}$$

Owing to the invariance of $N$ it follows immediately from (15.1) that $\omega$ *is an arithmetical invariant of the form* $G$.*

It is evident without further comment that the *integer* $\theta$ defined at the end of § 14 is also an *arithmetical invariant* of the form $G$.

On account of the invariance of the integer $\omega$ we may state the following theorem.

THEOREM (15.I). *Any second fundamental set of absolute scalar invariants* $I_i(H)$ *of order* $\omega+1$ *of the form* $G$ *constitutes a complete set of invariants of the form* $G$.

Another form of statement of Theorem (15.I) is as we know that this set of invariants $I_i(H)$ of order $\omega+1$ suffices for the complete algebraic characterization of the form $G$.†

16. **Special forms** $G$. It may happen that in addition to the complete set of identities given in § 9 the $H$-invariants satisfy certain relations which however, are not identities in the components $H_{\alpha\beta}$ and their derivatives but in the variables $x^i$. For example we may have

$$H^{\alpha\beta} H_{ij,\alpha\beta} = 0 \tag{16.1}$$

or its equivalent

$$B^{\alpha}_{ij\alpha} = 0 \tag{16.2}$$

which is the form of the Einstein equations of the gravitation field in free space when the cosmological term is neglected. If equations of the sort (16.1) are taken into account this will decrease the number of independent components of the $H$-invariants and it is obvious that in general this will result in a corresponding decrease in the number of functionally independent scalar invariants $I(H)$. However, it is always permissible to speak of the number of functionally independent scalar invariants $I(H)$ when only the identities of the complete set of identities in § 9 are taken into account and with this understanding the discussion and conclusion in any preceding paragraph involving the number of functionally independent scalar invariants $I(H)$ continue to hold even in the case of these special forms $G$. As a matter of fact the scalar invariants $I(H)$ are primarily functions of the $x^i$ variables and from this standpoint the general nature of the

* The invariance of the integer $\omega$ can obviously be shown directly on the basis of the absolute scalar invariants but we have wished to establish the relationship (15.1).

† In the affine case (loc. cit., § 18) the invariance of the corresponding integers $\omega\,(1 \leqq \omega \leqq n)$ and $\theta\,(0 \leqq \theta \leqq n)$ can be established in a similar manner. For this case the relationship $N = \omega$ holds which cuts down the upper bound of the invariant $N$ established in our affine paper.

dependence of the invariants $I$ on the $H$-components is not of so much importance.

For the case of any particular form $G$ it is possible that one of the scalar invariants $I$ of the complete set of invariants may become indefinitely large (i. e., $\infty$) or indeterminate when expressed in terms of the $x^i$ variables. Any difficulty arising by the invariant becoming infinite can be removed immediately by replacing it by its reciprocal in the invariants of the complete set. Then the invariant in question will vanish identically when expressed as a function of the $x^i$ variables; under this circumstance it is necessary that the corresponding invariant of the form $G$ likewise vanishes identically in the algebraic test of equivalence of the two forms in § 14, if these two forms are to be equivalent since otherwise equations (14.2) would be incompatible in the sense that they would furnish a functional condition on the set of variables $\overline{x}^i$. Should the invariant $I$ in the complete set of scalar invariants become indeterminate it is evident that the corresponding equation (14.2) will furnish no condition in the algebraic test of equivalence of the two forms. Such special behavior of the invariants of the two forms $G$ or $\overline{G}$ in no way interferes with their algebraic test of equivalence of § 14. This test is perfectly general and applies to all forms $G$.

An example of the case mentioned above is given by the form $G$ with constant coefficients $G_{\alpha\beta}$ or what amounts to the same thing by a form which can be reduced by a transformation $\mathfrak{G}$ to a form with constant coefficients $G_{\alpha\beta}$. Here it is seen that all scalar invariants $I$ of the form will either become zero, infinite, or indeterminate when we replace the $H$-components in the general form of the invariant $I(H)$ by their values as functions of the variables $x^i$. Moreover this behavior of the invariants $I(H)$, i. e., whether they become zero, infinite, or indeterminate must occur in a perfectly definite way depending on the invariant $I$ in question. Thus the invariant

$$H^{\alpha\beta}\, H^{\gamma\delta}\, H_{\alpha\beta,\,\gamma\delta} \tag{16.3}$$

must *always* become zero and never infinite or indeterminate. It is natural to say that all scalar invariants $I$ of the form $G$ *vanish identically* in case the form is reducible to one with constant coefficients $G_{\alpha\beta}$, and we may extend the significance of the phrase *vanish identically* to mean that the invariants become zero, infinite, or indeterminate in the perfectly definite manner described above. Then we can state the following theorem.

THEOREM (16.I). *A necessary and sufficient condition that a relative quadratic differential form $G$ be reducible to a form with constant coefficients $G_{\alpha\beta}$ is that all scalar invariants $I$ of a second fundamental set of order three vanish identically.*

The justification of this theorem is immediate on the basis of the algebraic test of equivalence of two forms $G$ and $\bar{G}$ given in § 14.

For the particular case of the form $G$ in two variables a sufficient condition that the form be reducible to a form with constant coefficients $G_{\alpha\beta}$ is that the single invariant (16.3) vanishes identically since this implies the vanishing of $H_{\alpha\beta,\gamma\delta}$.

17. **Differential equations of relative tensor invariants.** Although the absolute scalar differential invariants are sufficient for the characterization of the form $G$ it is not without interest to consider the differential equations satisfied by a relative scalar differential invariant or more generally by a relative tensor differential invariant.

From Theorem (3.I) and from the law of transformation of a relative scalar it follows that a necessary and sufficient condition that $J$ be a relative scalar differential invariant of order $p$ of the form $G$ is that

$$(17.1)\quad J(\bar{H}_{\alpha\beta};\bar{H}_{\alpha\beta,\gamma_1\gamma_2};\cdots;\bar{H}_{\alpha\beta,\gamma_1\cdots\gamma_p})\,(\bar{x}\,x)^{\chi} = J(H_{\alpha\beta};H_{\alpha\beta,\gamma_1\gamma_2};\cdots;H_{\alpha\beta,\gamma_1\cdots\gamma_p})$$

where

$$(\bar{x}\,x) = \left|\frac{\partial\,\bar{x}^i}{\partial\,x^k}\right|$$

and $k$ is the weight of the invariant $J(H)$. A necessary and sufficient condition that (17.1) hold is that

$$(17.2)\quad \left\{\frac{d\,[J(\bar{H})\,(\bar{x}\,x)^{\chi}]}{d\,\varepsilon}\right\}_{\varepsilon=0} = 0$$

under the family of transformations (10.2). Treating the equations (17.2) exactly as we have treated the equations (10.3) we find that a necessary and sufficient condition that the function $J(H)$ be a relative scalar invariant of order $p$ ($\geqq 0$) and weight $k$ is that it satisfies the differential equations

$$(17.3)\quad X^{\tau}_{\sigma}(p)\,J = k\,\delta^{\tau}_{\sigma}J$$

where the left hand member is defined by (10.8). In the determination of the equations (17.3) the only equation required in addition to those of § 10 is the equation

$$(17.4)\quad \left[\frac{d\,(\bar{x}\,x)}{d\,\varepsilon}\right]_{\varepsilon=0} = -\sum_{\alpha=1}^{n}\frac{\partial\,\xi^{\alpha}(x)}{\partial\,x^{\alpha}}$$

which is readily verified.

From the form of the equations (17.3), the most general relative scalar invariant $J$ of order $p$ and weight $k$ is given by

$$J = J_0 F(I_1, I_2, \cdots, I_{\mathfrak{R}(n,p)})$$

where $J_0$ is a particular invariant of weight $k$ and order $r (\leqq p)$ and $F$ is the most general absolute scalar invariant of order $p$. It is readily seen that we may take

$$J_0 = |H_{ij}|^{k/2}$$

Hence it follows that the most general scalar invariant of weight $k$ and order zero is

$$c\,|H_{ij}|^{k/2}$$

where $c$ is an arbitrary constant.

In a similar way the differential equations of the general relative tensor can be established.

THEOREM (17.I). *A necessary and sufficient condition that the set of functions $T^{ij\cdots s}_{\alpha\beta\cdots\lambda}(H)$ constitutes a relative tensor differential invariant of order $p$ and weight $k$ of the form $G$, i. e., that the functions $T(H)$ transform by the equations*

$$T(\bar{H})^{ij\cdots s}_{\alpha\beta\cdots\lambda} \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\cdots\frac{\partial x^t}{\partial \bar{x}^s}$$

(17.5)

$$= (x\bar{x})^k T^{pq\cdots t}_{\mu\nu\cdots\varphi}(H)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\cdots\frac{\partial x^\varphi}{\partial \bar{x}^\lambda}$$

*under the group of transformations $\mathfrak{G}$, is that the system of differential equations*

$$(17.6)\quad \bar{X}^\tau_\sigma(p)\, T^{ij\cdots s}_{\alpha\beta\cdots\lambda} = k\delta^\tau_\sigma T^{ij\cdots s}_{\alpha\beta\cdots\lambda} + \delta^\tau_\alpha T^{ij\cdots s}_{\sigma\beta\cdots\lambda} + \cdots + \delta^\tau_\lambda T^{ij\cdots s}_{\alpha\beta\cdots\sigma} - \delta^i_\sigma T^{\tau j\cdots s}_{\alpha\beta\cdots\lambda} - \cdots - \delta^s_\sigma T^{ij\cdots\tau}_{\alpha\beta\cdots\lambda}$$

*be satisfied.*

18. **A complete set of absolute scalar invariants of the binary form $G$.** As an example of the determination of absolute scalar invariants we shall construct a complete set of invariants of the form $G$ in two variables. By Theorem (15.I) and the inequality (14.17) a second fundamental set of absolute scalar invariants of this form of order four at most constitutes a complete set of absolute scalar invariants.

From § 13 the numbers of functionally independent scalar invariants in fundamental sets of orders 0, 1, 2, 3 and 4 are given by

$$\mathfrak{R}(2,0) = 0;\quad \mathfrak{R}(2,1) = 0;\quad \mathfrak{R}(2,2) = 1;\quad \mathfrak{R}(2,3) = 2;\quad \mathfrak{R}(2,4) = 5$$

respectively. It follows that a second fundamental set of scalar invariants of order four consists of one invariant of order two, one invariant of order three, and three invariants of order four. With this in mind let us construct the following scalar invariants:

$$I_1 = H^{\alpha\beta} H^{\gamma\delta} H_{\alpha\beta,\gamma\delta} \qquad \text{(indices} = 1, 2),$$

$$I_2 = H^{\alpha\beta} M_\alpha M_\beta; \qquad M_\alpha = H^{ij} H^{kl} H_{ij,kl\alpha} \qquad \text{(indices} = 1, 2),$$

$$I_3 = H^{\alpha\beta} M_{\alpha\beta}; \qquad M_{\alpha\beta} = H^{ij} H^{kl} H_{ij,kl\alpha\beta} \qquad \text{(indices} = 1, 2),$$

$$I_4 = M_\alpha M_\beta M^{\alpha\beta} \qquad \text{(indices} = 1, 2),$$

$$I_5 = M^{\alpha\beta} M^{\gamma\delta} H_{\alpha\beta,\gamma\delta} \qquad \text{(indices} = 1, 2),$$

where $M^{\alpha\beta}$ as usual is used to denote the cofactor of $M_{\alpha\beta}$ in the determinant $|M_{\alpha\beta}|$ divided by $|M_{\alpha\beta}|$. While it is fairly evident that the above invariants $I_1, \cdots, I_5$ are functionally independent, this can be shown directly by expressing them in terms of their independent components and then showing that by a proper choice of these components the invariants themselves can take on arbitrary values. *The invariants $I_1, \cdots, I_5$ therefore form a complete set of invariants for the form $G$ in two variables.* For a *special* form $G$ in two variables it may happen that the invariants $I_1$ and $I_2$ alone furnish a complete set of invariants (see § 16).

19. **Differential equations of parameters.** Consider an absolute scalar differential parameter $L$ of order $(p, q)$ involving $w$ arbitrary scalar functions $F^{(k)}$ (see § 3). By the replacement theorem (3.II) such a parameter $L$ can be given the form

$$L(H_{\alpha\beta};\ H_{\alpha\beta,\gamma_1\gamma_2};\ \cdots;\ H_{\alpha\beta,\gamma_1\cdots\gamma_p};\ F^{(k)};\ F^{(k)}_{\alpha_1};\ \cdots;\ F^{(k)}_{\alpha_1\cdots\alpha_q})$$

where $H_{\alpha\beta,\gamma_1\cdots\gamma_r}$ $(2 \leqq r \leqq p)$ and $F^{(k)}_{\alpha_1\cdots\alpha_s}$ $(1 \leqq s \leqq q)$ are the extensions of $H_{\alpha\beta}$ and $F^{(k)}$ respectively. The condition that the function $L$ be an absolute scalar parameter is that

$$\left[\frac{dL(\bar{H}, \bar{F})}{d\varepsilon}\right]_{\varepsilon=0} = 0 \tag{19.1}$$

under the one-parameter family of transformations (10.2). Since

$$\left[\frac{d\bar{F}^{(k)}(\bar{x})}{d\varepsilon}\right]_{\varepsilon=} = 0 \qquad (k = 1, 2, \cdots, w) \tag{19.2}$$

and

$$\left[\frac{d\bar{F}^{(k)}_{\alpha\beta\cdots\nu}}{d\varepsilon}\right]_{\varepsilon=0} = \begin{bmatrix} \tau\,(k) \\ \sigma\,\alpha\,\beta\,\cdots\,\nu \end{bmatrix} \frac{\partial \xi^\sigma}{\partial x^\tau} \tag{19.3}$$

where*

$$\begin{bmatrix} \tau\,(k) \\ \sigma\,\alpha\,\beta\,\ldots\,\nu \end{bmatrix} = \delta^{\tau}_{\alpha} F^{(k)}_{\sigma\beta\ldots\nu} + \delta^{\tau}_{\beta} F^{(k)}_{\alpha\sigma\ldots\nu} + \cdots + \delta^{\tau}_{\nu} F^{(k)}_{\alpha\beta\ldots\sigma}, \tag{19.4}$$

we see that the conditions (19.1) give

$$\sum_{\omega=2}^{p+2} \begin{pmatrix} \tau \\ \sigma\,\gamma_1\gamma_2\gamma_3\cdots\gamma_\omega \end{pmatrix} \frac{\partial L}{\partial H_{\gamma_1\gamma_2,\gamma_3\cdots\gamma_\omega}} + \sum_{k=1}^{w}\sum_{s=1}^{q} \begin{bmatrix} \tau\,(k) \\ \sigma\,\alpha_1\,\ldots\,\alpha_s \end{bmatrix} \frac{\partial L}{\partial F^{(k)}_{\alpha_1\cdots\alpha_s}} = 0 \tag{19.5}$$

(see § 10). The completeness of the equations (19.5) is readily deducible on the basis of the work of § 10. Let us write the equations (19.5) as

$$U^{\tau}_{\sigma}(p,\ w,\ q)\,L = X^{\tau}_{\sigma}(p)\,L + \sum_{k=1}^{w} W^{\tau}_{\sigma}(q,\ k)\,L = 0 \tag{19.6}$$

where

$$X^{\tau}_{\sigma}(p)\,L = \sum_{\omega=2}^{p+2} \begin{pmatrix} \tau \\ \sigma\,\gamma_1\gamma_2\cdots\gamma_\omega \end{pmatrix} \frac{\partial L}{\partial H_{\gamma_1\gamma_2,\cdots\gamma_\omega}} \tag{19.7}$$

and

$$W^{\tau}_{\sigma}(q,\ k)\,L = \sum_{s=1}^{q} \begin{bmatrix} \tau\,(k) \\ \sigma\,\alpha_1\,\ldots\,\alpha_s \end{bmatrix} \frac{\partial L}{\partial F^{(k)}_{\alpha_1\cdots\alpha_s}} \qquad (k \text{ not summed}). \tag{19.8}$$

We have

$$\begin{aligned} &(U^{\tau}_{\sigma}(p,\ w,\ q),\ U^{\lambda}_{\mu}(p,\ w,\ q))\,L \\ &\quad = (X^{\tau}_{\sigma}(p),\ X^{\lambda}_{\mu}(p))\,L + \sum_{k=1}^{w} (W^{\tau}_{\sigma}(q,\ k),\ W^{\lambda}_{\mu}(q,\ k))\,L. \end{aligned} \tag{19.9}$$

Since each term in the summation $\sum$ in the right hand member of (19.9) has the same form as the first term in the right hand member, it follows by (10.11) that

$$\begin{aligned} &(U^{\tau}_{\sigma}(p,\ w,\ q),\ U^{\lambda}_{\mu}(p,\ w,\ q))\,L \\ &\quad = \delta^{\tau}_{\mu} U^{\lambda}_{\sigma}(p,\ w,\ q)\,L - \delta^{\lambda}_{\sigma} U^{\tau}_{\mu}(p,\ w,\ q)\,L. \end{aligned} \tag{19.10}$$

Corresponding to Theorem (10.I) we can now state the following theorem.

THEOREM (19.I). *A necessary and sufficient condition that a function* $L(H, F^k)$ *be an absolute scalar differential parameter of order* $(p, q)$ *of*

* Loc. cit., p. 230.

*the form $G$ is that it satisfy the complete system of partial differential equations* (19.5).

We must next consider the independence of the differential equations (19.5). It is obvious that the following theorem is true from the form of these equations.

THEOREM (19.II). *If the $n^2$ differential equations* (19.5) *are independent for a particular $(n, p, q, w)$ they are also independent for $(n, p+1, q, w)$, $(n, p, q+1, w)$, and $(n, p, q, w+1)$.*

From the results already established in § 12 and contained in Theorem (12.III) we can say immediately that the equations (19.5) are independent if $n = 2$, $p \geqq 3$ or $n > 2$, $p \geqq 2$. It is found that we must still consider the following three particular cases of the equations (19.5) which we give together with the corresponding number of independent equations:

(a) $n \geqq 2$, $p = 0$, $q = 1$, $w \leqq n-1$ ($S$ independent equations),
(b) $n \geqq 2$, $p = 0$, $q = 1$, $w \geqq n-1$ ($n^2$ independent equations),
(c) $n \geqq 2$, $p = 0$, $q = 2$, $w = 1$ ($n^2$ independent equations)

where

$$S = \frac{n(n+1)}{2} + nw - \frac{w(w+1)}{2}. \tag{19.11}$$

When these results have been proved it will then follow by Theorem (19.II) and the above statement regarding the independence of the equations (19.5) on the basis of Theorem (12.III) that *except for the above case* (a) *the $n^2$ equations* (19.5) *are always independent.* However, we note that the equations for case (a) are all independent when $w = n-1$ since then $S = n^2$ as in fact demanded by case (b). The three particular cases (a), (b) and (c) will now be considered in detail.

*Case* (a). The equations (19.5) become

$$(\delta_\alpha^\tau H_{\sigma\beta} + \delta_\beta^\tau H_{\alpha\sigma}) \frac{\partial L}{\partial H_{\alpha\beta}} + \sum_{k=1}^{w} \delta_\alpha^\tau F_\sigma^{(k)} \frac{\partial L}{\partial F_\alpha^{(k)}} = 0. \tag{19.12}$$

Assume

$$H_{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha = \beta, \\ 0 \text{ if } \alpha \neq \beta \end{cases} \tag{19.13}$$

and

$$F_\alpha^{(k)} = \begin{cases} 1 & \text{if } k = \alpha, \\ 0 & \text{if } k \neq \alpha \end{cases} \qquad \begin{matrix} (k = 1, \cdots, w) \\ (k = 1, \cdots, w) \end{matrix} \tag{19.14}$$

and construct the square matrix from the coefficients of the derivatives in the equations (19.12) which is contained in Table VI. In this table the

element in the row headed by $(\tau, \sigma)$ and column headed by $\binom{\alpha}{\beta}$ or $\begin{bmatrix} k \\ \alpha \end{bmatrix}$ gives the coefficient of the derivative $\frac{\partial L}{\partial H_{\alpha\beta}}$ or $\frac{\partial L}{\partial F_\alpha^{(k)}}$ respectively in the equations (19.12). The blank sections above the square diagonal sections in this table are composed entirely of zero elements, while the elements in the sections below the diagonal sections have been left undetermined. The determinant of the matrix in the Table VI which is of order $S$ given by (19.11) has the value $2^n$. Consequently there are $S$ independent equations (19.12) *at least.* In other words there are *at most*

$$\frac{n(n+1)}{2} + w + nw - S = \frac{w(w+3)}{2}$$

independent solutions of the equations (19.12). But the $w(w+3)/2$ functions

$$Q^{(ij)} = H^{\sigma\tau} F_\sigma^{(i)} F_\tau^{(j)}; \qquad F^{(i)}$$

are functionally independent as can be seen quite readily by letting $F_\alpha^{(k)}$ have the particular values given by (19.14) in consequence of which the $Q^{(ij)}$ go over into the $w(w+1)/2$ independent quantities $H^{ij}$. Hence $S$ gives the actual number of independent equations (19.12).

*Case* (b). Since $S = n^3$ for case (*a*) when $w = n-1$, i. e., all equations (19.12) are independent, it follows by Theorem (19.II) that all equations (19.5) are independent for case (b).

*Case* (c). For this case the equations (19.5) become:

$$(\delta_\alpha^\tau H_{\sigma\beta} + \delta_\beta^\tau H_{\alpha\sigma}) \frac{\partial L}{\partial H_{\alpha\beta}} + \delta_\alpha^\tau F_\sigma^{(1)} \frac{\partial L}{\partial F_\alpha^{(1)}} + (\delta_\alpha^\tau F_{\sigma\beta}^{(1)} + \delta_\beta^\tau F_{\alpha\sigma}^{(1)}) \frac{\partial L}{\partial F_{\alpha\beta}^{(1)}} = 0. \tag{19.15}$$

If $n = 2$ we have from case (a) and Theorem (19.II) with regard to the order $q$ that the four equations (19.15) are all independent. To establish case (c) for $n > 2$ we shall prove the following theorem.

THEOREM (19.III). *If the equations* (19.15) *are all independent for the general form G in n variables $x^i$ they are all independent for the general form G in $(n+1)$ variables $x^i$.*

From our hypothesis there exists a determinant $\Theta$ of order $n^3$ formed from the coefficients of

$$\frac{\partial L}{\partial H_{\alpha\beta}}; \quad \frac{\partial L}{\partial F_\alpha^{(1)}}; \quad \frac{\partial L}{\partial F_{\alpha\beta}^{(1)}} \tag{19.16}$$

| | $\binom{1}{1}\binom{1}{2}\cdots\binom{1}{n}$ | $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\binom{2}{2}\cdots\binom{2}{n}$ | $\rightarrow$ | $\left[\begin{smallmatrix}1\\w+1\end{smallmatrix}\right]\cdots\left[\begin{smallmatrix}w\\w+1\end{smallmatrix}\right]\binom{w+1}{w+1}\cdots\binom{w+1}{n}$ | $\left[\begin{smallmatrix}1\\w+2\end{smallmatrix}\right]\cdots\left[\begin{smallmatrix}w\\w+2\end{smallmatrix}\right]\binom{w+2}{w+2}\cdots\binom{w+2}{n}$ | $\rightarrow$ | $\left[\begin{smallmatrix}1\\n\end{smallmatrix}\right]\cdots\left[\begin{smallmatrix}w\\n\end{smallmatrix}\right]\binom{n}{n}$ |
|---|---|---|---|---|---|---|---|
| $(1,1)$ | $2\ 0\ \cdots\ 0$ | | | | | | |
| $(1,2)$ | $0\ 1\ \cdots\ 0$ | | | | | | |
| $\vdots$ | | | | | | | |
| $(1,n)$ | $0\ 0\ \cdots\ 1$ | | | | | | |
| $(2,1)$ | | $1\ 0\ \cdots\ 0$ | | | | | |
| $(2,2)$ | $\star$ | $0\ 2\ \cdots\ 0$ | | | | | |
| $\vdots$ | | | | | | | |
| $(2,n)$ | | $0\ 0\ \cdots\ 1$ | | | | | |
| $\downarrow$ | | | $\searrow$ | | | | |
| $(w+1,1)$ | | | | $1\ \cdots\ 0\quad 0\ \cdots\ 0$ | | | |
| $\vdots$ | | | | | | | |
| $(w+1,w)$ | $\star$ | $\star$ | | $0\ \cdots\ 1\quad 0\ \cdots\ 0$ | | | |
| $(w+1,w+1)$ | | | | $0\ \cdots\ 0\quad 2\ \cdots\ 0$ | | | |
| $\vdots$ | | | | | | | |
| $(w+1,n)$ | | | | $0\ \cdots\ 0\quad 0\ \cdots\ 1$ | | | |
| $(w+2,1)$ | | | | | $1\ \cdots\ 0\quad 0\ \cdots\ 0$ | | |
| $\vdots$ | | | | | | | |
| $(w+2,w)$ | $\star$ | $\star$ | | $\star$ | $0\ \cdots\ 1\quad 0\ \cdots\ 0$ | | |
| $(w+2,w+2)$ | | | | | $0\ \cdots\ 0\quad 2\ \cdots\ 0$ | | |
| $\vdots$ | | | | | | | |
| $(w+2,n)$ | | | | | $0\ \cdots\ 0\quad 0\ \cdots\ 1$ | | |
| $\downarrow$ | | | | | | $\searrow$ | |
| $(n,1)$ | | | | | | | $1\ \cdots\ 0\quad 0$ |
| $\vdots$ | $\star$ | $\star$ | | $\star$ | $\star$ | | |
| $(n,w)$ | | | | | | | $0\ \cdots\ 1\quad 0$ |
| $(n,n)$ | | | | | | | $0\ \cdots\ 0\quad 2$ |

Table VI.

in the equations (19.15) for the general form $G$ in $n$ variables $x^i$ which does not vanish identically in the independent components $H_{\alpha\beta}$, $F^{(1)}_{\alpha}$, and $F^{(1)}_{\alpha\beta}$, i. e.,

$$\Theta \neq 0.$$

For the form $G$ in $(n+1)$ variables $x^i$ let us now put

$$H_{\alpha,n+1} = H_{n+1,\alpha} = \begin{cases} 1 & \text{if } \alpha = n+1, \\ 0 & \text{if } \alpha \neq n+1, \end{cases}$$

$$F^{(1)}_{n+1} = 0, \quad F^{(1)}_{\alpha,n+1} = F^{(1)}_{n+1,\alpha} = \begin{cases} 1 & \text{if } \alpha = n+1, \\ 0 & \text{if } \alpha \neq n+1 \end{cases}$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | $\Theta$ | | | | | |
| | | | | | * | * | ... | * | 0 |
| | | | | | * | * | ... | * | 0 |
| | | | | | * | * | ... | * | 0 |
| 0 | 0 | ... | 0 | | $\alpha$ | | $\beta$ | | 0 |
| 0 | 0 | ... | 0 | | | | | | 0 |
| | | | | | $\gamma$ | | $\delta$ | | |
| 0 | 0 | ... | 0 | | | | | | 0 |
| * | * | ... | * | | 0 | 0 | ... | 0 | 2 |

Table VII.

and consider the remaining independent components $H_{\alpha\beta}$, $F^{(1)}_{\alpha}$, and $F^{(1)}_{\alpha\beta}$ to be arbitrary. Then it is possible to form a non-vanishing determinant of order $(n+1)^2$ from the coefficients of the quantities (19.16) in the equations (19.15) for the form $G$ in $(n+1)$ variables $x^i$. This determinant is the determinant of the elements in Table VII. The rows of Table VII correspond to a particular pair of values $(\tau, \sigma)$ and the columns to the coefficients of the quantities (19.16). Asterisks are used to denote elements which have been left undetermined. The table is so arranged that in the upper left hand corner there is the above determinant $\Theta$ which does not vanish identically. Consequently the first $n^2$ rows correspond to values $\tau, \sigma = 1, \cdots, n$ and the first $n^2$ columns to coefficients of derivatives (19.16) depending only on subscripts having values $1, \cdots, n$. The remaining rows from top to bottom in Table VII are taken to correspond to the values $\tau = n+1$;

$\sigma = 1, \cdots, n$ and $\tau = 1, \cdots, n+1$; $\sigma = n+1$ while the remaining columns from left to right are taken to correspond to derivatives

$$\frac{\partial L}{\partial H_{i\,n+1}} \quad (i = 1, \cdots, n) \quad \text{and} \quad \frac{\partial L}{\partial F^{(1)}_{i\,n+1}} \quad (i = 1, \cdots, n+1).$$

With this selection of rows and columns there are four non-vanishing determinants labeled $\alpha$, $\beta$, $\gamma$ and $\delta$ in the lower right hand part of Table VII which are given by

$$\alpha = \begin{vmatrix} H_{11}, & \cdots, & H_{n1} \\ & & \\ H_{1n}, & \cdots, & H_{nn} \end{vmatrix}; \quad \beta = \begin{vmatrix} F^{(1)}_{11}, & \cdots, & F^{(1)}_{n1} \\ & & \\ F^{(1)}_{1n}, & \cdots, & F^{(1)}_{nn} \end{vmatrix}; \quad \gamma = \delta = \begin{vmatrix} 1, & 0, & \cdots, & 0 \\ 0, & 1, & \cdots, & 0 \\ & & & \\ 0, & 0, & \cdots, & 1 \end{vmatrix}.$$

The determinant formed from the elements of Table VII has the value

$$2\Theta \begin{vmatrix} H_{11} - F^{(1)}_{11}, & \cdots, & H_{n1} - F^{(1)}_{n1} \\ & & \\ H_{1n} - F^{(1)}_{1n}, & \cdots, & H_{nn} - F^{(1)}_{nn} \end{vmatrix}$$

and so does not vanish identically. Hence all equations (19.15) are independent for the general form $G$ in $(n+1)$ variables $x^i$. This proves Theorem (19.III) and completes the discussion of case (c).

20. **Fundamental sets of parameters.** The number of independent components

$$H_{\alpha\beta};\ H_{\alpha\beta,\gamma_1\gamma_2};\ \cdots;\ H_{\alpha\beta,\gamma_1\cdots\gamma_p};\ F^{(k)};\ F^{(k)}_{\alpha_1};\ \cdots;\ F^{(k)}_{\alpha_1\cdots\alpha_q}$$

where $k = 1, \cdots, w$ is given by

$$D(p, q, w, n) = \sum_{\alpha=0}^{p} N(n, \alpha) + w \sum_{\alpha=0}^{q} K(n, \alpha)$$

in which the quantities $N(n, \alpha)$ and $K(n, \alpha)$ are defined in § 9. Hence

$$\mathfrak{D}(p, q, w, n) = D(p, q, w, n) - n^2$$

gives the number of functionally independent solutions of the equations (19.5) except for the combinations*

* The combination $n \geqq 2$, $p = 0$, $q = 0$, $w \geqq 1$ need not be considered since there exist no parameters of order $(0, 0)$ which depend on the coefficients of the form $G$. See also last sentence in § 3.

$$\text{A:}\quad n=2,\quad p=2,\quad q=0,\quad w\geqq 1,$$
$$\text{B:}\quad n\geqq 2,\quad p=0,\quad q=1,\quad w<n-1.$$

For the combinations A and B the number of independent solutions is given by

$$\mathfrak{D}(2,0,w,2)=D(2,0,w,2)-3=w+1$$

and

$$\mathfrak{D}(0,1,w,n)=D(0,1,w,n)-S$$

respectively.

It can be shown by a consideration similar to that of § 13 that the number $D(p,q,w,n)$ is the number of differential parameters of order $(p,q)$ as defined in the following manner.

*Definition.* A fundamental set of absolute scalar differential parameters of order $(p,q)$ is a set of functionally independent absolute scalar differential parameters each of which is of order $(p,q)$ such that any other one of order $(p,q)$ is expressible as a function of the members of the set.

As thus defined the above fundamental set of differential parameters corresponds to the first fundamental set of scalar invariants of § 13. In considering the work of other writers a definition of a second fundamental set of parameters of order $(p-1,p)$ is given by us in the introduction of this paper. It does not seem to be necessary to attempt to extend this definition to the general case.

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